Calculus 221 Exam, Friday October 8, 1999

(In class: 55 Minutes)

I. (30 points.) Find the limit. Distinguish between an infinite limit and one which does not exist even if the values $\pm\infty$ are allowed. (If the limit does not exist, write DNE.)

(a) $\displaystyle \lim_{x\to 1} \frac{1-x^2}{1-x^3}$

Answer:

$$\lim_{x\to 1} \frac{1-x^2}{1-x^3} = \lim_{x\to 1} \frac{(1-x)(1+x)}{(1-x)(1+x+x^2)} = \lim_{x\to 1} \frac{(1+x)}{(1+x+x^2)} = \frac{2}{3}.$$

(b) $\displaystyle \lim_{x\to\infty} \frac{1-x^2}{1-x^3}$

Answer:

$$\begin{aligned}
\lim_{x\to\infty} \frac{1-x^2}{1-x^3} &= \lim_{x\to\infty} \frac{x^{-3}-x^{-1}}{x^{-3}-1} \\
&= \frac{\lim_{x\to\infty}(x^{-3}-x^{-1})}{\lim_{x\to\infty}(x^{-3}-1)} \\
&= \frac{\lim_{x\to\infty} x^{-3} - \lim_{x\to\infty} x^{-1}}{\lim_{x\to\infty} x^{-3} - 1} \\
&= \frac{0-0}{0-1} = 0.
\end{aligned}$$

II. (30 points.) Find the indicated derivative:

(a) $y = \sqrt{1+3x}$. $\qquad \dfrac{d^2y}{dx^2} = ?$

Answer:

$$y = (1+3x)^{1/2} \qquad \frac{dy}{dx} = \frac{1}{2}(1+3x)^{-1/2}(3) \qquad \frac{d^2y}{dx^2} = -\frac{1}{4}(1+3x)^{-3/2}(9).$$

(b) $f(x) = \dfrac{\sin(x^2)}{\sin(x^3)}$. $\qquad f'(x) = ?.$

Answer:

$$f(x) = \frac{u(x)}{v(x)}, \qquad \text{where} \qquad u(x) = \sin(x^2) \qquad \text{and} \qquad v(x) = \sin(x^3).$$

$$u'(x) = \big(\cos(x^2)\big)(2x), \qquad v'(x) = \big(\cos(x^3)\big)(3x^2),$$

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{\big[\big(\cos(x^2)\big)(2x)\big]\sin(x^3) - \sin(x^2)\big[\big(\cos(x^3)\big)(3x^2)\big]}{\sin^2(x^3)}$$

III. (30 points.) State and prove the chain rule for the derivative $(f \circ g)'(a)$ of the composition $f \circ g$ of the differentiable functions f and g. You may assume that $g'(a) \neq 0$. Justify each step with one or more of the following: HSA (High School Algebra), LL (Limit Law), DC (A differentiable function is continuous), DEF (Definition), DV (change of dummy variable).

Answer: The chain rule says that the limit

$$(f \circ g)'(a) = \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a}$$

exists and satisfies

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

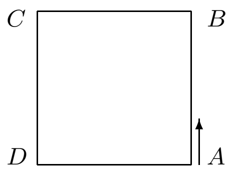
Condition DC says that

$$\lim_{x \to a} g(x) = g(a).$$

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{(f \circ g))(x) - (f \circ g)(a)}{x - a} && \text{DEF of } (f \circ g)' \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} && \text{DEF of } (f \circ g) \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} && \text{HSA}(\heartsuit) \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} && \text{LL} \\
&= \lim_{y \to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} && \text{DV} \\
&= f'(g(a))g'(a). && \text{DEF of } f'(g(a)) \text{ and } g'(a).
\end{aligned}$$

In step ($\heartsuit$) the high school algebra is justified by the fact that $g(x) \neq g(a)$ for $x \approx a$ but $x \neq a$ (since $g'(a) \neq 0$).

IV. (30 points.) A cyclist moves along side AB of a square $ABCD$ from A toward B. The square is 90 meters on a side and the speed of the cyclist is 24 meters per second.



(a) How fast is the distance from the cyclist to the corner C changing when the cyclist is 30 meters from A?

Answer: Let $b = 90$, $v = 24$, $a = 30$, t be the time, x be the distance from the cyclist to A, y be the distance from the cyclist to C, and z be the distance from the cyclist to D. Then b is the distance from B to C, b is the distance from D to A, $b - x$ is the distance from the cyclist to B, and

$$z^2 = b^2 + x^2, \qquad y^2 = (b - x)^2 + b^2, \qquad \frac{dx}{dt} = v = 24.$$

Hence

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} = 2xv, \qquad 2y\frac{dy}{dt} = -2(b - x)\frac{dx}{dt} = -2(b - x)v$$

When $x = a$, we have $y = \sqrt{(b-a)^2 + b^2}$ and $z = \sqrt{b^2 + a^2}$. Hence

$$\left.\frac{dy}{dt}\right|_{x=a} = -\frac{(b-a)v}{\sqrt{(b-a)^2 + b^2}}.$$

(The fact that the answer is negative indicates that the cyclist is getting nearer to C.)

(b) How fast is the distance from the cyclist to the corner D changing when the cyclist is 30 meters from A?

Answer: As above

$$\left.\frac{dz}{dt}\right|_{x=a} = \frac{av}{\sqrt{b^2 + a^2}}.$$

V. (30 points.) The graph of the equation $xy + 4 = x + y^2$ is a curve that crosses the y-axis at two points, P and Q. Find the point where the tangent lines to the curve at P and at Q cross.

Answer: The two points are $P(0, 2)$ and $Q(0, -2)$. By implicit differentiation

$$y + x\frac{dy}{dx} = 1 + 2y\frac{dy}{dx}$$

so

$$\frac{dy}{dx} = \frac{1-y}{x-2y}.$$

The slope of the tangent line at P is

$$m_P = \left.\frac{dy}{dx}\right|_{(x,y)=(0,2)} = \frac{1-2}{0-4} = \frac{1}{4}$$

The slope of the tangent line at Q is

$$m_Q = \left.\frac{dy}{dx}\right|_{(x,y)=(0,-2)} = \frac{1+2}{0+4} = \frac{3}{4}$$

The equation for the tangent line at P is

$$y = 2 + \frac{x}{4}.$$

The equation for the tangent line at Q is

$$y = -2 + \frac{3x}{4}.$$

The two lines intersect at $(x, y) = (8, 4)$.