

Math 221 Second Exam 5:30-7:00 P.M. WEDS. NOV 30

Answers

I. (35 points.) Evaluate.

(a) $\int_1^x \frac{t^2 + 1 + t^{-2}}{t} dt$

Answer:

$$\begin{aligned} \int_1^x \frac{t^2 + 1 + t^{-2}}{t} dt &= \int_1^x (t + t^{-1} + t^{-3}) dt \\ &= \left(\frac{t^2}{2} + \ln(t) - \frac{t^{-2}}{2} \right) \Big|_1^x = \\ &= \left(\frac{x^2}{2} + \ln(x) - \frac{x^{-2}}{2} \right) - \left(\frac{1}{2} + \ln(1) - \frac{1}{2} \right) \\ &= \frac{x^2}{2} + \ln(x) - \frac{1}{2x^2} - 0 \end{aligned}$$

(b) $\int_0^x \frac{e^t dt}{1 + e^{2t}}$

Answer: Take $u = e^t$ so $du = e^t dt$, $u = 1$ when $t = 0$, $u = e^x$ when $t = x$. Then

$$\int_0^x \frac{e^t dt}{1 + e^{2t}} = \int_1^{e^x} \frac{du}{1 + u^2} = \tan^{-1}(u) \Big|_{u=1}^{u=e^x} = \tan^{-1}(e^x) - \tan^{-1}(1) = \tan^{-1}(e^x) - \frac{\pi}{4}.$$

(c) $\frac{d}{dx} \int_0^{x^5} \frac{e^t dt}{1 + e^{2t}}$

Answer:

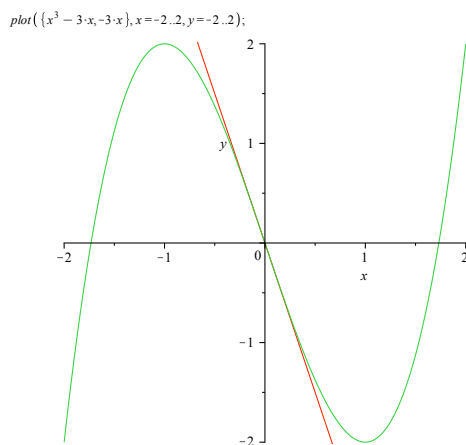
$$\frac{d}{dx} \int_0^{x^5} \frac{e^t dt}{1 + e^{2t}} = \frac{d}{du} \left(\int_0^u \frac{e^t dt}{1 + e^{2t}} \right) \Big|_{u=x^5} \frac{d}{dx} x^5 = \frac{e^{x^5}}{1 + e^{2x^5}} \cdot 5x^4.$$

II. (35 points.) Find the intervals on which the curve $y = x^3 - 3x$ is increasing, decreasing, concave up, concave down. Sketch the curve showing inflection points and local maxima and minima. Draw the tangent line at each point of inflection

Answer: $y = x^3 - 3x$, $y' = 3(x^2 - 1)$, $y'' = 6x$.

x	y	y'	y''	
$-\infty$	$-\infty$			
		> 0	< 0	$\curvearrowright, \uparrow$
-1		0		local max
		< 0	< 0	$\curvearrowleft, \downarrow$
0			0	inflection
		< 0	> 0	$\curvearrowleft, \downarrow$
1		0		local min
		> 0	> 0	$\curvearrowright, \uparrow$
∞	∞			w

Note that the tangent line crosses the curve at the point of inflection.



III. (35 points.) An open box with a square base, four sides, and no top is to be made out of cardboard. Find the maximum volume of the box if the total area (base plus four sides) is 25 square feet. Give the side length of the base and the height of the box which has the largest volume.

Answer: Let x denote the side length of the square base and y denote the height of the box. Then the total area of the box is $25 = x^2 + 4xy$ and the volume is $V = x^2y$. Solving the former equation for y in terms of x gives

$$y = \frac{25 - x^2}{4x}, \quad \text{so} \quad V = x^2y = \frac{x^2(25 - x^2)}{4x} = \frac{25x - x^3}{4}.$$

We have $V_{x=0} = V_{x=5} = 0$ and $V > 0$ for $0 < x < 5$ so at the maximum we must have

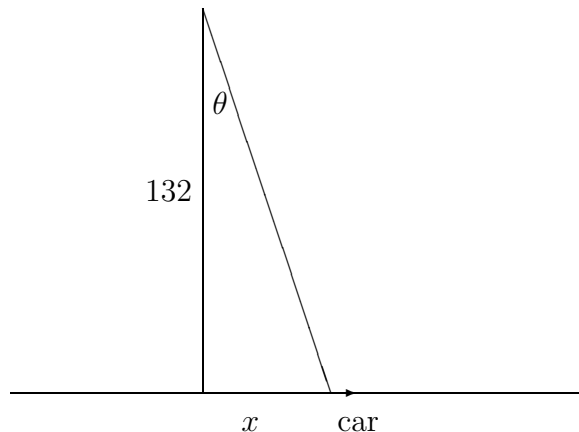
$$0 = \frac{dV}{dx} = \frac{25 - 3x^2}{4}$$

i.e.

$$x = \sqrt{\frac{25}{3}}, \quad y = \frac{25 - x^2}{4x} = \frac{25 - 25/3}{4\sqrt{25/3}}.$$

IV. (35 points.) You are videotaping a race from a stand 132 feet from the track, following a car that is moving at a constant velocity v along a straight track. When the car is directly in front of you, the camera angle θ is changing at a rate of

$$\left. \frac{d\theta}{dt} \right|_{\theta=0} = \frac{60}{43} \text{ radians per second.}$$



