Chapter 22
Borrowing Models

For All Practical Purposes: Effective Teaching

- When material involves a lot of formulas or multiple methods, determine if you want to allow students to use a formula or note sheet prior to presenting the material. If they will be required to memorize any formula or method, let students know this at the onset. However, if you plan to either give them formulas or some other information, wait until close to exam time or if asked directly by a student during lecture. Even if you had already planned to furnish such information or allow them to bring notes to an exam, let students see it as a reaction to their request. It shows your flexibility and should be greatly appreciated by the students.
- Students in the same course, but different instructors often compare notes. Be aware of what materials are covered in other sections, if there is a comprehensive final, and what the other instructors' grading policies are. Showing that you are aware that "Professor X is not covering that *hard* topic", or "X doesn't need to take a comprehensive final, so *why are we*?" will help to disarm student concerns and remind them that they are in your class and this is how you constructed it.

Chapter Briefing

In this chapter, you will be mainly examining applications where you find an optimal solution. Financial institutions would not be able to offer interest-bearing accounts such as savings accounts unless they had a way to make money on them. To do this, financial institutions use the money from savings accounts to make loans. The loans provide interest for the loaning institution and also enable individuals to make major purchases such as buying a house or a car. Using a credit card is also a way of taking out a loan. This chapter looks at some of the common types of loans.

Being well prepared for class discussion with examples and knowledge that students may need help using their calculators (or spreadsheets) is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Simple Interest**, **Compound Interest**, **Conventional Loans**, and **Annuities**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

Since you may demonstrate some techniques of this chapter using graphing calculator, the *Teaching Guide* includes the feature **Teaching the Calculator**. It includes brief calculator instructions with screen shots from a TI-83.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide Questions**. These are the complete solutions to the four questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Simple Interest

The initial amount borrowed for a loan is the **principal**. **Interest** is the money charged on the loan, based on the amount of the principal and the type of interest charged.

Simple interest uses a fixed amount of interest, which is added to the account for each period of the loan. The amount of interest owed after t years for a loan with principal P and annual rate of interest r is given by $I = Prt$. The total amount A of the loan after this time is $A = P(1 + rt)$.

Example

A bank offers a \$3000 loan and charges 4% simple interest. How much interest will be charged after five years? What is the amount of the loan after five years?

Solution

Because $P = \$3000$, $r = 0.04$, and $t = 5$ years, we have $I = Prt = \$3000(0.04)(5) = \$600$. The amount of the loan after 5 years is $\$3000 + \$600 = \$3600$.

An **add-on loan** is borrowed at an amount P that is to be repaid with n payments in t years. The interest is simple interest at an annual rate of r. Since the total amount of the loan is $A = P(1 + rt)$, each payment, d, will be $d = \dfrac{P(1 + rt)}{n}$.

Example

A 4% add-on loan is to be repaid in monthly installments over 5 years. The total amount borrowed was \$3000. How much is the monthly payment?

Solution

This loan will have 60 payments, because it is paid 12 times per year for 5 years. The amount of the loan will be $A = \$3600$. Dividing this by the 60 payments, we have a monthly payment of $d = \dfrac{\$3600}{60} = \$60.00$.

A discounted loan is borrowed at an amount P that is to be repaid in t years. The interest is simple interest at an annual rate of r, just as with an add-on loan. However, the interest is subtracted from the amount given to the borrower at the time the loan is made. So the borrower only gets $P - I = P - Prt = P(1 - rt)$, but still needs to pay back the principal P.

Teaching Tip

In borrowing models, show students how much the interest rate and length of loan can affect not only the monthly payment, but how much interest is paid back.

Example

A 7% discounted loan is to be repaid in monthly installments over 5 years. The total amount borrowed was $6000.

a) How much is the monthly payment?

b) How much money does the borrower actually get at the beginning of the loan?

Solution

a) The total amount to be paid over 5 years is P = $6000. Because it is being paid 12 times per year, the monthly payment is $\frac{\$6000}{60} = \$100.00$.

b) The amount the borrower actually gets at the beginning of the loan is the following.

$$P(1-rt) = \$6000(1-(0.07)(5)) = \$6000(0.65) = \$3900.00$$

Example

How much would the discount-loan borrower in the last example need to borrow to get $6000 at the start of the loan?

