Chapter 21
Savings Models

For All Practical Purposes: Effective Teaching

Doing self-evaluation can be a useful technique to continue the evolving process of becoming an effective instructor. Questions you can ask yourself and reflect on could include the following.

- *How do I typically begin a class?*
- *Do I typically ask students questions during lecture or do I strictly lecture to students?*
- *How would a student describe me as their professor/instructor (or teaching assistant)?*
- *How do I typically react to student questions or corrections to my errors?*
- *How do I typically end a class?*

Chapter Briefing

In this chapter, you will be mainly modeling situations of growth in economics and finance, examining the increase or decrease in value of investments and various economic assets. These models will apply similarly to the growth of biological populations, money in a bank account, pollution levels, and so on. In the same vein, managing a financial entity like a trust fund is similar to managing a renewable biological resource.

Being well prepared for class discussion with examples is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Arithmetic Growth and Simple Interest**, **Geometric Growth and Compound Interest**, **A Limit to Compounding**, **A Model for Investment**, **Exponential Decay and the Consumer Price Index**, and **Real Growth and Valuing Investments**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the three questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

➾Arithmetic Growth and Simple Interest

The initial balance of an account, such as a savings account, is the **principal**. At the end of a fixed amount of time, **interest** is added to the account.

If a population, measured at regular time intervals, is experiencing **arithmetic growth** (also called **linear growth**), then it is gaining (or losing) a constant amount with each measurement.

Example
If an account has an initial value of \$10,000 and gains \$200 at each interval, write the sequence of population values.

Solution
Starting with the initial value \$10,000, successively add \$200. The result is: \$10000, \$10200, \$10400, \$10600, \$10800, … .

☝Teaching Tip
Relay to students that they have been introduced to arithmetic sequences when they learned their multiplication tables.

A financial account which is paying **simple interest** will grow arithmetically in value. If we let I be the interest earned, P be the principal, r be the annual rate of interest, A be the total amount (including principal and interest), and t be the number of years, the formulas involving simple interest are as follows.

$$I = Prt \text{ and } A = P + I = P + Prt = P(1 + rt)$$

☝Teaching Tip
Relay to students that formulas are case sensitive. The symbols "I" and "i" will have different meanings.

➾Geometric Growth and Compound Interest

🗝 Key idea
A savings account which earns **compound interest** is growing geometrically. At the end of the first year, the initial balance, or principal, is increased by the interest payment. Each successive year, the new balance is the previous balance plus interest, paid as a fixed percentage of that balance. The value of the savings account is determined by the **principal** (or **initial balance**), the **rate of interest**, and the **compounding period**.

Example
Suppose you deposit a \$5000 principal to start up a bank account with an annual interest rate of 8%, compounded quarterly. How much money will you have in the account at the end of the first year?

Solution
Each quarter the value of the account grows by 1/4 of the annual interest of 4%. This means the account balance is multiplied by 1.01 four times in the course of the year. Here is a table of account values.

initial deposit	quarter 1	quarter 2	quarter 3	quarter 4
\$5000	\$5050	\$5100.50	\$5151.51	\$5203.03

There will be \$5203.03 at the end of the year.

☝Teaching Tip

Point out to students that the amount of interest gained from one compounding period to the next increases in an example such as the last one. Ask students as to why this is the case.

The annual interest rate of a savings account is called the **nominal rate**. With compounding, the actual realized percentage is higher; it is called the **effective annual rate**, or **EAR**.

The nominal rate *i* for a period during which no compounding is done is given by $i = \frac{r}{n}$, where *r* is the nominal annual rate and *n* is the number of times interest is compounded per year.

Example

For an account earning 5% interest compounded quarterly, what is the nominal rate *i* for a one-quarter period?

Solution

Because $r = 5\%$ and $n = 4$, $i = \frac{0.05}{4} = 0.0125$ or 1.25% per quarter.

The **compound interest formula** for the value of a savings account after compounding periods is as follows.

$$A = P(1+i)^n$$

Here, *P* is the principal and *i* is the interest rate per compounding period.

Example

If $5000 is deposited in an account earning 6% interest compounded annually, what will be the value of the account:

a) after 10 years?

b) after 20 years?

Solution

a) With annual compounding, we have one interest payment each year, or five in five years. Because $i = 6\% = 0.06$, the formula for the value of the account after 10 years is $A = 5000(1+0.06)^{10} = \$8954.24$.

b) After 20 years we have $A = 5000(1+0.06)^{20} = \$16,035.68$.

Example

If $5000 is deposited in a different account earning 6% interest compounded monthly, what will be the value of the account:

a) after 10 years?

b) after 20 years?

Solution

a) With monthly compounding, we have 12 interest payments per year, or 120 in ten years. Now the formula for the value of the account after 10 years is $A = 5000\left(1+\frac{0.06}{12}\right)^{120} = \$9096.98$.

b) After 20 years, we have 240 interest periods, so $A = 5000\left(1+\frac{0.06}{12}\right)^{240} = \$16,551.02$.

