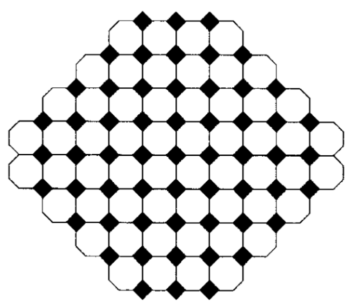
Chapter 20
Tilings

For All Practical Purposes: Effective Teaching

- With this day and age of technology, most students are adept at using E-mail as a form of communication. Many institutions automatically assign an E-mail address to active students. If students are assigned an address, encourage them to activate it and work with each other. If it seems feasible, create an E-mail distribution list.
- Assigning projects or papers that require teamwork helps to create an atmosphere where students come to class knowing one another. They tend to want to help each other more and form study groups.

Chapter Briefing

In this chapter, you will be mainly examining some traditional and modern ideas about *tiling* (*tessellation*), the covering of an area or region of a surface with specified shapes. The beauty and complexity of such designs come from the interesting nature of the shapes themselves, the repetition of those shapes, and the symmetry or asymmetry of arrangement of the shapes.



Being well prepared for class discussion with examples is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Tilings with Regular Polygons**, **Tilings with Irregular Polygons**, **Using Translations**, **Using Translations Plus Half-Turns**, and **Nonperiodic Tilings**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the two questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

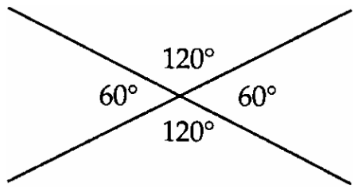
Chapter Topics to the Point

Tilings with Regular Polygons

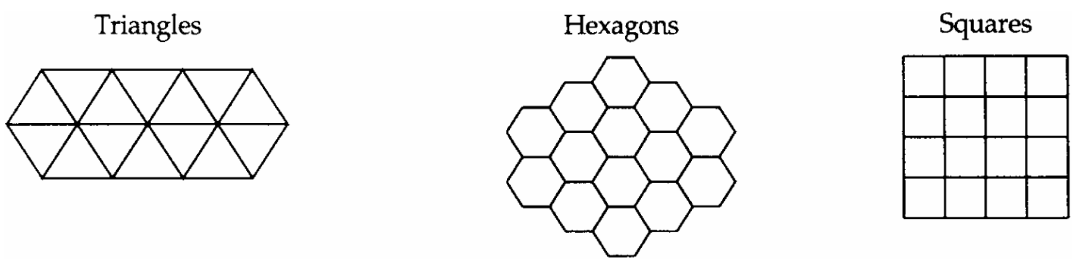
A tiling is **monohedral** if all the tiles are the same shape and size; the tiling would consist of repetitions of one figure laid down next to each other.

A **tiling** of the plane is a covering of that flat surface with non-overlapping figures.

In an **edge-to-edge tiling**, the interior angles at any vertex add up to 360°. An example is the following.



A **regular tiling** uses one tile, which is a regular polygon. Here are only three possible regular tilings.



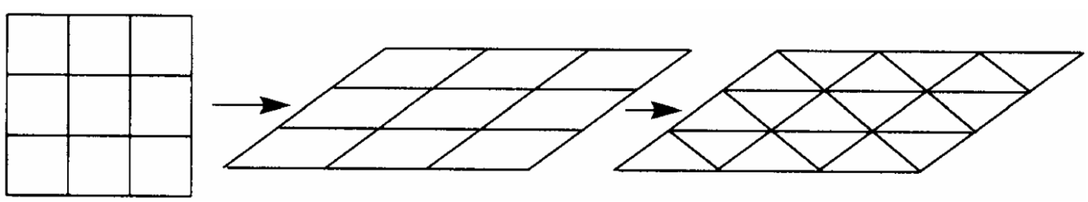
Teaching Tip

Review the names of polygons at the beginning of this chapter. Students should be comfortable with quadrilateral up to decagon.

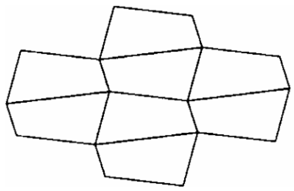
There are eight additional **semi-regular tilings**, using a mix of regular polygons with different numbers of sides.

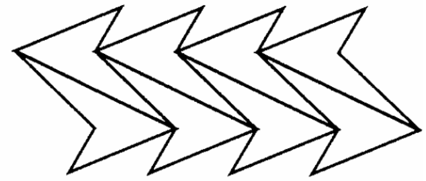
Tilings with Irregular Polygons

It is easy to adapt the square tiling into a monohedral tiling using a **parallelogram**. Since two triangles together form a parallelogram, any triangle can tile the plane.



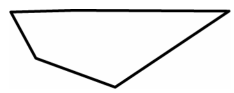
Any **quadrilateral** (four-sided figure), even one that is not convex, can tile the plane. A figure is **convex** if any two points on the figure (including the boundary) can be connected and the line segment formed does not go out of the figure.



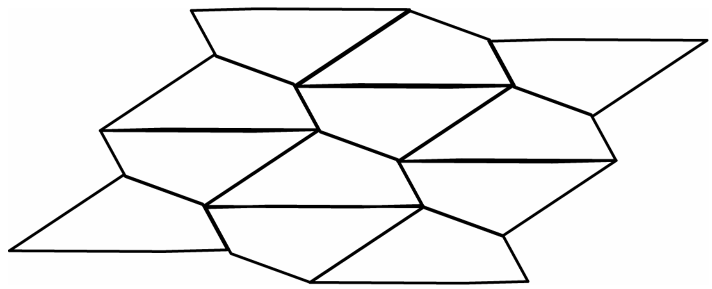


Also, any triangle can tile the plane. A scalene triangle has no two sides the same measure.

