

Math 221 – Exam III (50 minutes) – Friday April 19, 2002

Answers

I. (10 points.) Fill in the boxes so as to complete the following statement:

A definite integral can be approximated by a Riemann sum. More precisely, if

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

$$\Delta x_k = \boxed{x_k - x_{k-1}},$$

$$\boxed{x_{k-1}} \leq c_k \leq \boxed{x_k},$$

and

$$\boxed{\max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}} \approx 0,$$

then

$$\sum_{k=1}^n f(c_k) \Delta x_k \approx \int_a^b f(x) dx.$$

(This problem was on quiz 5.)

*Grader's Comments*

*I think that though students may understand the concept of Riemann sum well, there is a gap between their understanding of it and their use of notations. I felt many answers came from the memorization of the answers from the quiz 5. Some answers suggest confusion in understanding of the meaning of  $x_k$ ,  $c_k$ ,  $\Delta$ , etc. In other words, they may understand what's going on, but they don't understand what  $x_k$ ,  $c_k$ , and so on, represent in the definition.*

II. (20 points.) A continuous function  $f$  satisfies

$$f(1) = 3, \quad f(1.6) = 6.7, \quad f(2.8) = 11.2, \quad f(3.3) = 9.9, \quad f(4) = 3,$$

$f(x)$  is increasing for  $1 \leq x \leq 2.8$  and  $f(x)$  is decreasing for  $2.8 \leq x \leq 4$ . Find a Riemann sum  $S$  such that

$$3 < S < \int_1^4 f(x) dx.$$

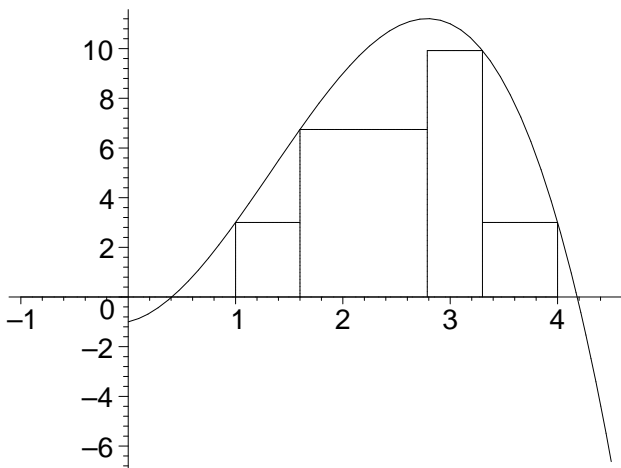
Sketch a possible graph and also draw the area represented by your Riemann sum. (You should leave the addition and multiplication undone so as to make your work easier to grade.)

**Answer:** Take the partition

$$x_0 = 1 < x_1 = 1.6 < x_2 = 2.8 < x_3 = 3.3 < x_4 = 4$$

of the interval  $[1, 4]$  and in each interval  $[x_{k-1}, x_k]$  let  $c_k$  be the point where  $f(x)$  is smallest. Thus  $c_1 = x_0 = 1$ ,  $c_2 = x_1 = 1.6$ ,  $c_3 = x_3 = 3.3$  and  $c_4 = x_4 = 4$ . The Riemann sum is

$$\begin{aligned} S &= \sum_{k=1}^4 f(c_k)(x_k - x_{k-1}) \\ &= f(1)(1.6 - 1) + f(1.6)(2.8 - 1.6) + f(3.3)(3.3 - 2.8) + f(4)(4 - 3.3) \\ &= 4(1.6 - 1) + 6.7(2.8 - 1.6) + 9.9(3.3 - 2.8) + 3(4 - 3.3) \end{aligned}$$



*Grader's Comments*

*Most students did well on this problem, having seen it before on a quiz. However, every once in a while i'd see such atrocities as rectangles floating above the x-axis, or worse, people not knowing how to compute areas of rectangles.*

**III.** (30 points.) Evaluate the integral:

(1)  $\int x \sin(2x^2) dx$ .

**Answer:** Let  $u = 2x^2$  so  $du = 4x dx$ . Then

$$\int x \sin(2x^2) dx = \int \sin(u) \frac{du}{4} = \frac{\cos(u)}{4} + C = \frac{\cos(2x^2)}{4} + C.$$

(This is problem 3 on page 187.)

(2)  $\int_0^2 \sqrt{4x+1} dx$ .

**Answer:** Let  $u = 4x + 1$ . Then  $du = 4 dx$ ,  $u = 1$  when  $x = 0$ , and  $u = 9$  when  $x = 2$ . Then

$$\int_0^2 \sqrt{4x+1} dx = \int_1^9 \sqrt{u} \frac{du}{4} = \frac{u^{3/2}}{6} \Big|_1^9 = \frac{27}{6} - 0.$$

(This is problem 9 on page 210.)

*Grader's Comments*

*I graded 15 points each part. On the second integral I took 3 points off for doing the substitution without changing the limits of integration, and 3 more if they actually use them to evaluate the integral. I also took points for missing factors of integration. Some students gave the answer right away, I took 1 point for this, giving them the benefit of doubt.*

**IV.** (20 points.) Evaluate the derivative:

(1)  $F(x) = \int_1^x \frac{dt}{t}$ .       $F'(x) = ?$

**Answer:** By the Fundamental Theorem

$$F'(x) = \frac{1}{x}.$$

(This is problem 19 on page 210.)