The present value P of an amount A to be paid in the future, after earning compound interest or n compounding periods at a rate i per compound period is as follows.

$$P = \frac{A}{(1+i)^n}$$

If a population is experiencing **geometric** (**exponential**) **growth**, then it is increasing or decreasing by a fixed proportion of its current value with each measurement. The proportion is called the growth rate of the population.

Accounts earning compound interest will grow more rapidly than accounts earning simple interest. In general, geometric growth (such as compound interest) is much more dramatic than arithmetic growth (such as simple interest).

A Limit to Compounding

For a nominal interest rate r compounded n times per year, **the annual effective interest rate**, or **APY**, is $\left(1+\frac{r}{n}\right)^n - 1$.

Example

What is the APY for 6% compounded monthly?

Solution

Because $r = 0.06$ and $n = 12$, the APY is $\left(1+\frac{r}{n}\right)^n - 1 = \left(1+\frac{0.06}{12}\right)^{12} - 1 = (1.005)^{12} - 1 = 0.0617$ or 6.17%.

Compounding interest more often results in a higher value in the account because interest is earned earlier and is included with the principal in the next compounding period. However, the more often interest is compounded, the less significant this increase becomes. The limit is reached when interest is **compounded continuously**. The formula for finding the account balance when interest is compounded continuously is $A = Pe^{rt}$, where r is the nominal interest rate and t is the time in years. e is a constant which is approximately equal to 2.718281828. The pattern of decimals do not repeat and does not yield a rational number.

Example

If \$5000 is deposited into an account earning 6% interest compounded continuously, find the value of the account after

a) 10 years.

b) 20 years.

Solution

a) Using a calculator we find that $A = 5000e^{(0.06)(10)} = 5000e^{0.6} = \$9110.59$.

b) Similar to what was done in part a), we have $A = 5000e^{(0.06)(20)} = 5000e^{1.2} = \$16{,}600.58$.

There is virtually no difference whether a bank treats a year as 365 days or 360 days. The **365 over 365 method** with a daily nominal rate of $\frac{r}{365}$ is the usual method for daily compounding. The **360 over 360 method** with a daily nominal rate of $\frac{r}{360}$ is the usual method for loans with equal monthly installments.

Note: In the *Instructor's Manual with Full Solutions*, the solutions to Exercises 13 and 14 should use the formulas $P\left(1+\frac{r}{360}\right)^{360}$ and $P\left(1+\frac{r}{365}\right)^{365}$.

A Model for Investment

A **geometric series** with first term 1, common ratio *x*, and *n* terms is the following.

$$1 + x + x^2 + ... + x^{n-1}$$

The sum of these terms is $\frac{x^n - 1}{x - 1}$.

Teaching Tip

Emphasize to students that the sum contains *n* terms, as opposed to $n-1$ terms.

We can accumulate a desired amount of money in a savings account by a fixed date by making regular deposits at regular intervals – a **sinking fund**. With a uniform deposit of *d* dollars at the end of each interval, and an interest rate of *i* per interval, the **savings formula** predicts that the value of the account after *n* intervals will be as follows.

$$A = d\left[\frac{(1+i)^n - 1}{i}\right]$$

This formula is obtained by summing a geometric series of accumulated deposits and interest.

Example

John and Janna Samons will need $10,000 five years from now to make a down payment on a new house. How much would they have to deposit each month in a sinking fund with a 3% annual interest rate to accumulate this amount?

Solution

Here, d is the unknown amount to be deposited, $i = \frac{0.03}{12} = 0.0025$ is the monthly interest rate and $n = 5 \times 12 = 60$, the number of deposits in 5 years. Then we can solve for d in the savings formula as follows.

$$A = d\left[\frac{(1+i)^n - 1}{i}\right]$$

$$10,000 = d\left[\frac{(1+0.0025)^{60} - 1}{0.0025}\right]$$

$$10,000 = d(64.64671262)$$

$$d = \frac{10,000}{64.64671262} = \$154.69$$

They need to deposit $154.69 at the end of each month for 5 years.

An **annuity** pays a specified number of equal payments at equally spaced time intervals. A sinking fund is the reverse of an annuity. With a sinking fund you make payments, and with an annuity you receive payments.

Exponential Decay and the Consumer Price Index

Geometric growth with a negative growth rate is called **exponential decay**. Examples are depreciation of the value of a car and decay in the level of radioactivity of a given quantity of a radioactive isotope. The quantity is declining at a rate that is negative and proportional to its size; the proportion l is called the decay constant.

With inflation, the value of currency declines. If the rate of inflation is *a*, then the present value of a dollar in one year is given by the formula $\frac{\$1}{1+a}=\$1-\frac{\$a}{1+a}$.

Example

Assuming constant 3.2% inflation, what would the present value of a $20 bill be in ten years?

Solution

The present value of one dollar after one year is given by $\frac{\$1}{1+0.032}=0.96899.$ or 96.9 cents. After ten years, the present value has shrunk by this factor ten times. Thus, after ten years one dollar has a value of $\left(\frac{\$1}{1+0.032}\right)^{10}=0.7298,$ or 73.0 cents. Multiplying this by 20 we have $14.60.

The **Consumer Price Index** (**CPI**) is the measure of inflation. The CPI compares the current cost of certain goods, including food, housing, and transportation, with the cost of the same (or comparable) goods in a base period.

