Math 221 – Exam III – Friday December 8, 2000
Answers

I. (30 points.) Find the velocity $v = \dfrac{ds}{dt}$ and the position s of a particle if the acceleration is given by $a = \dfrac{d^2s}{dt^2} = t$ and $v = v_0$ and $s = s_0$ when $t = 0$.

Answer: (This is problem 8 on page 184.) Since $a = dv/dt = t$ we have

$$\frac{ds}{dt} = v = \int t\,dt = \frac{t^2}{2} + C_1.$$

Since $v = v_0$ when $t = 0$ this becomes

$$\frac{ds}{dt} = v = \frac{t^2}{2} + v_0.$$

Now do it again:

$$s = \int \left(\frac{t^2}{2} + v_0\right) dt = \frac{t^3}{6} + v_0 t + C_2.$$

Since $s = s_0$ when $t = 0$ this becomes

$$s = \frac{t^3}{6} + v_0 t + s_0.$$

Grader's Comments. Question #1 is about distance and velocity. It seems many students had forgotten how to find the C in the indefinite integral using the initial conditions. They simply say $v = v_0$ instead of $v = t^2/2 + C$ and $C = v0$. Their reason is the problem said $v = v_0$, but actually the problem said $v = v_0$ when $t = 0$. We must remind them the concept of initial condition – it's an equation only for the initial point.

II. (40 points.) (a) Express the area to the right of the y-axis and between the curves $y = x^2$ and $y = 18 - x^2$ both as a definite integral and as a limit of a (Riemann) sum of areas of rectangles. (Parts (a) and (b) are problem 17 on page 210. Part (c) is similar to Example 2 on page 195.)

Answer: The parabolas intersect at $(3, 9)$. Take $\Delta x = 3/n$ and $x_k = 3k/n$ so the area of the kth rectangle is

$$\Delta A_k = (18 - 2x_k^2)\,\Delta x.$$

The Riemann sum is the sum

$$S_n = \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \left(18 - 2\left(\frac{9k^2}{n^2}\right)\right)\frac{3}{n}$$

of the areas of the rectangles. It is the approximate area. The exact area is the limit

$$\int_0^3 (18 - x^2)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \left(18 - 2\left(\frac{9k^2}{n^2}\right)\right)\frac{3}{n}.$$

(b) Find this limit (the area) using calculus.

Answer: The indefinite integral is

$$\int (18 - 2x^2)\,dx = 18x - \frac{2x^3}{3} + C$$

so the definite integral is

$$\int_0^3 (18 - 2x^2)\,dx = \left(18(3) - \frac{2(3^3)}{3}\right) - \left(18(0) - \frac{2(0^3)}{3}\right) = 36.$$

(c) Find this limit (the area) using the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Answer: The nth Riemann sum is

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \left(18 - 2\left(\frac{9k^2}{n^2}\right)\right)\frac{3}{n} \\
&= \frac{54}{n}\left(\sum_{k=1}^{n} 1\right) - \frac{54}{n^3}\left(\sum_{k=1}^{n} k^2\right) \\
&= 54 - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6},
\end{aligned}$$

so the limit is

$$\begin{aligned}
\lim_{n\to\infty} S_n &= 54 - \lim_{n\to\infty}\frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= 54 - \lim_{n\to\infty}\frac{54}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= 54 - 18 = 36.
\end{aligned}$$

Grader's Comments. #2 was an easy half credit the way my anonymous partner and i graded it. If the students could find the area between the two curves over the given interval by an integral, then they got 20 out of 40 points. Hardly anyone got full credit, because hardly anyone could represent the area as a limit of Riemann sums, and less still could evaluate the limit. I'm not surprised, because they didn't see many examples of such a feat in class or section, and i don't think many students read the book. READ THE BOOK! It helps.

III. (20 points.) (a) Find dy/dx if $y = \displaystyle\int_2^{x^5} \cos(t)\,dt$.

Answer: This problem can be done two ways. First

$$y = \int_2^{x^5} \cos(t)\,dt = \sin(t)\big|_{t=2}^{t=x^5} = \sin(x^5) - \sin(2).$$

Hence, by the chain rule,

$$\frac{dy}{dx} = \cos(x^5)\,5x^4.$$

The second method is the same as the method used in part (b) below.

(b) Find dz/dx if $z = \displaystyle\int_2^{x^5} \sqrt{1+t^3}\,dt$.

