Math 221 – Final Exam – December 18, 2000
Answers

I. (30 points.) Find dy/dx:

i) $x^3 - xy + y^3 = 1$.

Answer: Use implicit differentiation and solve.

$$3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}.$$

(This is problem 8 on page 83.)

ii) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Answer: $\dfrac{dy}{dx} = \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \dfrac{4}{(e^x + e^{-x})^2}$. (This is problem 7 on page 315.)

iii) $y = \ln(\ln x)$.

Answer: $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$. (This is problem 18 on page 307.)

iv) $y = \ln e^x$

Answer: $y = x$ so $\dfrac{dy}{dx} = 1$. (This is problem 39 on page 329.)

II. (30 points.) Find the integral:

(1) $\displaystyle\int \sqrt{2x+1}\, dx$.

Answer: $\displaystyle\int \sqrt{2x+1}\, dx = \frac{(2x+1)^{3/2}}{3} + C$. (This is Example 4 on page 181.)

(2) $\displaystyle\int \frac{x\, dx}{(4x^2+1)^2}$.

Answer: $\displaystyle\int \frac{x\,dx}{(4x^2+1)^2} = \frac{-1}{8(4x^2+1)} + C$. (This is Problem 21 on page 307.)

(3) $\displaystyle\int \frac{x\,dx}{4x^2+1}$.

Answer: $\displaystyle\int \frac{x\,dx}{4x^2+1} = \frac{\ln(4x^2+1)}{8} + C$. (This is Problem 21 on page 307.)

(4) $\displaystyle\int_3^6 \frac{dx}{x}$.

Answer: $\displaystyle\int_3^6 \frac{dx}{x} = \ln 6 - \ln 3 = \ln 2$ (This is from the review sheet.)

III. (30 points.) Evaluate the limit or say why it does not exist. Be sure to distinguish between an infinite limit and one which does not exist.

(1) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^5}$

Answer: By l'Hopital's Rule

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}}.$$

Applying this for $n = 5, 4, 3, 2, 1$ gives

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty.$$

(2) $\displaystyle\lim_{x\to 0+} x \ln x$

Answer: Do some algebra to move x to the denominator and then apply l'Hopital's rule:

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0+} -x = 0.$$

(3) $\displaystyle\lim_{x\to 2} \frac{x^2-1}{x^3-1}$

Answer: This is not an indeterminate form so l'Hopital's rule does not apply. By the Limit Laws

$$\lim_{x\to 2}\frac{x^2-1}{x^3-1}=\frac{2^2-1}{2^3-1}=\frac{3}{7}.$$

(4) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Answer: If $\ln\left(\lim_{n\to\infty}F(n)\right)=L$ then $\lim_{n\to\infty}F(n)=e^L$. Now

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n=$$

$$=\lim_{n\to\infty}n\ln\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\ln(1+n^{-1})}{n^{-1}}.$$

By l'Hopital's rule

$$\lim_{n\to\infty}\frac{\ln(1+n^{-1})}{n^{-1}}=\lim_{n\to\infty}\frac{(1+n^{-1})^{-1}(-n^{-2})}{-n^{-2}}=\lim_{n\to\infty}(1+n^{-1})^{-1}=1$$

so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

(5) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

Answer: Do the algebra and apply l'Hopital's rule twice:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\sin x-x}{x\sin x}=$$

$$=\lim_{x\to 0}\frac{\cos x-1}{\sin x+x\cos x}=\lim_{x\to 0}\frac{-\sin x}{2\cos x-x\sin x}=0$$

(6) $\displaystyle\lim_{x\to 0}\frac{2^x-1}{3^x-1}$

Answer: By l'Hopital's rule

$$\lim_{x\to 0}\frac{2^x-1}{3^x-1}=\lim_{x\to 0}\frac{2^x\ln 2}{3^x\ln 3}=\frac{\ln 2}{\ln 3}.$$

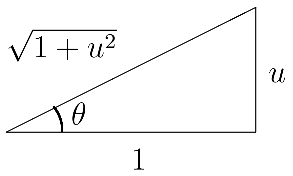
IV. (30 points.) **(a)** Assume $u = \dfrac{\sin\theta}{\cos\theta}$. What is $\dfrac{du}{d\theta}$?

Answer: By the quotient rule

$$\frac{du}{d\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

(b) What is $\cos(\tan^{-1}(u))$? Draw a picture to show that you understand why.

Answer:



A right triangle with an height u and base 1 has an angle θ with $\tan\theta = u/1 = u$. The hypotenuse is $\sqrt{1+u^2}$ so

$$\cos(\tan^{-1}(u)) = (1+u^2)^{-1/2}.$$

(c) Assume $\theta = \tan^{-1}(u)$. What is $\dfrac{d\theta}{du}$? (Express your answer as a function of u.)

Answer: From parts (a) and (b) and the rule for the derivative of an inverse function we get

$$\frac{d\theta}{du} = \cos^2\theta = \cos^2(\tan^{-1} u) = \frac{1}{1+u^2}.$$

(d) Evaluate $\displaystyle\int_0^1 \frac{dt}{1+9t^2}$.

Answer: Make the substitution $u = 3t$ so $du = 3\,dt$, $u = 0$ for $t = 0$, and $u = 3$ for $t = 1$. By the Fundamental Theorem

$$\int_0^1 \frac{dt}{1+9t^2} = \int_0^3 \frac{du}{3(1+u^2)} = \frac{\tan^{-1}3 - \tan^{-1}0}{3} = \frac{\tan^{-1}3}{3}.$$

V. (30 points.) Find Y as a function of t if

$$\frac{dY}{dt} = 3Y$$

and $Y = 100$ when $t = 0$.