(a) How fast is the car going? (i.e. find v).¹

Answer: Choose coordinates so that the car moves along the x -axis and the viewing stand is on the y -axis at the point $Q(0, 132)$. Let $P(x, 0)$ be the position of the car. Let t be the time in seconds and suppose $t = 0$ is the time when the car is directly in front of you. Let θ be the angle between the y -axis and QP . We are given that

$$\frac{dx}{dt} = v \text{ is constant, and } \left. \frac{d\theta}{dt} \right|_{t=0} = \frac{60}{43}.$$

Since

$$x = 132 \tan \theta$$

we have

$$v = \frac{dx}{dt} = (132 \sec^2 \theta) \cdot \frac{d\theta}{dt}$$

so evaluating at $t = 0$ gives:

$$v = 132 \cdot \left. \frac{d\theta}{dt} \right|_{t=0} = 132 \cdot \frac{60}{43}.$$

(b) How fast will the camera angle θ be changing a half second later?

Answer: The position of the car at time t is

$$x = vt$$

since v is constant. Thus

$$\theta = \tan^{-1} \left(\frac{vt}{132} \right).$$

By the chain rule:

$$\frac{d\theta}{dt} = \frac{1}{1 + \left(\frac{vt}{132}\right)^2} \cdot \left(\frac{v}{132}\right).$$

Evaluate at $t = 1/2$ and plug in the value of the constant v which we found above:

$$\left. \frac{d\theta}{dt} \right|_{t=1/2} = \frac{1}{1 + \left(\frac{v}{2 \cdot 132}\right)^2} \cdot \left(\frac{v}{132}\right) = \frac{1}{1 + \left(\frac{30}{43}\right)^2} \cdot \left(\frac{60}{43}\right).$$

¹ In this problem leave the arithmetic undone to make it easier to grade.

Answer:

V. (35 points.) Polonium-210 has a half life of 140 days. (The half life is the length of time it takes half the sample to decay.)

(a) If a sample has a mass of 200 mg find a formula for the mass that remains after t days.

Answer: $M = M_0 a^{-t} = 200 \cdot 2^{-t/140}$.

(b) Find the mass after 100 days.

Answer: $M_{t=100} = 200 \cdot 2^{-100/140} = 200 \cdot 2^{-5/7}$.

(c) When will the mass be reduced to 10 mg?

Answer: $10 = 200 \cdot 2^{-t/140}$ so $t = -140 \log_2(10/200) = 140(\log_2(200) - \log_2(10))$.

Answer:

VI. (30 points.) The Tangent Concavity and Secant Concavity Theorems both say that under a certain hypothesis a function $f(x)$ satisfies certain inequalities on an interval $[a, b]$.

1. State the hypothesis and draw a picture illustrating the two theorems.

Answer: The picture is Figure 1 Section 22 Page 81 Of the notes. The hypothesis is that f is concave up (i.e. $f''(x) > 0$) for $a \leq x \leq b$ and that c and x are any two points in the interval $[a, b]$.

2. Write the inequality expressing the Secant Concavity Theorem.

Answer: The graph of $f(x)$ on the interval $[a, b]$ lie below the graph of the secant line $y = W(x)$, i.e.

$$f(x) \leq W(x) := f(a) + \left(\frac{f(b) - f(a)}{b - a} \right) (x - a).$$

3. Write the inequality expressing the Tangent Concavity Theorem.

Answer: The graph of $f(x)$ on the interval $[a, b]$ lie above the graph of the tangent line $y = L(x)$ at the point $(c, f(c))$, i.e.

$$L(x) := f(c) + f'(c)(x - c) \leq f(x).$$

VII. (30 points.) Let $A = \int_0^2 \sqrt{1+x^3} dx$. Using the partition

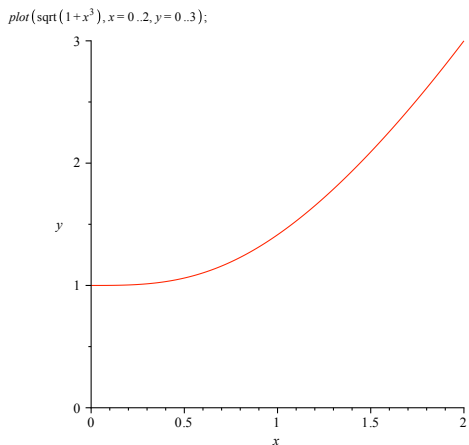
$$x_0 = 0, \quad x_1 = 1, \quad x_2 = 1.8, \quad x_3 = 2$$

write a Riemann sum S_{lower} which is smaller than A and another Riemann sum S_{upper} which is larger than A . Do not try to evaluate the integral. *Leave your answer in a form that can be evaluated by a calculator which can add, subtract, multiply, divide, and do square roots. To make it easier to grade the problem do not do any arithmetic.*

Answer:

$$\begin{aligned} S_{\text{lower}} &= f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + f(x_2)(x_3 - x_2) \\ &= \sqrt{1+0^3}(1-0) + \sqrt{1+1^3}(1.8-1) + \sqrt{1+1.8^3}(2-1.8) \end{aligned}$$

$$\begin{aligned} S_{\text{upper}} &= f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + f(x_3)(x_3 - x_2) \\ &= \sqrt{1+1^3}(1-0) + \sqrt{1+1.8^3}(1.8-1) + \sqrt{1+2^3}(2-1.8) \end{aligned}$$



CURVE

grade	range	count	percent
A	195 ... 235	25	14%
B	155 ... 199	55	31%
C	120 ... 159	36	20%
D	90 ... 119	28	16%
F	0 ... 89	32	18%

There are 176 scores. Mean score = 141.97. Mean grade = 2.07.