Answer: This problem is similar to Problems 18-24 on page 210 of the text. We cannot use the method of part (a) since we cannot evaluate the definite integral. The trick is to write

$$z = F(x^5) \qquad \text{where} \qquad F(u) = \int_2^{u} \sqrt{1+t^3}\,dt.$$

By the Fundamental Theorem

$$F'(u) = \sqrt{1+u^3}.$$

Hence by the Chain Rule

$$\frac{dz}{dx} = F'(x^5)5x^4 = \left(\sqrt{1+x^{15}}\right)5x^4.$$

Grader's Comments. Very few students seemed to understand what was required. Many computed the first integral (4 points). Most also made attempts to get the second integral in closed form.

Since I did quite a few problems like this in my sections, it seemed to me that many people would get it correct. Unfortunately, that was not the case.

I have tried to be as consistent as possible in grading. However, if two persons came up with the same answers, but one with no explanations and the other showing the wrong working in detail, I naturally gave many more points to the latter.

IV. (40 points.) In each of the following the region bounded by the parabola $y = x^2$ and the line $y = 4$ is rotated about the indicated axis. Write the volume of the resulting surface of revolution as a definite integral using the indicated method. DO NOT EVALUATE THE INTEGRAL. DO INDICATE THE LIMITS OF INTEGRATION. (This is problem 18 page 246 and also quiz 7.)

(a) The axis is the line $y = 4$; use disks.

Answer: $V = \displaystyle\int_{x=-2}^{2} \pi\big((4 - x^2)^2\big)\,dx.$

(b) The axis is the line $y = 4$; use shells.

Answer: $V = \displaystyle\int_{y=0}^{4} 2\pi(4 - y)\big(2\sqrt{y}\big)\,dy.$

(c) The axis is the line $y = -1$; use washers.

Answer: $V = \displaystyle\int_{x=-2}^{2} \pi\big(5^2 - (1 + x^2)^2\big)\,dx.$

(d) The axis is the line $y = -1$; use shells.

Answer: $V = \displaystyle\int_{y=0}^{4} 2\pi(y + 1)\big(2\sqrt{y})\big)\,dy.$

Grader's Comments. I graded the volume problem. I graded it somewhat harshly, as they had been given the solutions the week before. The biggest problem I saw was that people wanted to use the formula $2\pi x\, f(x)$ for shells, and if you're revolving around a line other than the axes, that won't work.

V. (20 points.) A thin homogeneous wire is bent to form a semicircle of radius 5 described parametricly by the equations

$$x = 5\cos\theta, \qquad y = 5\sin\theta, \qquad 0 \le \theta \le \pi.$$

Find its center of mass $(\bar{x}, \bar{y})$. (Evaluate the integrals.)

Answer: (This is example 2 on page 268 of the text.) The word *homogeneous* means that the mass density dm is proportional to the arclength ds, i.e.

$$dm = c\,ds$$

where c is a constant. Hence the formula for the center of mass simplifies to

$$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int x\,ds}{\int ds}, \qquad \bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int y\,ds}{\int ds};$$

i.e. the constant factor c cancels in numerator and denominator. Now from the parametric equations

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \sqrt{5^2\cos^2\theta + 5^2\sin^2\theta}\,d\theta = 5\,d\theta,$$

and the limits of integration are $\theta = 0$ and $\theta = \pi$. Hence

$$\bar{x} = \frac{\int_0^\pi 25\cos\theta\,d\theta}{\int_0^\pi 5\,d\theta} = \frac{-25\sin(\pi) + 25\sin(0)}{5\pi} = 0$$

and

$$\bar{y} = \frac{\int_0^\pi 25\sin\theta\,d\theta}{\int_0^\pi 5\,d\theta} = \frac{-25\cos(\pi) + 25\cos(0)}{5\pi} = \frac{10}{\pi}.$$

Grader's Comments. I did not expect many students to get this one because I never did one like it in lecture. Lu did a similar problem in his section and his students did seem to do somewhat better. I put it on the exam because I am trying to reinforce the idea that the students should read the book. (This problem is an example from the book.)

The problem deals with finding the center of mass of a homogeneous wire in the shape of a semicircle. Some students found the center of mass of a semicircular plate - if they did it correctly I gave half credit. Most students do not understand the difference between dx, dy, and $d\theta$.

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There are  189 scores
grade       range        count       percent
   A      110...150      19          10.1%
  AB      105...109       7           3.7%
   B       90...104      25          13.2%
  BC       85... 89      19          10.1%
   C       60... 84      67          35.4%
   D       40... 59      39          20.6%
   F        0... 39      13           6.9%
Mean score =  76.1. Mean grade = 2.10.
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