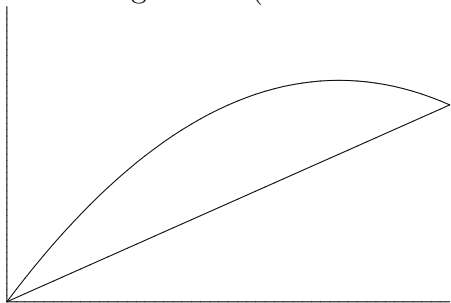
(2)  $G(x) = \int_1^{2x} \cos(t^2) dt$ .       $G'(x) = ?$

**Answer:** By the Fundamental Theorem and the Chain Rule

$$G'(x) = \cos((2x)^2) \cdot 2 = 2 \cos(4x^2).$$

(This is problem 22 on page 210.)

**V.** (30 points.) Find the volume generated when the area in the first quadrant bounded by the parabola  $y = 3x - x^2$  and the line  $y = x$  is rotated about the  $x$ -axis. Set up a definite integral for the answer. Do not evaluate the integral. Do specify the limits of integration. (Note: the  $x$ -axis is the line  $y = 0$ .)



**Answer:** The curves intersect at  $(x, y) = (0, 0)$  and at  $(x, y) = (2, 2)$ . Using the method of washers with  $f_1(x) = 3x - x^2$  and  $f_2(x) = x$  we get

$$dV = \pi(f_1(x)^2 - f_2(x)^2) dx = \pi((3x - x^2)^2 - x^2) dx$$

so

$$V = \int_0^2 \pi((3x - x^2)^2 - x^2) dx.$$

(This is problem 3 on page 245.)

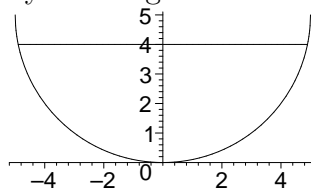
*Grader's Comments*

*This question was pretty well handled. Most students were aware that the problem required using washers, and almost everyone found the correct limits of integration. Attempted solutions that presented the correct limits of integration, but involved mysterious and unexplained integrands received only 15 points. The commonest error in writing down the integrand arose from students confusing  $\pi(r_{outer}^2 - r_{inner}^2)$  with  $\pi(r_{outer} - r_{inner})^2$ .*

**VI.** (40 points.) A hemispherical bowl is obtained by rotating the semicircle

$$x^2 + (y - a)^2 = a^2, \quad y \leq a$$

about the  $y$ -axis. It is filled with water to a depth of  $h$ , i.e. the water level is the line  $y = h$ .



(1) Find the volume of the water in the bowl as a function of  $h$ . (Set up a definite integral for the answer. Do not evaluate the integral. Do specify the limits of integration.)

**Answer:** The right half of the semicircle has equation

$$x = \sqrt{a^2 - (y - a)^2} = \sqrt{2ay - y^2}.$$

Using the method of disks

$$dV = \pi(\sqrt{2ay - y^2})^2 dy = \pi(2ay - y^2) dy$$

so

$$V = \int_0^h \pi(2ay - y^2) dy.$$

(2) Water runs into a hemispherical bowl of radius 5 ft at the rate of  $0.2 \text{ ft}^3/\text{sec}$ . How fast is the water level rising when the water is 4 ft deep? (Hint: Use the method of related rates and the Fundamental Theorem.)

**Answer:** By the previous formula with  $a = 5$  we have

$$V = \int_0^h \pi(10y - y^2) dy.$$

By the Fundamental Theorem

$$\frac{dV}{dh} = \pi(10h - h^2).$$

We are given that

$$\frac{dV}{dt} = 0.2.$$

By the chain rule

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt},$$

so

$$\left. \frac{dh}{dt} \right|_{h=4} = \left. \frac{dV/dt}{dV/dh} \right|_{h=4} = \frac{0.2}{\pi(40 - 16)}.$$

(This is problem 11 on page 239, but the wording has been changed to suggest how to set up the problem. Here is the original wording: (1) A hemispherical bowl of radius  $a$  contains water to a depth  $h$ . Find the volume of the water in the bowl. (2) Water runs into a hemispherical bowl of radius 5 ft at the rate of 0.2 ft<sup>3</sup>/sec. How fast is the water level rising when the water is 4 ft deep?)

*Grader's Comments*

*This question was a disaster. Perhaps I shouldn't have changed the wording from the book. I thought I was making the problem easier by setting up the equations. Also I will ask a question like this on the final (with similar wording) where the integral cannot be done explicitly and the student must use the Fundamental Theorem. Here are some of the comments I wrote on student papers.*

- *$h$  is a free variable, not the dummy variable (variable of integration).*
- *Are you using shells or disks? (I can't tell.)*
- *Rotate about  $y$ -axis not the  $x$ -axis.*
- *Use  $h, a$  in part (1),  $h = 4, a = 5$  in part (2).*
- *Incorrect limits of integration.*
- *$x$  should not appear in the integrand if  $y$  is the dummy variable.*
- *Do not write  $\int$  without  $d$ .*
- *Incorrect use of notation: finite = infinitesimal.*
- *The limits of integration depend on  $h$ .*
- *$\sqrt{P^2 + Q^2} \neq P + Q$ .*