

Answers

I. (40 points.) One of the three parts of the Monotonicity Theorem says what happens when the derivative of a function is positive on an interval. State and prove it. In your proof you may use (without proof) the Mean Value Theorem.

Answer: Theorem. *If $f'(x) > 0$ for all x in an interval I , then f is increasing on that interval, i.e.*

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for any two points x_1, x_2 of the interval.

Proof. Choose x_1 and x_2 in the interval I with $x_1 < x_2$. By the Mean Value Theorem there is a c with $x_1 < c < x_2$ and

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c). \quad (\#)$$

Since c is between x_1 and x_2 it lies in the interval I and hence $f'(c) > 0$. Hence the ratio on the left in equation (#) is positive. Since $x_1 < x_2$ the denominator $x_2 - x_1$ is positive and hence the numerator $f(x_2) - f(x_1)$ is positive, i.e. $f(x_2) - f(x_1) > 0$. Thus $f(x_1) < f(x_2)$. This theorem and proof appeared (in slightly more generality) on the handout entitled *Possible questions for Exam II*.

Grader's comments

The responses on this question ranged from none at all to utter confusion to perfectly clear and correct. I suppose some students did not get the handout or come to the lecture where I told the students what to expect. Some students proved the wrong theorem indicating (to me at least) that they were memorizing rather than understanding. Other students wrote misguided things like f' is increasing rather than f is increasing; perhaps these were just dumb mistakes and these students really understand. I did not penalize the following two common errors heavily:

- (i) *Not saying that there is a number c between x_1 and x_2 satisfying (#), i.e. leaving the impression that the Mean Value Theorem holds for all c .*
- (ii) *Saying that the interval was $a \leq x \leq b$ and proving that $f(a) < f(b)$. This mistake could be defended by pointing out that since the proof works for all intervals, the result is the same.*

I don't know if asking students to produce proofs on an exam has any value. My opinion is that students have trouble reading and learning mathematics on their own and this kind of question will force them to read and write carefully. These are two of the principal objectives of a college education.

II. (20 points.) Evaluate the following limits. If the limit does not exist write DNE and say why. Distinguish between a limit which is infinite and one which does not exist.

(i) $\lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\pi - \theta}$.

Answer: Plugging in gives the indeterminate form $0/0$ so by l'Hôpital's Rule

$$\lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\pi - \theta} = \lim_{\theta \rightarrow \pi} \frac{\cos \theta}{-1} = 1.$$

This also follows directly from the definition of the derivative. Since $\sin \pi = 0$ we have

$$\lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\pi - \theta} = - \lim_{\theta \rightarrow \pi} \frac{\sin \theta - \sin \pi}{\theta - \pi} = - \sin'(\pi) = - \cos(\pi) = 1.$$

(This is Problem 11 on page 165.)

(ii) $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.

Answer: Plugging in gives the indeterminate form $\infty - \infty$ so we need to do some high school algebra first.

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x}.$$

Now plugging in gives the indeterminate form $0/0$ so by l'Hôpital's Rule twice

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2 - 0} = 0.$$

(This is Example 7 on page 165.)

Grader's comments

The most common mistakes I saw were 1) the limit of a quotient equals the quotient of the limits, despite the fact that both the numerator and denominator tend to zero; 2) not connecting statements with equal signs; 3) saying $0/2$ is infinite.

III. (30 points.) A point moves along the curve $y^2 = x^3$ in such a way that its distance from the origin increases at a constant rate of two units per second. Find dx/dt at the point $(x, y) = (2, 2\sqrt{2})$.

Answer: Call the distance from the origin r . Then $r = \sqrt{x^2 + y^2}$ by the Pythagorean Theorem and we are given that $dr/dt = 2$ for all t . Since $y = x^{3/2}$ near the point $(x, y) = (2, 2\sqrt{2})$ we have

$$r = \sqrt{x^2 + x^3}.$$

By the Chain Rule

$$\frac{dr}{dt} = \frac{(2x + 3x^2)}{2\sqrt{x^2 + x^3}} \cdot \frac{dx}{dt}$$

so

$$\frac{dx}{dt} = \frac{2\sqrt{x^2 + x^3}}{(2x + 3x^2)} \cdot \frac{dr}{dt}$$

and hence

$$\left. \frac{dx}{dt} \right|_{x=2} = \frac{2\sqrt{4+8}}{(4+12)} \cdot 2 = \frac{\sqrt{3}}{2}.$$

(This is problem 23 on page 172.)