Solution

Call the loan amount x. The borrower will receive $x(1-rt) = x(1-(0.07)(5)) = 0.65x$. We want this to be equal to $6000, so solve $0.65x = \$6000$ to get $x = \frac{\$6000}{0.65} = \$9230.77$.

Compound Interest

When interest is compounded, the interest that is charged is added to the principal. The next interest calculation is based on the new amount, so the interest from the previous period is now earning interest as well.

If a loan has a principal P with an interest rate of i per compounding period, then the amount owed on the loan after n compounding periods is $A = P(1+i)^n$. This is assuming no repayments.

Example

Jivan borrows $1000 at 2% interest per month, compounded monthly. If he pays the loan back in three years, how much will he owe?

Solution

Because the interest is compounded monthly for three years, $n = 12 \times 3 = 36$. So the amount owed is $A = P(1+i)^n = \$1000(1+0.02)^{36} = \$2039.89$.

Teaching Tip

Since many students taking this course may have never applied for a loan, in examples like the last one, point out that over half of the amount paid back is in interest.

The **nominal rate** for a loan is the stated interest rate for a particular length of time. The nominal rate does not take compounding into account.

The effective rate for a loan does reflect compounding. The **effective rate** is the actual percentage that the loan amount increases over a length of time.

When the effective rate is given as a rate per year, it is called the **effective annual rate** (**EAR**).

The nominal rate for a length of time during which no compounding occurs is denoted as *i*. The **annual percentage rate** (**APR**) is given by the rate of interest per compounding period, *i*, times the number of compounding periods per year, *n*. Thus, $\text{APR} = i \times n$.

Example

A credit card bill shows a balance due of \$875 with a minimum payment of \$17 and a monthly interest rate of 1.89%. What is the APR?

Solution

Because the monthly interest rate is $i = 0.0189$ and the account is compounded monthly, the APR is $i \times n = (0.0189)(12) = 0.2268 = 22.68\%$! The balance due and minimum payment information is not needed to solve this.

Conventional Loans

When the regular amounts *d* are payments on a loan, they are said to **amortize** the loan. The **amortization formula** equates the accumulation in the savings formula with the accumulation in a savings account given by the compound interest formula. This is a model for saving money to pay off a loan all at once, at the end of the loan period.

For a loan of *A* dollars requiring n payments of *d* dollars each, and with interest compounded at rate *i* in each period, the amortization formula is as follows.

$$A = d\left[\frac{1-(1+i)^{-n}}{i}\right] \text{ or } d = \frac{Ai}{1-(1+i)^{-n}}$$

Example

Dan Dreibelbis takes out a conventional loan to purchase a car. The interest rate is 3.75% compounded monthly and Dan has five years to repay the \$22,000 he borrowed. What are Dan's monthly payments?

Solution

In this case, $A = \$22{,}000$. Because there are twelve compounding periods for each of the five years, $n = 12 \times 5 = 60$. To find the interest rate per compounding period, divide by twelve because there are twelve compounding periods per year: $i = \frac{0.0375}{12} = 0.003125$. So given $d = \frac{Ai}{1-(1+i)^{-n}}$ we have the following.

$$d = \frac{\$22{,}000(0.003125)}{1-(1+0.003125)^{-60}} = \frac{\$68.75}{0.1707284716} = \$402.69$$

The **effective annual rate** (**EAR**) takes into account monthly compounding. For a loan with n compounding periods per year, with interest compounded at rate i in each compounding period, the EAR can be found using EAR $= (1 + i)^n - 1$.

Example

A credit card bill shows a balance due of \$875 with a minimum payment of \$17 and a monthly interest rate of 1.89%. What is the EAR?

Solution

Because the monthly interest rate is i = 0.0189 and the account is compounded monthly, the EAR is as follows.

$$(1 + 0.0189)^{12} - 1 = 0.2519 = 25.19\%$$

The balance due and minimum payment information is not needed to solve this.

After owning a home for a period of time, one builds **equity** in the home. One can use the amortization formula, $A = d\left[\frac{1-(1+i)^{-n}}{i}\right]$, to determine the amount of equity given the number of payments originally required, the number of payments made, the amount of each payment, the original loan amount, and the interest per payment period. Here, n is the number of payments left to be made on the life of the loan (number of payments originally required minus the number of payments made).

☝Teaching Tip

At the end of this chapter discuss with students as to how loan repayments should be rounded. We have been using pure rounding, but if we round down the loan would not be paid in full.

