Calculus 221 Exam, Tuesday October 26, 1999, 5:30-7 PM

I. (30 Points.) Find the indicated derivative.

(a) $y = \ln(1+x^2)$. $\quad \dfrac{dy}{dx} = ?$

Answer:

$$\frac{dy}{dx} == \frac{2x}{1+x^2}.$$

(b) $f(x) = e^{-x^2/2}$. $\quad f''(x) = ?$ (Note: 2nd derivative.)

Answer:

$$f'(x) = -xe^{-x^2/2} \qquad f''(x) = -e^{-x^2/2} + x^2e^{-x^2/2}.$$

II. (30 Points.) (a) Find the equation of the tangent line to the graph

$$x + y + e^y = 3 + e$$

at the point $(x, y) = (2, 1)$.

Answer: The slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)}$$

By implicit differentiation we have that

$$1 + \frac{dy}{dx} + e^y\frac{dy}{dx} = 0$$

at any point of the curve. Evaluating at $(x, y) = (2, 1)$ gives

$$1 + m + em = 0, \qquad \text{so} \qquad m = -\frac{1}{1+e}.$$

The tangent line is the line through $(1, 2)$ with slope m; its equation is

$$y = 1 + m(x-2), \qquad \text{i.e.} \qquad y = 1 - \frac{x-2}{1+e}.$$

(b) The function $y = f(x)$ is implicitly defined by the above equation, i.e.

$$x + f(x) + e^{f(x)} = 3 + e.$$

Use linear approximation to find $f(2.1)$ approximately. **Answer:** The graph of the linear approximation $y = L(x)$ is tangent line. The value at $x = 2.1$ is

$$L(2.1) = 1 - \frac{2.1-2}{1+e} = 1 - \frac{0.1}{1+e}.$$

III. (25 Points.) The count in a bacteria culture grows exponentially. If the count is 200 after two hours and 600 after six hours, when will the count be 900?

Answer: The general formula for the bacteria count $N(t)$ at time t is

$$N(t) = N_0e^{kt}$$

where N_0 is the count when $t = 0$. We are given that $N(2) = 200$ and $N(6) = 600$ and we must find a value of t for which $N(t) = 900$.

$$200 = N_0e^{2k}, \qquad 600 = N_0e^{6k}, \qquad 900 = N_0e^{kt}.$$

Divide the second equation by the first to get $3 = e^{4k}$ so $k = \ln(3)/4$. Divide the third equation by the second to get $3/2 = e^{k(t-6)}$. Hence $\ln(3/2) = kt - 6k$ so

$$t = \frac{\ln(3/2)}{k} + 6 = \frac{4(\ln(3/2)}{\ln(3)/4} + 6 = \frac{16(\ln(3/2)}{\ln(3)} + 6 = 22 - \frac{16\ln 3}{\ln 2}.$$

IV. (25 Points.) State and prove a formula for the derivative $f'(x)$ of the function

$$f(x) = 2^x.$$

Justify each step. You may use without proof the fact that the limit

$$\ln 2 = \lim_{h\to 0} \frac{2^h - 1}{h} \qquad (\diamondsuit)$$

exists. You should not use (without proof) the formulas for the derivative of e^x or $\ln x$. **Answer:**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{definition of } f'(x) \\
&= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} && \text{definition of } f(x) \\
&= \lim_{h\to 0} \frac{2^x \cdot 2^h - 2^x}{h} && \text{high school algebra} \\
&= \lim_{h\to 0} \frac{2^x(2^h - 1)}{h} && \text{high school algebra} \\
&= 2^x \lim_{h\to 0} \frac{2^h - 1}{h} && \text{limit law} \\
&= 2^x \ln 2 && (\diamondsuit)
\end{aligned}$$

V. (30 Points.) (a) $\displaystyle\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = ?$

Answer: $\displaystyle\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$

(b) $\displaystyle\lim_{x\to\infty} \left((\cosh(x))^2 - (\sinh(x))^2\right) = ?$ Hint: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ and $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Answer: $(\cosh(x))^2 - (\sinh(x))^2 = 1$ for all x; hence the limit is 1.