Grader's comments

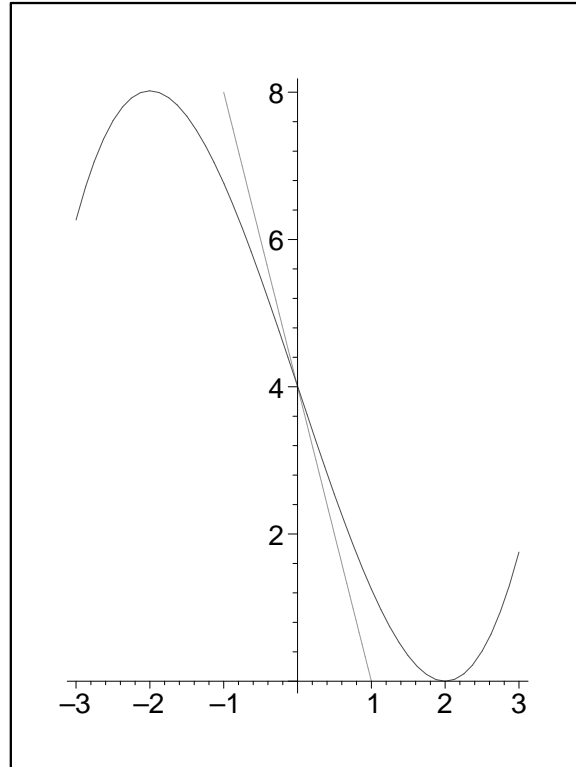
Only few students solved this problem correctly. Most students did not use the distance formula and started differentiating both sides of the equation $y^2 = x^3$. As a result, the average score on this problem was very low.

Professors's Response

Some students complained that they did not know I would take problems from the miscellaneous problems at the end of each chapter, but this is clearly stated in the policy statement of the syllabus.

IV. (20 points.) Sketch the graph $y = f(x)$ of a function f which is twice continuously differentiable, and has the following characteristics: $f'(x) < 0$ for $|x| < 2$, $f'(x) > 0$ for $|x| > 2$, $f''(x) < 0$ for $x < 0$, $f''(0) > 0$ for $x > 0$, $f(-2) = 8$, $f(0) = 4$, $f(2) = 0$. Draw the tangent line at each point of inflection. (That f is *twice continuously differentiable* means that the second derivative $f''(x)$ is defined for all x and is continuous.)

Answer:



The function plotted is $f(x) = 4(x^3/3 - 4x + 16/3)/3$ which has all the above properties, but the only three points that we know for sure are on the graph of the problem as stated are the three given points $(-2, 8)$, $(0, 4)$, and $(2, 0)$. (This is problem 19 from page 138.)

V. (30 points.) An isosceles triangle is drawn with its vertex at the origin and its other two vertices symmetrically placed on the curve $12y = 36 - x^2$ and above the x -axis. Determine the area of the largest such triangle.

Answer: The vertices of the triangle are at the points

$$(x, y) = (0, 0), \left(x, \frac{36 - x^2}{12}\right), \left(-x, \frac{36 - x^2}{12}\right).$$

Hence the base is $2x$, the altitude is $(36 - x^2)/12$, so the area is

$$A = \frac{x(36 - x^2)}{12} = 3x - \frac{x^3}{12}.$$

Clearly $A(0) = A(6) = 0$ and the problem asks us to consider only those triangles with $(36 - x^2)/12 \geq 0$. i.e. $0 \leq x \leq 6$. Hence $A(x) \geq 0$ so the maximum occurs at an interior point so at the maximum

$$0 = \frac{dA}{dx} = 3 - \frac{x^2}{4}.$$

Therefore the maximum occurs at $x = \sqrt{12}$ and the maximum value is

$$A(\sqrt{12}) = \frac{\sqrt{12}(36 - 12)}{12}.$$

(This is problem 40 page 172.)

Grader's comments

The problem most students had was to set up a (useful) formula for the area of the triangle. Many students didn't get to this point, but the ones who did were able (usually) to get the relevant critical point and the maximal area asked for (a common mistake was to get just half the area, because they set up the problem like that). What very few students did was explain why the maximum (exists and) occurs when the derivative was zero. I took off 5 points for this lack of explanation, which I think serious.

Professors's Response

The problem would have been much easier if I had provided a picture.

VI. (30 points.) Use the Mean Value Theorem to estimate $127^{1/3} - 5$. Hint: $5^3 = 125$. The word *estimate* means to find a number which you can prove is bigger but is still relatively small.