Example

A couple buys a house for \$120,000 at 3.75% interest. After making payments for two years, they decide to sell their home. If the original mortgage was for 30 years, how much equity do they have?

Solution

We must first determine how much their monthly payment was during the 24 months. We have A = \$120,000. Because there are twelve compounding periods for each of the thirty years, $n = 12 \times 30 = 360$. To find the interest rate per compounding period, divide by twelve because there are twelve compounding periods per year: $i = \frac{0.0375}{12} = 0.003125$. So given $d = \frac{Ai}{1-(1+i)^{-n}}$ we have the following.

$$d = \frac{\$120,000(0.003125)}{1-(1+0.003125)^{-360}} = \frac{\$375}{0.6747775408} = \$555.7387$$

The monthly payment would be \$555.74. We need to determine how much is left on the life of the loan. Since the couple has made 24 payments, they still have $360 - 24 = 336$ payments left. We can use the amortization formula, $A = d\left[\frac{1-(1+i)^{-n}}{i}\right]$, where $d = \$555.74$, $i = 0.003125$, and $n = 336$.

$$A = d\left[\frac{1-(1+i)^{-n}}{i}\right] = \$555.74\left[\frac{1-(1+0.003125)^{-336}}{0.003125}\right] = \$115,503.02$$

The amount of equity would be $\$120,000.00 - \$115,503.02 = \$4496.98$.

☝Teaching Tip

Ask students in an example like the last one to determine how much of the total payments went to build equity in the house and how much went to interest.

Annuities

Example

An annuity has a value of \$100,000 and will be paid in equal monthly payments over 25 years at 3% annual interest. How much would each monthly payment be?

Solution

Because the annual rate is 3%, $i = \frac{0.03}{12} = 0.0025$. Set up the annuity formula with $n = 12 \times 25 = 300$.

$$d = \frac{Ai}{1-(1+i)^{-n}} = \frac{\$100,000 \times 0.0025}{1-(1+0.0025)^{-300}} = \$474.21$$

Teaching the Calculator

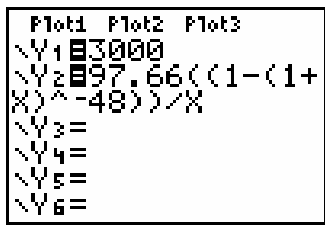
In Exercises 31 – 33 in this chapter, students will need to solve $A = d\left[\frac{1-(1+i)^{-n}}{i}\right]$ for i using some form of technology. One method for doing this is with a graphing calculator.

Example

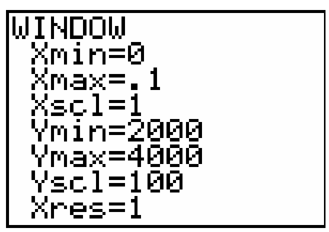
In Exercise 32 we must first solve $3000 = 97.66\left[\frac{1-(1+i)^{-48}}{i}\right]$. Use the graphing calculator to solve for *i*.

Solution

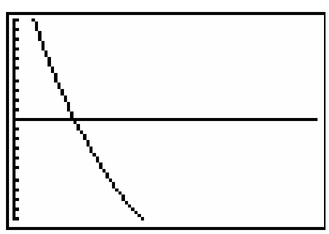
Press [Y=] and enter the left hand side of the equation to be Y_1 and the right to be Y_2. We have the following screen.



By pressing [WINDOW] you will enter an appropriate window for Xmin, Xmax, Ymin and Ymax. Since *x* represents an interest rate (in decimal form) and we need a point of intersection involving the horizontal line $y = 3000$, an appropriate window would be the following.

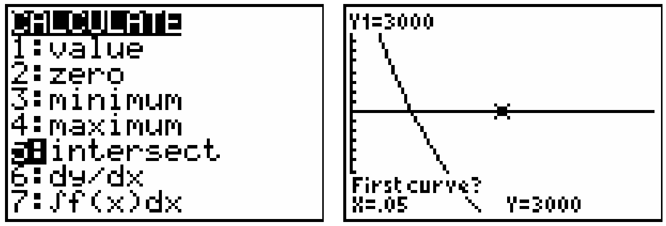


By pressing [GRAPH] we obtain the following.

