141 Sample Exam I

You should show exactly what to type into a calculator to get a numerical answer, but you may leave the arithmetic undone.

1a. Define rate of depreciation.

Answer. That the annual rate of depreciation for an asset is r means that the value (in dollars) of the asset in n years is $(1-r)$ times what it is today.

1b. If a car cost \$20,000 and depreciates at 10% per year what is price in two years?

Answer. $\$20{,}000(0.9)^2$

2a. Define rate of inflation.

Answer. That the annual rate of inflation is a means that the cost of a market basket in n years is $(1+a)^n$ times what it is today.

2b. If a market basket of goods costs \$123 and the inflation rate is 3% per year, what does it cost in two years?

Answer. $\$123(1.03)^2$

3. Starting today you deposit \$200 every month into an account that pays interest at 5% per year compounded daily. Exactly much money is in the account in fifteen and a half months? (Assume one month = 30 days, one year = 12 months = 360 days.)

Answer. The equivalent monthly rate is $x-1$ where

$$x = \left(1+\frac{.05}{30}\right)^3 0.$$

The \$200 deposited at the beginning of first month grows to $\$200x^{15.5}$ at 15.5 months. The \$200 deposited at the beginning of second month grows to $\$200x^{14.5}$ at 15.5 months. The \$200 deposited at the beginning of third month grows to $\$200x^{13.5}$ at 15.5 months. The \$200 deposited at the beginning of sixteenth month (=end of fifteenth month) grows to \$200$x^{0.5}$ at 15.5 months. The total amount in the account is

$$\begin{aligned}\$200x^{15.5} + \$200x^{14.5} + \cdots + \$200x^{0.5} \\ = \$200x^{0.5}(x^{15} + x^{14} + \cdots + 1) = \\ = \$200x^{0.5}\left(\frac{x^{16}-1}{x-1}\right)\end{aligned}$$

where x was defined above.

4. An account which pays 10% per year compunded monthly pays as much interest as an account paying what % compounded daily? (Assume one year = 12 months = 360 days.)

Answer. The daily rate is i such that

$$\left(1+\frac{.1}{12}\right)^{12} = (1+i)^{30}.$$

Hence

$$\left(1+\frac{.1}{12}\right)^{12/30} = 1+i$$

so

$$i = \left(1+\frac{.1}{12}\right)^{12/30} - 1$$

5. You have a fifteen year \$70,000 mortgage at a rate of 7.3% per year compounded monthly. What is your monthly payment? How much do you owe after 9 years, i.e. just after the 108th payment? What about just before the 108th payment?

Answer. First method. Imagine you set up a parallel savings account which pays 7.3% per year compounded monthly and will deposit d dollars every month so that at the end of 180 months the amount in the account is exactly enough to pay of your loan. In 180 months your debt has grown to

$$y^{180}\$70,000 \qquad \text{where } y = 1+\frac{.073}{12}$$

and the amount in the savings account is

$$dy + dy^2 + \cdots dy^{180} = dy(1 + y + \cdots + y^{179}) = dy\left(\frac{y^{180}-1}{y-1}\right)$$

Equating this two gives

$$y^{180}\$70,000 = dy\left(\frac{y^{180}-1}{y-1}\right)$$

so

$$d = \frac{y^{180}\$70,000}{y(y^{180}-1)/(y-1)} = \frac{y^{180}\$70,000(y-1)}{y(y^{180}-1)}$$

After 108 months your debt has grown to

$$y^{108}\$70,000 \qquad \text{where } y = 1 + \frac{.073}{12}$$

and the amount in the savings account is

$$dy + dy^2 + \cdots dy^{108} = dy(1 + y + \cdots + y^{107}) = dy\left(\frac{y^{108}-1}{y-1}\right)$$

so you still owe

$$y^{108}\$70,000 - dy\left(\frac{y^{108}-1}{y-1}\right)$$

where d and y are as above.

Second method. Imagine that you divide the \$70,000 into 180 different loans

$$\$70,000 = L_1 + L_2 + cdots + L_{180}$$

and that the kth loan L_k is paid off with the same monthly payment d at the end of the kth month. Then

$$\left(1 + \frac{0.073}{12}\right)^k L_k = d$$

so

$$L_k = x^k d \qquad \text{where } x = \left(1 + \frac{0.073}{12}\right)^{-1}.$$

Hence

$$\begin{aligned}\$70,000 &= xd + x^2d + \cdots x^{180}d \\ &= xd(1 + x + \cdots + x^{179}) \\ &= xd\left(\frac{x^{180}-1}{x-1}\right)\end{aligned}$$

so the monthly payment is

$$d = \frac{\$70,000}{(x^{180}-1)/(x-1)} = \frac{\$70,000(x-1)}{(x^{180}-1)}$$

After 108 months the first 108 loans have been paid and the remaining 72 loans have grown to a total debt of

$$D = \left(1 + \frac{0.073}{12}\right)^{108} (L_{109} + L_{110} + \cdots + L_{180}).$$

Using the above formula for L_k this is

$$D = \left(1 + \frac{0.073}{12}\right)^{108} (x^{109}d + x^{110}d + \cdots + x^{180}d).$$

Since $x = \left(1 + \frac{0.073}{12}\right)^{-1}$ so the debt is

$$D = x^1d + x^2d + \cdots + x^{72}d = xd(1 + x + \cdots + x^{71}) = xd\left(\frac{x^{72}-1}{x-1}\right)$$

where x and d are as above.

Both methods give the same answer since $x = 1/y$.

6. You have $2^{64} - 1$ kernels of wheat. $1024 = 2^{10}$ kernels of wheat fit into a tiny box which is one inch on each side. How many of these tiny boxes are required to hold the wheat? How many tiny boxes fit into a large box which is one foot on each side? (Hint: One foot = 12 inches.) How many large boxes are required to hold the wheat?

Answer. $2^{64} - 1$ kernels $= (2^{64} - 1)/1024$ tiny boxes $= (2^{64} - 1)/(1024 \cdot 12 \cdot 12 \cdot 12)$ large boxes.