Answer: Let $f(x) = x^{1/3}$ so $f(125) = 5$. By the Mean Value Theorem

$$\frac{127^{1/3} - 5}{2} = \frac{f(127) - f(125)}{127 - 125} = f'(c)$$

for some c satisfying $125 < c < 127$. Since $125 < c$ we have

$$f'(c) = \frac{c^{-2/3}}{3} < \frac{125^{-2/3}}{3} = \frac{1}{75}$$

Hence

$$0 < 127^{1/3} - 5 = f(127) - f(125) = 2f'(c) < \frac{2}{75}.$$

(This problem is like Problem 7 page 159. Something like this was done in lecture when we showed that $|\sin(\pi/7) - 1/2| < \pi/42$.)

Grader's comments

A small, but statistically significant, minority of students did very well on this problem, scoring between 28 and 30 on it. A large majority, however, handled the problem poorly.

In my opinion, what students should receive credit for is writing the out the correct expression for the linear approximation around a of $f(b)$, where b is close to a . Yet, every student who gave a reasonably legible statement of the mean value theorem, whether or not they explained the meanings of f , a , b and c that occur in the statement very clearly, got 3 points right away.

Students who were able to write down an expression of f relevant to the problem, the values of a and b compatible to this choice of f AND were able to differentiate f correctly received 15 points straight away. The absence of some item from this list, that WASN'T implicitly supplied later on, was penalized heavily. In hindsight, I feel that the heavy penalty was justified, because a lot of students stated choices for f , a , and b that were simply not compatible - thus betraying a lack of understanding of what was required.

Two final observations : First, a small minority of students realized that the value of $f'(a)$, for the appropriate a , was important; but merely putting down a disembodied $f'(a)$, bearing no connection to the rest of the solution, is NOT likely to fetch too many points in partial credit. Secondly, I think that the way in which Thomas & Finney spoon-feeds its audience on problems like VI - literally spelling out the recommended f , a and b , may be the reason why so many students were ill-prepared to tackle this problem.

VII. (30 points.) (i) Write the polynomial of degree four which best approximates $\cos t$ for t near zero. Make explicit the general formula you are using.

Answer: This is the Taylor Polynomial

$$P(t) = \sum_{k=0}^4 \frac{f^{(k)}(0)t^k}{k!}$$

of degree four about zero. We evaluate the coefficients by differentiating:

$$f^{(0)}(t) = \cos t, \quad f^{(1)}(t) = -\sin t, \quad f^{(2)}(t) = -\cos t,$$

$$f^{(3)}(t) = \sin t, \quad f^{(4)}(t) = \cos t,$$

evaluating:

$$f^{(0)}(0) = 1, \quad f^{(1)}(0) = 0, \quad f^{(2)}(0) = -1,$$

$$f^{(3)}(0) = 0, \quad f^{(4)}(0) = 1,$$

and plugging in:

$$P(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24}.$$

(Continued on next page)

(ii) Evaluate $\lim_{t \rightarrow 0} \frac{\cos t - 1}{t^2}$.

Answer: We could use l'Hôpital's Rule but it is more instructive to use Taylor's Theorem. Taylor's Theorem tells us that

$$\lim_{t \rightarrow 0} \frac{\cos t - P(t)}{t^4} = 0$$

where

$$P(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24}$$

is the Taylor polynomial we computed in part (i). This says that $\cos t$ and $P(t)$ agree to order four near zero and hence certainly to order two. This means we can replace $\cos t$ by its Taylor approximation, i.e.

$$\lim_{t \rightarrow 0} \frac{\cos t - 1}{t^2} = \lim_{t \rightarrow 0} \frac{P(t) - 1}{t^2} = \lim_{t \rightarrow 0} \frac{1}{t^2} \cdot \left(-\frac{t^2}{2} + \frac{t^4}{24} \right) = -\frac{1}{2}.$$

(This is Problem 11 on page 165.)

Grader's Comments

Many students didn't write a correct formula for the Taylor polynomial, especially in the general case; some of them did, but applied it incorrectly for the case at hand, namely $n = 4$, $a = 0$. There was a lot of confusion of variables, to the point that the derivative of $\cos x$ was $-\sin t$. Part b) was better, I have the impression that most of them got it right, notational problems were present to the same extent, for example, not writing the limit, I took a couple of points for that.

----- Wed Mar 20 15:51:48 2002 --

There are 216 scores

grade	range	count	percent
A	155...200	24	11.1%
AB	145...154	13	6.0%
B	120...144	40	18.5%
BC	105...119	23	10.6%
C	80...104	50	23.1%
D	60... 79	37	17.1%
F	0... 59	29	13.4%

Mean score = 103.1. Mean grade = 2.11.

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