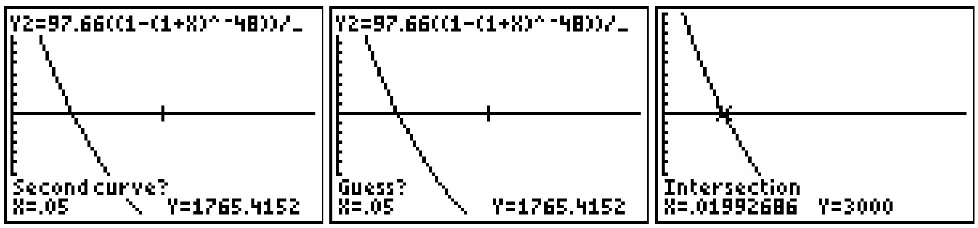


Continued on next page

We need to find the point of intersection between the two lines. To do this, you will need to press [2nd] followed by [TRACE]. You then need to toggle down or press [5] followed by [ENTER].



Press [ENTER] three times and the following three screens will be displayed.



We are interested in the x-component of the point of intersection. Thus, $i = 0.01992686$.

Solutions to Student Study Guide ✏ Questions

Question 1
A 9% add-on loan is to be repaid in monthly installments over 14 years. The total amount borrowed was $4500. How much is the monthly payment?

Solution
This loan will have 168 payments, because it is paid 12 times per year for 14 years. The amount of the loan will be $A=\$4500\left(1\ +\ 0.09(14)\right)=\$4500(2.26)=\$10{,}170.$ Dividing this by the 168 payments, we have a monthly payment of $d=\dfrac{\$10{,}170}{168}=\$60.54.$

Question 2
A 2.5% discounted loan is to be repaid in monthly installments over 5 years. The total amount borrowed was $6000.

a) How much is the monthly payment?

b) How much money does the borrower actually get at the beginning of the loan?

Solution
a) The total amount to be paid over 5 years is P = $6000. Because it is being paid 12 times per year, the monthly payment is $\dfrac{\$6000}{60}=\$100.00.$

b) The amount the borrower actually gets at the beginning of the loan is the following.

$$P(1-rt)=\$6000\left(1-(0.025)(5)\right)=\$6000(0.875)\ =\$5250.00$$

Question 3
Ivan takes out a conventional five-year loan to purchase a $20,000 car. The interest rate is 3.6% compounded monthly. What are Ivan's monthly payments?

Solution
In this case, A = $20,000. Because there are twelve compounding periods for each of the five years, $n=12\times 5=60.$ To find the interest rate per compounding period, divide by twelve, because there are twelve compounding periods per year: $i=\dfrac{0.036}{12}=0.003$. So given $d=\dfrac{Ai}{1-(1+i)^{-n}}$ we have the following.

$$d=\frac{\$20{,}000(0.003)}{1-(1+0.003)^{-60}}=\frac{\$60}{0.1645046853}=\$364.731253$$

The payments would be $364.73 (with pure rounding) or $364.74 (with rounding up).

Question 4

A couple buys a house for $250,000 at 4.5% interest. After making payments for a year and a half, they decide to sell their home. If the original mortgage was for 30 years, how much equity do they have? Round to the nearest $100.

Solution

We must first determine how much their monthly payment was during the 18 months. We have A = $250,000. Because there are twelve compounding periods for each of the thirty years, $n = 12 \times 30 = 360$. To find the interest rate per compounding period, divide by twelve, because there are twelve compounding periods per year: $i = \frac{0.045}{12} = 0.00375$. So given $d = \frac{Ai}{1-(1+i)^{-n}}$ we have the following.

$$d = \frac{\$250{,}000(0.00375)}{1-(1+0.00375)^{-360}} = \frac{\$937.5}{0.7401043463} = \$1266.713275$$

With pure rounding, the monthly payment would be $1266.71, rounding up to the next penny would be $1266.72. Since we will round to the nearest $100 at the end of the problem either monthly payment is acceptable to use.

We need to determine how much is left on the life of the loan. Since the couple has made 18 payments, they still have $360 - 18 = 342$ payments left. We can use the amortization formula, $A = d\left[\frac{1-(1+i)^{-n}}{i}\right]$, where $d = \$1266.71$, $i = 0.00375$, and $n = 342$.

$$A = d\left[\frac{1-(1+i)^{-n}}{i}\right] = \$1266.71\left[\frac{1-(1+0.00375)^{-342}}{0.00375}\right] = \$243{,}880.81$$

The amount of equity would be $\$250{,}000.00 - \$243{,}880.81 = \$6119.19$. Rounding to the nearest hundreds, the amount of equity would be $6100.