Answer: We use separation of variables. $\dfrac{dY}{Y} = 3\,dt$ so

$$\ln Y = \int \frac{dY}{Y} = 3t + C.$$

Exponentiate

$$Y = e^{3t+C} = e^C e^{3t} = Y_0 e^{3t}$$

where $Y_0 = e^C$. Evaluate at $t = 0$; we get $100 = Y_0 e^0 = Y_0$, so $Y = 100e^{3t}$.

VI. (25 points.) Suppose that \$621 is deposited in an account with interest at the rate of 6 percent and the interest is compounded continuously for eight years. Find the amount in the account at the time $t = 8$ years.

Answer: As in problem V the amount in the account after t years is

$$B = B_0 e^{rt} = 621e^{0.06t}.$$

Thus $B_{t=8} = 621e^{0.48}$. (This is Example 2 on page 326.)

VII. (25 points.) A Riemann sum is an expression of form

$$S = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

i) Write a Riemann sum with four terms ($n = 4$) whose value is less than $\displaystyle\int_1^3 \frac{dx}{x}$. Illustrate with a picture.

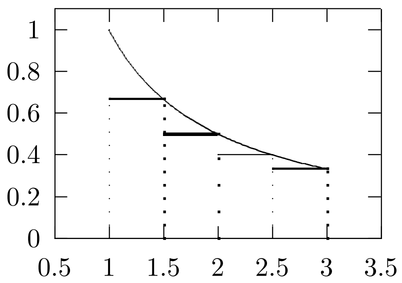
ii) Write a Riemann sum with four terms ($n = 4$) whose value is greater than $\displaystyle\int_1^3 \frac{dx}{x}$. Illustrate with a picture.

Answer: We take $x_k = 1 + (k/2)$ so $x_0 = 1$, $x_4 = 3$, and $x_k - x_{k-1} = 1/2$ for $k = 1, 2, 3, 4$. The function $f(x) = 1/x$ is decreasing so we take $c_k = x_k$ for a Riemann sum which is too small and $c_k = x_{k-1}$ for a Riemann sum which is too big. The Riemann sums are

$$\frac{1}{1.5}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2.5}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\le\int_1^3\frac{dx}{x}$$

and

$$\int_1^3\frac{dx}{x}\le\frac{1}{1}\cdot\frac{1}{2}+\frac{1}{1.5}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2.5}\cdot\frac{1}{2}.$$



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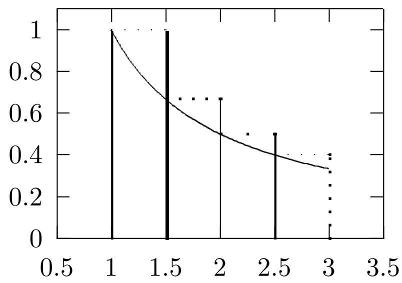

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VIII. (20 points.) Complete the definition of the derivative:

$$f'(a) = \lim_{x\to a} \boxed{}$$

Answer: $$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

IX. (25 points.) Find the interval on which the curve $y = \int_0^x \frac{dt}{1+t+t^2}$ is concave upward.

Answer: By the Fundamental Theorem

$$\frac{dy}{dx} = \frac{1}{1+x+x^2} \qquad \text{so} \qquad \frac{d^2y}{dx^2} = \frac{-1-2x}{(1+x+x^2)^2}.$$

This is positive when $x < -1/2$.

X. (30 points.) **(1)** Find the length of the curve $x = \frac{y^2}{4} - \frac{\ln y}{2}$ between $y = 1$ and $y = 3$. (DO evaluate the definite integral.)

Answer:

$$\frac{dx}{dy} = \frac{y}{2} - \frac{1}{2y}$$

so

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{y}{2} - \frac{1}{2y}\right)^2 = \left(\frac{y}{2} + \frac{1}{2y}\right)^2$$

The arc length is

$$\int_{y=1}^{y=3} ds = \int_1^3 \left(\frac{y}{2} + \frac{1}{2y}\right) dy = \frac{y^2}{4} + \frac{\ln(2y)}{2}\bigg|_1^3 = \frac{9}{4} + \frac{\ln 6}{2} - \frac{1}{4} - \frac{\ln 2}{2}.$$

(2) A homogeneous wire is bent to the shape of of the curve in part i). Find its center of mass $(\bar{x}, \bar{y})$. Do NOT evaluate the definite integrals.

Answer:

$$\bar{x} = \frac{\int x\,ds}{\int ds} = M^{-1} \int_1^3 \left(\frac{y^2}{4} - \frac{\ln y}{2}\right)\left(\frac{y}{2} + \frac{1}{2y}\right) dy$$

and

$$\bar{y} = \frac{\int y\,ds}{\int ds} = M^{-1} \int_1^3 y \left(\frac{y}{2} + \frac{1}{2y}\right) dy$$

where $M = \int ds$ is the answer to part (1).

XI. (30 points.) (1) Find the polynomial of degree 5 which best approximates e^x for x near 0.

Answer: This is the Taylor polynomial

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

with $n = 5$, $f(x) = e^x$, and $a = 0$. Since the exponential is its own derivative and $e^0 = 1$ this is

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$

(2) Estimate the error if you use part (1) to approximate e^{-1}.

Answer: The error is given by

$$f(x) - P(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is between a and x. For the exponential with $n = 5$ this is

$$e^x - P(x) = \frac{e^c x^6}{720}.$$

Since $e^c < 1$ for $-1 < c < 0$ we get

$$|e^{-1} - P(-1)| < \frac{1}{720}.$$