Math 221 First Exam 5:30-7:00 P.M. WEDS. OCT 12

Answers.

I. (20 points.) True or false? (No reason required. Two points for each correct answer, three points off for each incorrect answer.)

(i) **T** **F** $c(a+b) = ca + cb$?

Answer: True. This is the distributive law.

(ii) **T** **F** $\cos(a+b) = \cos(a) + \cos(b)$?

Answer: False: if $a = b = 0$ then $\cos(a+b) = \cos(0+0) = 1$ but $\cos(a) + \cos(b) = 1 + 1 = 2$.

(iii) **T** **F** $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$?

Answer: True: $\dfrac{a+b}{c} = c^{-1}(a+b) = c^{-1}a + c^{-1}b = \dfrac{a}{c} + \dfrac{b}{c}$.

(iv) **T** **F** $\dfrac{c}{a+b} = \dfrac{c}{a} + \dfrac{c}{b}$?

Answer: False: If $a = b = c = 1$, then $\dfrac{c}{a+b} = 1/2$ but $\dfrac{c}{a} + \dfrac{c}{b} = 2$

(v) **T** **F** $\dfrac{a+b}{c+d} = \dfrac{a}{c} + \dfrac{b}{d}$?

Answer: False: if $a = b = c = d = 1$ then $\dfrac{a+b}{c+d} = 1$, but $\dfrac{a}{c} + \dfrac{b}{d} = 2$

(vi) **T** **F** $\dfrac{a \cdot b}{c \cdot d} = \dfrac{a}{c} \cdot \dfrac{b}{d}$?

Answer: True: $(cd)^{-1} = c^{-1}d^{-1}$ so both sides $= c^{-1}d^{-1}a \cdot b$ by the commutative law for multiplication.

(vii) **T** **F** $(a \cdot b)^c = a^c + b^c$?

Answer: False: if $a = b = c = 1$ then $(a \cdot b)^c = 1$ but $a^c + b^c = 2$

(viii) **T** **F** $(a+b)^c = a^c \cdot b^c$?

Answer: False: If $a = b = 1$ and $c = 2$, $(a+b)^c = 4$ but $a^c \cdot b^c = 2$.

(ix) **T** **F** $c^{ab} = c^a \cdot c^b$?

Answer: False: If $a = b = 1$, then $c^{ab} = c$, but $c^a \cdot c^b = c^2$.

(x) **T** **F** $\sin(\theta)\cos(h) + \cos(\theta)\sin(h) - \sin(\theta) = \sin(\theta)(\cos(h) - 1) + \cos(\theta)\sin(h)$?

Answer: True: $AB + CD - A = A(B-1) + CD$.

II. (40 points.) State and prove the formula for the derivative of the product of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable function is continuous, and high school algebra.

Answer: Suppose that $f(x) = u(x)v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)}\\
&= \lim_{x\to a} \frac{u(x)\cdot v(x) - u(a)\cdot v(a)}{x-a} && \text{(hypothesis)}\\
&= \lim_{x\to a} \left(\left(\frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(x) \cdot \left(\frac{v(x)-v(a)}{x-a} \right) \right) && \text{(hsa)}\\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot v(a) + u(a) \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(limit laws)}\\
&= u'(a)v(a) + v'(a)u(a) && \text{(definition)}
\end{aligned}$$

(In the fourth step the theorem that a differentiable function is continuous is also used. “hsa” means high school algebra.)

III. (40 points.) Find the limit. Distinguish between an infinite limit and one which does not exist even if the values $\pm\infty$ are allowed. (If the limit does not exist, write DNE.)

(a) $\displaystyle\lim_{x\to 1} \frac{1-x^2}{1-x^3}$

Answer: If $x = 1+h$ then $1 - x^2 = 1 - (1+h)^2 = 1 - (1 + 2h + h^2) = -h(2+h)$ and $1 - x^3 = 1 - (1+h)^3 = 1 - (1 + 3h + 3h^2 + h^3) = -h(3 + 3h + h^2)$ so

$$\lim_{x\to 1} \frac{1-x^2}{1-x^3} = \lim_{h\to 0} \frac{1-(1+h)^2}{1-(1+h)^3} = \lim_{h\to 0} \frac{-h(2+h)}{-h(3+3h+h^2)} = \lim_{h\to 0} \frac{2+h}{3+3h+h^2} = \frac{2}{3}.$$

Another method is by factoring:

$$1 - x^2 = (1-x)(1+x), \qquad 1 - x^3 = (1-x)(1+x+x^2)$$

so

$$\lim_{x\to 1} \frac{1-x^2}{1-x^3} = \lim_{x\to 1} \frac{(1-x)(1+x)}{(1-x)(1+x+x^2)} = \lim_{x\to 1} \frac{1+x}{1+x+x^2} = \frac{2}{3}.$$

