Math 221 – Exam I (50 minutes) – Friday Feb 22, 2002

Answers

I. (30 points.) Find the indicated derivative.

(a) $y = \left(\dfrac{x+1}{x-1}\right)^2$. $\qquad \dfrac{dy}{dx} = ?$

Answer: By the chain rule and the quotient rule

$$\frac{dy}{dx} = 2\left(\frac{x+1}{x-1}\right) \cdot \frac{1(x-1)-(x+1)1}{(x-1)^2}.$$

(This is problem 8 on page 75 of the text.)

(b) $x = \dfrac{t}{1+t}$ and $y = \dfrac{t^2}{1+t}$. $\qquad \dfrac{dy}{dx} = ?$

Answer: By the quotient rule

$$\frac{dx}{dt} = \frac{(1+t)-t}{(1+t)^2} = \frac{1}{(1+t)^2}, \qquad \frac{dy}{dt} = \frac{2t(1+t)-t^2}{(1+t)^2} = \frac{t^2+2t}{(1+t)^2}$$

so by the chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = t^2 + 2t.$$

From $x = 1/(1+t)$ we get $x + tx = t$ and then $t = x/(1-x)$ so

$$\frac{dy}{dx} = \left(\frac{x}{1-x}\right)^2 + 2\left(\frac{x}{1-x}\right).$$

(This is problem 4 on page 93 of the text.)

(c) $y = \sec^2 5x = (\cos 5x)^{-2}$. $\qquad \dfrac{dy}{dx} = ?$

Answer: By the chain rule

$$\frac{dy}{dx} = -2(\cos 5x)^{-3}\frac{d}{dx}\cos(5x) = -2(\cos 5x)^{-3}(-5\sin 5x).$$

(This is Example 4 on page 105 of the text.)

Grader's comments

Some students did not have good understanding of the chain rule and hence wrote completely wrong answers for (a) and (c). When the chain rule *was followed by a non-trivial differentiation as in (a) and (c), many students failed to use appropriate differentiation rules.*

In (b), many students wrote too much which made their work hard to follow. I was also able to see the fact that they don't have to simplify their answers has some undesirable effects:

1. *Many people solved for t in x and substituted that to express y in terms of x and then tried to find dy/dx. But while they were doing that, they did not simplify anything in the process hence applying the quotient rule many times inside the quotient rule. It was very hard to follow and most of those students ended up making mistakes.*

2. *In some cases, there were obvious cancellations and they just simply wouldn't cancel them. Again that made it very hard to follow their answers. Things like* $t(-1 \cdot (1+t^{-1})^{-2})$ *or* $-2(cos5x)^{-3}(-sin5x \cdot 5)$ *as final answers were not visually pleasing to see.*

I think it's a good policy that they don't have to simplify expressions when there's absolutely no benefit of doing that. But I think they should simplify to reasonable extent. I also personally think that they should be capable of simple algebra.

Lastly, some students wrote $-sin$ *in their answer hence exhibited that they do not have good understanding of cosine or sine functions.*

Professor's response

I have wrestled with the problem of how much simplification and what algebraic skills to require ever since I began teaching calculus over thirty years ago. My problem is that I don't know how to word a question which forces simplification. My current approach is to tell students that they needn't simplify unless this is necessary to complete the problem, but I certainly do believe that algebra skills are important. Any ideas?

II. (20 points.) Find an equation for the tangent line to the curve

$$x^2 + xy - y^2 = 1$$

at the point $P_0(2,3)$.

Answer: By implicit differentiation

$$2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = \frac{2x+y}{2y-x}.$$

The slope of the tangent line at P_0 is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)} = \frac{4+3}{6-2} = \frac{7}{4}$$

and the equation of the tangent line is

$$y - 3 = \frac{7}{4}(x-2)$$

(This is Problem 30 on page 83 of the text.)

Grader's Comments

Students did very well on this problem, almost all of them got 20 or a little less in case they had an arithmetical or notational mistake. I assigned 10/20 for finding the derivative and 5/10 for each of finding the slope and the equation. The following error was common: $dy/dx = 2x + y + x\,dy/dx + 2y\,dy/dx = 0$*, I took off a point for that.*

III. (30 points.) Two functions f and g are *inverse functions* if the equations $y = f(x)$ and $x = g(y)$ have the same graph, i.e. if $x = g(f(x))$ and $y = f(g(y))$. What is the inverse function g of the function $f(x) = x^3 + 1$ and what is the derivative of g?

Answer: Since $y = x^3 + 1 \iff x = (y-1)^{1/3}$ The inverse function is $g(y) = (y-1)^{1/3}$ so

$$g'(y) = \frac{(y-1)^{-2/3}}{3}.$$

This is a consequence of the Chain Rule: $g(f(x)) = x$ so $g'(f(x))f'(x) = 1$ so if $y = f(x)$, then

$$g'(y) = g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{3x^2} = \frac{1}{3(y-1)^{2/3}}.$$

(This is Example 3 on page 78.)

