Calculus 221 Exam
Friday November 19, 1999

Answers

I. (50 points.) Consider the function $f(x) = (x^2 - 2x)e^x$.

(a) Find $f'(x)$ and $f''(x)$. (Hint: Simplify your answers so the rest of the problem is easy. Don't forget the product rule.)

Answer:

$$f'(x) = (2x-2)e^x + (x^2-2x)e^x = (x^2-2)e^x$$
$$f''(x) = 2xe^x + (x^2-2)e^x = (x^2+2x-2)e^x.$$

(b) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Answer:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(x^2-2x)\cdot\lim_{x\to\infty} e^x = \infty\cdot\infty = \infty,$$

and by l'Hopital's rule twice:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x) = \lim_{x\to\infty}\frac{x^2+2x}{e^x} = \lim_{x\to\infty}\frac{2x+2}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$$

(c) Find all points x where the tangent line to the graph $y = f(x)$ is horizontal.

Answer: $f'(x) = 0 \iff x = \pm\sqrt{2}$.

(d) Find all inflection points x for the graph $y = f(x)$.

Answer:

$$f''(x) = 0 \iff x^2 + 2x - 2 = 0 \iff x = \frac{-2\pm\sqrt{4+8}}{2} = -1\pm\sqrt{3}.$$

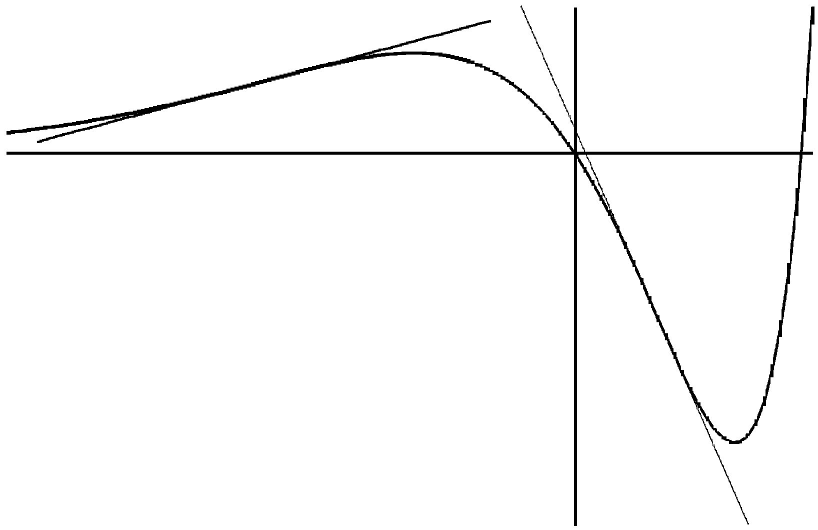
(I continued) **(e)** On which intervals is the function f increasing?

Answer: $f'(x) > 0 \iff x < -\sqrt{2}$ or $x > \sqrt{2}$.

(f) On which intervals is the function f concave up? **Answer:** $f''(x) > 0 \iff x < -1-\sqrt{3}$ or $x > -1+\sqrt{3}$.

(g) Sketch the graph on the axes. Indicate the point(s) of inflection. (Your graph need not be drawn to scale.)

Answer:



II. (30 points.) A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. How high does it go? When does it hit the ground? Hint: The acceleration $a = -32$ ft /s^2 is constant. Be sure your answer is expressed in terms of height above the ground and not height above the cliff.

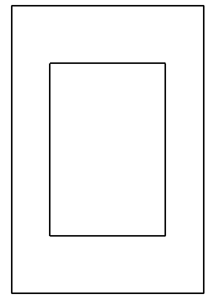
Answer: Let y be the height of the ball above the ground at time t, $v = \dfrac{dy}{dt}$ be the velocity of the ball at time t and $a = \dfrac{dv}{dt} = \dfrac{d^2y}{dt^2}$ be the acceleration. Then

$$v = at + v_0, \qquad v_0 = 48, \qquad a = -32$$

$$y = \frac{at^2}{2} + v_0 t + y_0, \qquad y_0 = 432.$$

At the highest point $v = \dfrac{dy}{dt} = 0$ so $t = 48/32 = 3/2$ so $y = -16(3/2)^2 + 48(3/2) + 432 = -36 + 72 + 432 = 468$. It hits the ground when $y = 0$ (and $t > 0$) so $-16t^2 + 48t + 432 = 0$ so $t^2 - 3t - 27 = 0$ so $t = (3 + \sqrt{9 + 108})/2$.