(b) $\displaystyle\lim_{x\to\infty} \frac{1-x^2}{1-x^3}$

Answer:

$$\lim_{x\to\infty} \frac{1-x^2}{1-x^3} = \lim_{x\to\infty} \frac{(1-x^2)x^{-3}}{(1-x^3)x^{-3}} = \lim_{x\to\infty} \frac{x^{-3}-x^{-1}}{x^{-3}-1} = \frac{0+0}{0+1} = 0.$$

IV. (40 points.) Find the indicated derivative:

(a) $y = \sqrt{1+3x}$. $\qquad \dfrac{d^2y}{dx^2} = ?$

Answer: $y = u^{1/2}$ where $u = 1 + 3x$ so $\frac{du}{dx} = 3$ and hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \tfrac{1}{2}u^{-1/2}3 = \frac{3}{2\sqrt{1+3x}}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\,\tfrac{3}{2}u^{-1/2} = -\tfrac{1}{2}\cdot\tfrac{3}{2}u^{-3/2}\frac{du}{dx} = -\frac{9}{4(1+3x)^{3/2}}$$

(b) $f(x) = \dfrac{\sin(x^3)}{\sin(x^2)}$. $\qquad f'(x) = ?$.

Answer: Let $u(x) = \sin(x^3)$ and $v(x) = \sin(x^2)$. Then

$$f(x) = \frac{u(x)}{v(x)}, \qquad u'(x) = \cos(x^3)3x^2, \qquad v'(x) = \cos(x^2)2x,$$

so

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{\cos(x^3)3x^2\cdot\sin(x^2) - \sin(x^3)\cdot\cos(x^2)2x}{(\sin(x^2))^2}.$$

The denominator $(\sin(x^2))^2$ on the right may be written $\sin^2(x^2)$.

V. (30 points.) Find an equation for the tangent line to the curve $y^5 + y - x = 0$ at the point $(x_0, y_0) = (2, 1)$.

Answer: The derivative satisfies

$$5y^4\frac{dy}{dx} + \frac{dy}{dx} - 1 = 0 \qquad \text{so} \qquad \frac{dy}{dx} = \frac{1}{1+5y^4}$$

on the curve. The point $(x_0, y_0) = (2, 1)$ is on the curve as at that point we have

$$y_0^5 + y_0 - x_0 = 1^5 + 1 - 2 = 0$$

and the slope of the tangent line at (x_0, y_0) is

$$m := \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{1}{1+5} = \frac{1}{6}.$$

The equation for the tangent line is $y = y_0 + m(x - x_0)$ i.e.

$$y = 1 + \tfrac{1}{6}(x - 2).$$

VI. (30 points.) Suppose that $u = \tan(\theta)$. Find a formula for $\dfrac{d\theta}{du}$ in terms of u. (The phrase "in terms of u" means you should give a formula with u in it but not θ.)

Answer: Differentiating gives

$$\frac{du}{d\theta} = \sec^2(\theta) = 1 + \tan^2(\theta) = 1 + u^2$$

so

$$\frac{d\theta}{du} = \frac{1}{1+u^2}.$$

The identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ follows from $1 = \cos^2(\theta) + \sin^2(\theta)$ by dividing both sides by $\cos^2(\theta)$.

The above reasoning is correct even if it is not the case that $-\pi/2 < \theta < \pi/2$, i.e. even if $\theta \neq \tan^{-1}(u)$. For example, if $\pi/2 < \theta < 3\pi/2$, then $\theta = \tan^{-1}(u) + \pi/2$, so θ and $\tan^{-1}(u)$ differ by a constant and therefore have the same derivative.

VII. (35 points.) Find all points on the curve $x^2 + xy + y^2 = 3$ where the tangent line is horizontal (i.e. parallel to the x-axis).

Answer: On the curve we have

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

by implicit differentiation. The tangent line is horizontal when its slope $\frac{dy}{dx}$ is zero. At such points both equations

$$x^2 + xy + y^2 = 3 \qquad \textit{and} \qquad 2x + y = 0$$

hold. Solving we get $y = -2x$ so $x^2 + x(-2x) + (-2x)^2 = 3$ so $3x^2 = 3$ so $x = \pm 1$. The points on the curve where the tangent line is horizontal are $(x, y) = (1, -2)$ and $(x, y) = (-1, 2)$. (Both points lie on the curve as $1^2 + 1(-2) + (-2)^2 = (-1)^2 + (-1)2 + 2^2 = 3$.)

Grades

grade	range	count	percent
A	205 ... 235	35	19%
B	170 ... 204	67	35%
C	140 ... 169	36	19%
D	94 ... 139	29	15%
F	0 ... 93	22	12%

There are 189 scores.

Mean score = 161.72. Mean grade = 2.3399999999999999.