To convert the cost of an item in dollars for one year to dollars in a different year, use the following proportion.

$$\frac{\text{Cost in year A}}{\text{Cost in year B}}=\frac{\text{CPI for year A}}{\text{CPI for year B}}$$

You will need to use Table 21.5 on page 816 of your text to obtain the CPI for a particular year.

Example

Adam bought a house in 1991 for $61,000 and sold it in 2005. How much would the house be worth in 2005 dollars? Round to the nearest hundred dollars.

Solution

Set up the following proportion using values from Table 21.5.

$$\frac{\text{Cost in 1991}}{\text{Cost in 2005}}=\frac{\text{CPI for year 1991}}{\text{CPI for year 2005}}$$

$$\frac{\$61{,}000}{\text{Cost in year 2005}}=\frac{136.2}{195.0}$$

$$\$61{,}000(195.0)=136.2(\text{Cost in year 2005})$$

$$\text{Cost in year 2005}=\frac{\$11{,}895{,}000}{136.2}=\$87{,}334.80176$$

The house is worth around $87,300.

Real Growth and Valuing Investments

An investment is affected by the rate of inflation. An investment that grows at, say, 6.5% per year will not actually gain purchasing power at 6.5% per year if inflation is considered.

If an investment grows at an annual rate r and the rate of inflation is a, the real growth rate g is given by the following.

$$g = \frac{r-a}{1+a}$$

Example
In late 2001 the inflation rate was about 2.9%. If you invested in a savings account with an annual interest rate of 7.2%, what was the real growth rate of this investment?

Solution
In this case, $r = 7.2\% = 0.072$ and a = 2.9% = 0.029, so we have the following.

$$g = \frac{r-a}{1+a} = \frac{0.072-0.029}{1+0.029} = \frac{0.043}{1.029} = 0.041788 \text{ or about } 4.2\%$$

Buyers of stock in a company receive part of the profits of the company in the form of cash payments, known as **dividends.**

An **overvalued** stock is one that has a current price that is too high compared to the company's expected earnings.

Modeling the value of a stock requires considering the growth rate of the annual dividend as well as the **discount rate**. The discount rate for a stock takes into account the effects of inflation, the level of risk for that particular stock, and the value of interest on alternate investments.

Suppose you want to buy a stock at the beginning of the year and D was last year's dividends, paid at the beginning of this year to those that owned stock on a particular date, g is the annual rate the company expands and prospers and r is the discount rate per year, then the present value of next year's dividends is $D\frac{1+g}{1+r}$.

An option to buy a stock at a certain price by a certain time is an example of a "financial derivative." The true value of a derivative depends on the current value and the probabilities that the stock will go up or down within the option's time frame. The famous "**Black-Scholes formula**" is often used to value financial derivatives.

Solutions to Student Study Guide ✎ Questions

Question 1

If \$5000 is invested at an annual rate of 2.4% with simple interest, how much money is in the account after 4 years?

Solution

P is \$5000, r is 2.4% = 0.024, and t is 4 years. Since $A = P(1+rt)$, we have the following.

$$A = P(1+rt) = \$5000(1+0.024\cdot 4) = \$5000(1.096) = \$5480$$

Question 2

If \$5000 is deposited in an account earning 2.4% interest compounded quarterly, what will be the value of the account:

a) after 5 years?

b) after 20 years?

Solution

P is \$5000. Since the money is compounded quarterly, $i = \frac{0.024}{4} = 0.006$. We need to find A, where $A = P(1+i)^n$.

a) After 5 years, we have 20 interest periods, so $A = \$5000(1+0.006)^{20} = \$5635.46$.

b) After 20 years, we have 80 interest periods, so $A = \$5000(1+0.006)^{80} = \$8068.79$.

Question 3

If \$10,000 is deposited into an account with annual rate of 2.7%, what is the amount in the account after 1 year if it is compounded

a) continuously?

b) using the 365 over 365 method?

c) using the 360 over 360 method?

Solution

a) P is \$5000. Since the money is compounded continuously we find A, where $A = Pe^{rt}$, r is the nominal interest rate, 2.7% = 0.027, and t is 1 year.

$$A = Pe^{rt} = \$10,000e^{0.027(1)} = \$10,273.68$$

b) P is \$5000. Since the money is compounded using the 365 over 365 method we find A, where $A = P\left(1+\frac{r}{365}\right)^{365}$, and $\frac{r}{365} = \frac{0.027}{365}$ is the daily nominal interest rate.

$$A = P\left(1+\tfrac{r}{365}\right)^{365} = \$10,000\left(1+\tfrac{0.027}{365}\right)^{365} = \$10,273.67$$

c) P is \$5000. Since the money is compounded using the 360 over 360 method we find A, where $A = P\left(1+\frac{r}{360}\right)^{360}$, and $\frac{r}{360} = \frac{0.027}{360}$ is the daily nominal interest rate.

$$A = P\left(1+\tfrac{r}{360}\right)^{360} = \$10,000\left(1+\tfrac{0.027}{360}\right)^{360} = \$10,273.67$$