III. (30 points.) The top and bottom margins of a poster are each 3 inches tall and the side margins are each 2 inches wide. If the area of the printed material is to be 200 square inches find the dimensions of the poster of least area. (Hint: The margin is the part of the poster where there is no printing.)



Answer: Let x be the width of the printed matter so the height of the printed matter is $200/x$. Then the area of the poster is

$$f(x) = (4+x)\left(6+\frac{200}{x}\right) = 224 + 6x + \frac{800}{x}.$$

The derivative is $f'(x) = 6 - 800/x^2$ which vanishes at $x = \sqrt{\frac{800}{6}} = 20/\sqrt{3}$. Since $\lim_{x\to 0+} f(x) = \lim_{x\to\infty} f(x) = \infty$, this critical point is the minimum. The width of the poster is $4 + x = 4 + 20/\sqrt{3}$ and the height is $6 + 200/x = 6 + 10\sqrt{3}$.

IV. (20 points.) Express $\int_1^2 x^2\,dx$ as a limit of Riemann sums and evaluate this limit. Hint:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{ik=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Answer: Take $f(x) = x^2$ and $x_i^* = x_i = 1 + i/n$ so $x_i - x_{i-1} = 1/n$ and $x_0 = 1$, $x_n = 2$. Then

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\frac{1}{n} = \sum_{i=1}^{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\frac{1}{n} =$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n} 1\right) + \frac{2}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{1}{n^2}\left(\sum_{i=1}^{n} i^2\right) =$$

$$= \frac{1}{n}(n) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) =$$

$$= 1 + \frac{n+1}{n} + \frac{(n+1)(2n+1)}{6n^2}$$

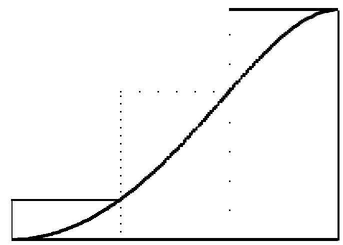
Hence

$$\int_1^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\frac{1}{n} = \lim_{n\to\infty}\left(1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2}\right) = \frac{7}{3}.$$

V. (20 points.) **(a)** Let $f(x) = x^7 + x^3 - 1$. How many solutions does the equation $f(x) = 0$ have? (You must justify your answer.)

Answer: $f'(x) = 7x^6 + 3x^2 > 0$ except when $x = 0$. Therefore $f(x)$ is strictly increasing so has at most one zero. Since $f(0) = -1$ and $f(1) = 1$ there is a zero between 0 and 1 by the Intermediate Value Theorem.

(b) The function $f(x) = \sin(x^2)$ is increasing for $0 \le x \le \sqrt{\pi/2}$. Write a Riemann sum with three terms which is bigger than $\int_0^{\sqrt{\pi/2}} \sin(x^2)\, dx$. (There are many correct answers.) Hint: "three terms" means $n = 3$.



Answer: Take $x_0 = 0$, $x_1 = (\sqrt{\pi/2})/3$, $x_2 = (2\sqrt{\pi/2})/3$, $x_3 = \sqrt{\pi/2}$. Then

$$\int_0^{\sqrt{\pi/2}} f(x)\, dx < \sum_{i=1}^{3} f(x_i)(x_i - x_{i-1}) = \left(\sin(\tfrac{\pi}{18}) + \sin(\tfrac{4\pi}{18}) + \sin(\tfrac{\pi}{2})\right)\frac{(\sqrt{\pi/2})}{3}.$$