Math 221 – Exam I – Friday October 6, 2000

Answers

I. (25 points.) Let $y = \dfrac{\sin x}{\cos x}$. Find dy/dx and d^2y/dx^2.

Answer:

$$\frac{dy}{dx} = \frac{\sin'(x)\cos(x) - \cos'(x)\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = (\cos x)^{-2}$$

so

$$\frac{d^2y}{dx^2} = -2(\cos x)^{-3}(-\sin x) = \frac{2\sin x}{\cos^3 x}.$$

Grader's Comments. *In regards to students' performance on Problem I of the exam, the big mistake was just not knowing the Quotient Rule. That's a shame, because I thought it was an easy 25 points.*

II. (25 points.) If a point traces the circle $x^2 + y^2 = 25$ and if $dx/dt = 7$ when the point reaches $(3,4)$ find dy/dt there.

Answer: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$ so $2 \cdot 3 \cdot 7 + 2 \cdot 4 \cdot \left.\dfrac{dy}{dt}\right|_{(x,y)=(3,4)} = 0$ so $\left.\dfrac{dy}{dt}\right|_{(x,y)=(3,4)} = -\dfrac{21}{4}$.

III. (25 points.) Two functions f and g are *inverse functions* if the equations $y = f(x)$ and $x = g(y)$ have the same graph, i.e. if $x = g(f(x))$ and $y = f(g(y))$. What is the inverse function g of the function $f(x) = x^3$ and what is the derivative of g?

Answer: The inverse function is $g(y) = y^{\frac{1}{3}}$ so

$$g'(y) = \frac{y^{-\frac{2}{3}}}{3}.$$

This is a consequence of the Chain Rule: $g(f(x)) = x$ so $g'(f(x))f'(x) = 1$ so if $y = f(x)$, then

$$g'(y) = g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{3x^2} = \frac{1}{3y^{\frac{2}{3}}}.$$

Grader's Comments. *More than half of the students did my question – find the inverse of $f(x) = x^3$ and its derivative – correctly. For the others, about half claimed the inverse as x^{-3}, another half as $x = y^3$. The notation – whether to use x or y as the variable of the inverse function – is part of the reason the students got confused. We'd better make a clear rule on it to the students.*

Remark. *The function g defined by the formula*

$$g(y) = y^{\frac{1}{3}}$$

is the same as the function g defined by the formula

$$g(x) = x^{\frac{1}{3}}.$$

IV. (25 points.) State and prove the formula for the derivative of the quotient of two functions. In your proof you may use (without proof) the limit laws, the theorem that a differentiable function is continuous, and high school algebra.[1]

[1] This is one of five proofs the students were told to prepare.

Answer: Suppose that $f(x) = u(x)/v(x)$ for all x where u and v are differentiable and $v(a) \neq 0$. Then

$$
\begin{aligned}
f'(a) = & \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} && \text{(definition)} \\
&= \lim_{x\to a} \frac{\big(u(x)/v(x)\big) - \big(u(a)/v(a)\big)}{x-a} && \text{(hypothesis)} \\
&= \lim_{x\to a} \frac{u(x)v(a) - u(a)v(x)}{v(x)v(a)(x-a)} && \text{(hsa)} \\
&= \lim_{x\to a} \left(\frac{u(x)-u(a)}{x-a} \cdot \frac{v(a)}{v(x)v(a)} - \frac{u(a)}{v(x)v(a)} \cdot \frac{v(x)-v(a)}{x-a} \right) && \text{(hsa)} \\
&= \left(\lim_{x\to a} \frac{u(x)-u(a)}{x-a} \right) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot \left(\lim_{x\to a} \frac{v(x)-v(a)}{x-a} \right) && \text{(lim law)} \\
&= u'(a) \cdot \frac{v(a)}{v(a)^2} - \frac{u(a)}{v(a)^2} \cdot v'(a) && \text{(definition)} \\
&= \frac{u'(a)v(a) - u(a)v'(a)}{v(a)^2} && \text{(hsa)}
\end{aligned}
$$

(In the fifth step the theorem that a differentiable function is continuous is also used.)