Remark. The function g defined by the formula

$$g(y) = (y-1)^{\frac{1}{3}}$$

is the same as the function g defined by the formula

$$g(x) = (x-1)^{\frac{1}{3}}.$$

Grader's Comments

Many students (probably because they were taught to in an earlier course) immediately went from the equation $y = x^3+1$ *to the equation* $x = y^3+1$*. This is contrary to the approach of our text book, the wording of the question, and the approach I took in lecture. After a student writes this the reader cannot know whether* dy/dx *means* $f'(x)$ *or* $g'(x)$*. I deducted three points (I think it is a serious error) and wrote: "Do not write both equations* $y = x^3+1$ *and* $x = y^3+1$ *in the same problem.* $y \neq (y^3+1)^3+1$*." Anticipating a general riot, I only deducted one point from the actual score I wrote on the first page of the exam. It was also common for students to write things like* $g'(y) = (x-1)^{-2/3}/3$ *rather than* $g'(y) = (y-1)^{-2/3}/3$ *or* $g'(x) = (x-1)^{-2/3}/3$ *indicating that they don't understand functional notation.*

IV. (30 points.) State and prove the formula for the derivative of the product of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable function is continuous, and high school algebra.[1]

Answer: Suppose that $f(x) = u(x)v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)} \\
&= \lim_{x\to a} \frac{u(x)\cdot v(x) - u(a)\cdot v(a)}{x-a} && \text{(hypothesis)} \\
&= \lim_{x\to a} \left(\left(\frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(x) \cdot \left(\frac{v(x)-v(a)}{x-a} \right) \right) && \text{(hsa)} \\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(a) \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(limit laws)} \\
&= u'(a)v(a) + v'(a)u(a) && \text{(definition)}
\end{aligned}$$

[1]This is one of five proofs the students were told to prepare.

(In the fourth step the theorem that a differentiable function is continuous is also used.)

Grader's comments

I would say half of the students got the problem essentially correct, even though it was clear that 90% of the time they just memorized the proof, as they wrote down exactly what was on the handout. For the rest, some didn't bother with the proof and most of them tried to reproduce the proof as well as they could remember, and the result was typically having an intermediate step that was completely inconsistent with the rest of the steps, or something like starting to prove the product rule and by some mysterious algebra ending up with the quotient rule, or the addition rule. Nevertheless, there were exceptions to this, and students that seemed to understand what they were doing, and why.

As to the grading, I took off three points if the hypothesis of the two functions being multiplied be differentiable was not mentioned, and seven for each serious inconsistent statement.

Professor's response

Was it as low as half? My philosophy in asking this kind of question is that without some degree of understanding simple memorization will not work. Also it helps us distinguish the serious students from the others.

V. (40 points.) Find the limit. Distinguish between an infinite limit and one which doesn't exist. (Give reasons!)

(a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$

Answer: By a change of the dummy variable, the limit laws, and the theorem on page 101

$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim_{x\to 0} \frac{\sin 3x}{3x} = 3 \lim_{u\to 0} \frac{\sin u}{u} = 3.$$

(b) $\lim\limits_{x\to \infty} \dfrac{\sin 3x}{x}$

Answer: Since $-\dfrac{1}{x} \leq \dfrac{\sin(3x)}{x} \leq \dfrac{1}{x}$, the limit is zero by the Sandwich Theorem.

(c) $\lim\limits_{x\to 0+} \dfrac{\sin 3}{x}$

Answer: $0 < \sin 3$ since $0 < 3 < \pi$. Hence $\lim_{x\to 0+} \frac{\sin 3}{x} = \infty$

(d) $\lim_{h\to 0} \frac{\sin(3+h) - \sin 3}{h}$

Answer: By the definition of the derivative this is $\sin'(3)$, the derivative of the sine function evaluated at 3. In other words,

$$\lim_{h\to 0} \frac{\sin(3+h) - \sin 3}{h} = \cos 3.$$

Remark. This could also be written as

$$\lim_{x\to 3} \frac{\sin x - \sin 3}{x - 3} = \cos 3.$$

Grader's comments

This problem wasn't well handled. The most common errors were

(1) *In part (a) : students writing sin*$(3x) = 3$*sin*(x) *to obtain "the correct answer";*

(2) *In part (c) : students forgetting (or rather, I'd suspect, not realizing) that one needs to check/state that sin*(3) *is positive, yet claiming that the relevant limit is* $+\infty$.

Each part in V was worth 10 points. Although sorely tempted to award a 0/10 for the error (1), I finally decided to award a 2/10 if students committed error (1) BUT stated 3 as the answer to that part. There's the commonplace, routine type of incompetence, and then, there's stunningly gross incompetence. Error (1) falls into the latter class, because each student qualified to register for 221 is supposed to have seen basic trigonometry.

I decided not to penalize error (2) heavily : just 2 points off. A student could come to me and validly argue that s/he did not explicitly state that sin(3) is positive because "it's an obvious fact". But I suspect they won't try to do this for a mere 2 points!

Professor's Comment

I suppose you all know the common trick that you can correctly evaluate 64/16 by cancelling the 6's. The next time I ask a question like this I will use $\sin 4$ *so that the answer is* $-\infty$.

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_______ Tue Feb 26 09:00:48 2002
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There are  223 scores
grade      range       count      percent
   A     135...150     25         11.2%
  AB     120...134     49         22.0%
   B     105...119     40         17.9%
  BC      90...104     40         17.9%
   C      75... 89     28         12.6%
   D      60... 74     16          7.2%
   F       0... 59     25         11.2%
Mean score = 100.6. Mean grade = 2.53.
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    0 *
    7 **
   14 **
   21
   28 *****
   35 ****
   42 ****
   49 ****
   56 ****
   63 *********
   70 *********
   77 ************
   84 **************
   91 ****************
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  105 ****************
  112 ******************
  119 **********************
  126 ************************
  133 ********************
  140 ************
  147 **
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