(c) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x} = ?$

Answer:

$$\ln\left[\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x}\right] = \lim_{x\to\infty} \ln\left[\left(1 + \frac{2}{x}\right)^{3x}\right] = \lim_{x\to\infty} 3x \ln\left(1 + \frac{2}{x}\right) = \lim_{x\to\infty} 3\frac{\ln\left(1 + 2x^{-1}\right)}{x^{-1}}.$$

By l'Hôpital's rule:

$$\lim_{x\to\infty} 3\frac{\ln\left(1 + 2x^{-1}\right)}{x^{-1}} = \lim_{x\to\infty} 3\frac{\left(1 + 2x^{-1}\right)^{-1} \cdot \left(-2x^{-2}\right)}{-x^{-2}} = \lim_{x\to\infty} 6\left(1 + \frac{2}{x}\right)^{-1} = 6.$$

Hence

$$\ln\left[\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x}\right] = 6$$

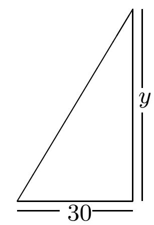
so

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x} = e^6.$$

VI. (30 Points.) I am observing a meteor through a telescope. The meteor is falling straight downward toward an impact point 30 miles away from me. When the meteor is 40 miles above the ground, it is falling at a speed of 1 mile per second. At that moment, how fast must I be changing the angle my telescope makes with the horizontal in order to keep the meteor in view?

Answer: Let y be the height of the meteor above the ground and θ be the angle the telescope makes with the ground. Then $\theta = \tan^{-1}(y/30)$. Then

$$\left.\frac{d\theta}{dt}\right|_{y=40} = \frac{1}{1+(y/30)^2}\cdot\frac{1}{30}\left.\frac{dy}{dt}\right|_{y=40} = \frac{1}{1+(40/30)^2}\cdot\frac{1}{30}\cdot(-1) = -\frac{30}{50^2} = -\frac{3}{250}.$$



VII. (15 Points.) Suppose that f and g are inverse functions, that $f'(1) = 5$, $f'(2) = 7$, $f'(3) = 9$, $f(1) = 2$, $f(2) = 3$ and $f(3) = 4$. Find $g'(2)$. Justify your answer. If there is insufficient information to do this problem, say why.

Answer: $f(g(y)) = y$ so $f'(g(y))g'(y) = 1$ by the chain rule, so $g'(y) = 1/f'(g(y)$ so $g'(2) = 1/f'(g(2))$. But $f(1) = 2$ so $g(2) = 1$. Hence $g'(2) = 1/f'(1) = 1/5$.

VIII. (15 Points.) Let $F(x) = x\tan^{-1}(1/x)$ if $x \neq 0$ and $F(0) = 0$.

(a) Is F continuous at 0? (Justify your answer.)

Answer:

$$\lim_{x\to 0+} F(x) = \lim_{x\to 0+} x\tan^{-1}(1/x) = \lim_{x\to 0+} x \lim_{y\to\infty} \tan^{-1}(y) = 0\pi = 0 = F(0)$$

and similarly

$$\lim_{x\to 0-} F(x) = \lim_{x\to 0-} x\tan^{-1}(1/x) = \lim_{x\to 0-} x \lim_{y\to -\infty} \tan^{-1}(y) = 0(-\pi) = 0 = F(0)$$

so $\lim_{x\to 0} F(x) = F(0)$ and therefore F is continuous.

(b) Is F differentiable at 0? If so, what is $F'(0)$? (Justify your answer.)

Answer:

$$\lim_{x\to 0+} \frac{F(x)-F(0)}{x-0} = \lim_{x\to 0+} \tan^{-1}(1/x) = \lim_{y\to\infty} \tan^{-1}(y) = \pi$$

and similarly

$$\lim_{x\to 0-} \frac{F(x)-F(0)}{x-0}) = \lim_{x\to 0-} \tan^{-1}(1/x) = \lim_{y\to -\infty} \tan^{-1}(y) = -\pi$$

so the derivative $F'(0)$ does not exist.