Grader's Comments. *The most common mistake I saw was the incorrect use of free and dummy variables as in*

$$f'(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

I also saw the expression $f'(u/v)$ where $(u/v)'$ would have been better. Another frequent error was mixing x and a as in

$$f'(a) = \frac{u'(a)v(x) - u(a)v'(a)}{v(a)^2}$$

These errors tell me that the students do not know how to use the notation correctly.

V. (25 points.) The function

$$f(x) = x^{1/2} + x^{-1/2}$$

is to be approximated near $x = 4$ by a quadratic function

$$Q(x) = C_0 + C_1(x-4) + C_2(x-4)^2$$

by choosing coefficients C_0, C_1, C_2 in such a way that

$$f(4) = Q(4), \qquad f'(4) = Q'(4), \qquad f''(4) = Q''(4).$$

Find the coefficients.

Answer: $Q(4) = C_0$, $Q'(4) = C_1$, and $Q''(4) = 2C_2$ while $f(4) = 2\frac{1}{2}$, $f'(4) = \frac{1}{2}4^{-\frac{1}{2}} - \frac{1}{2}4^{-\frac{3}{2}} = \frac{3}{16}$, and $f''(4) = -\frac{1}{4}4^{-\frac{3}{2}} + \frac{3}{4}4^{-\frac{5}{4}} = -\frac{1}{128}$. Hence

$$C_0 = \frac{5}{2}, \qquad C_1 = \frac{3}{16}, \qquad C_2 = -\frac{1}{256}.$$

Grader's Comments. *Basically, those who knew how to do the problem got it right and only made a few minor algebraic mistakes those who got it wrong had mainly these problems: 1) not knowing how to differentiate with C_0, C_1, C_2...didn't treat them as constants, 2) algebra (hsa) for exponents...ie, not knowing that $4^{-1/2} = 1/2$. I mostly saw $(1/4)\cdot(4)^{-1/2} = 1$???...however, those who left it in more complicated terms and choose not to simplify did not make those mistakes and got full credit, so...the moral is?*

VI. (25 points.) Find the limit. Distinguish between an infinite limit and one which doesn't exist. (Give reasons!)

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$

Answer: By a change of the dummy variable $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{u\to 0+} \frac{\sin u}{u} = 1$.

(b) $\displaystyle\lim_{x\to \infty} \frac{\sin(x^2)}{x^2}$

Answer: Since $\displaystyle -\frac{1}{x^2} \le \frac{\sin(x^2)}{x^2} \le \frac{1}{x^2}$, the limit is zero by the Sandwich Theorem.

(c) $\displaystyle\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$

Answer: $\displaystyle\sin\left(\frac{1}{x^2}\right) = 1$ if $x = 1/\sqrt{2n\pi + \pi/2}$ and $\displaystyle\sin\left(\frac{1}{x^2}\right) = -1$ if $x = 1/\sqrt{2n\pi - \pi/2}$ where n is an integer. Hence as $x \to 0$ the value $\displaystyle\sin\left(\frac{1}{x^2}\right)$ oscillates between -1 and 1 and so the limit does not exist.

(d) $\displaystyle\lim_{x\to 0} \frac{\sin 1}{x^2}$

Answer: $\displaystyle\lim_{x\to 0} \frac{\sin 1}{x^2} = (\sin 1) \lim_{x\to 0} \frac{1}{x^2} = \infty$. (Note that $\sin 1 > 0$ since $0 < 1 < \pi$.)

(e) $\displaystyle\lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h}$

Answer: By the definition of the derivative this is $\sin'(a)$, the derivative of the sine function evaluated at a. In other words,

$$\lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h} = \cos a.$$

Grader's Comments. *On the limit problems, most people seemed to be guessing. Many people said that if you have zero in the denominator, then the limit DNE. Also many people believed that* $\lim(\sin x/x) = 1$*, regardless of whether* $x \to 0$ *or not. The most egregious mistakes however were things like :*

$$\sin(a+h) = \sin a + \sin h, \qquad \sin(x^2) = (\sin x)\cdot x, \qquad \sin 1 = 0.$$