Example
Draw a tiling of the plane with the following figure.



Solution
Two copies of the quadrilateral, one of which is rotated by 180°, fit together, forming an easy tiling.



Only certain classes of convex pentagons and hexagons can be used to tile the plane. There are exactly three classes of convex hexagons that can tile a plane. A convex polygon with seven or more sides cannot tile the plane.

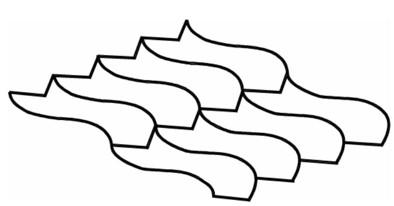
The work of artist **M. C. Escher,** famous for his prints of interlocking animals, demonstrates an intimate link between art and mathematics.

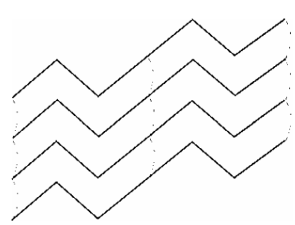
Using Translations

The simplest way to create an Escher-like tiling is through the use of translation. The boundary of each tile must be divisible into matching pairs of opposing parts that interlock.

A single tile can be duplicated and used to tile by **translation** in two directions if certain opposite parts of the edge match each other.

These two are based on a parallelogram tiling.





This one is based on a hexagon tiling.

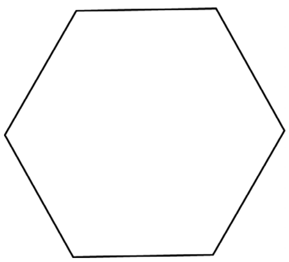
Example

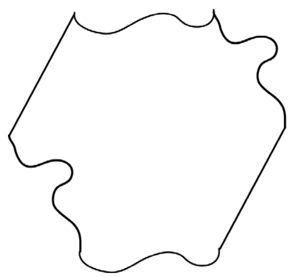
a) Draw a hexagon, replace at least two sets opposite sides with congruent curved edges, and draw a tiling by translation with the resulting figure.

b) Can you do the same thing with a pentagon?

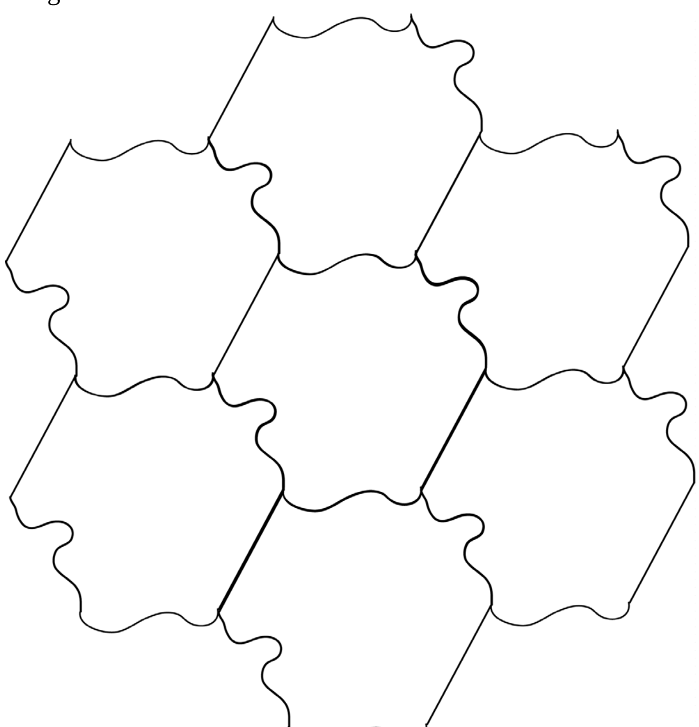
Solution

a)





Translating horizontally and vertically, opposite sides match up. Here is an illustration of the process for a hexagon-based tile.



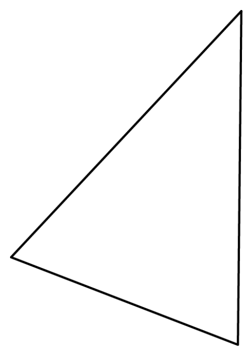
b) You cannot pair up “opposite” sides in a pentagon, or any polygon with an odd number of sides. This cannot be done.

Using Translations Plus Half-Turns

If you replace certain sides of a polygon with matching **centrosymmetric** segments, it may be possible to use the resulting figure to tile the plane by translations and half-turns. The **Conway criterion** can be used to decide if it is possible. The Conway criterion is given on page 770 of your text.

Example

Start with the following triangle.

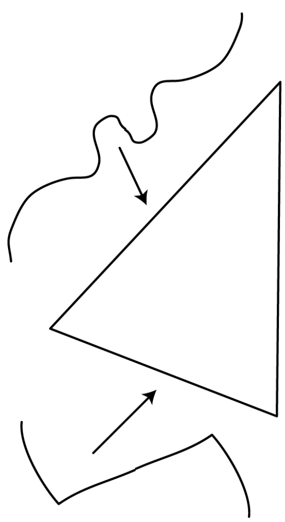


Replace two sides with centrosymmetric curves and sketch a tiling by translations and half-turns.

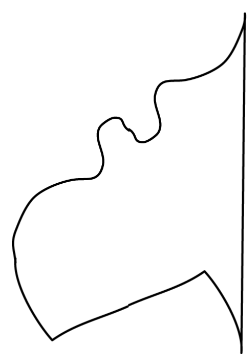
Solution

This shows the process and the resulting tiling:

Here are the new sides.



This is the tile.



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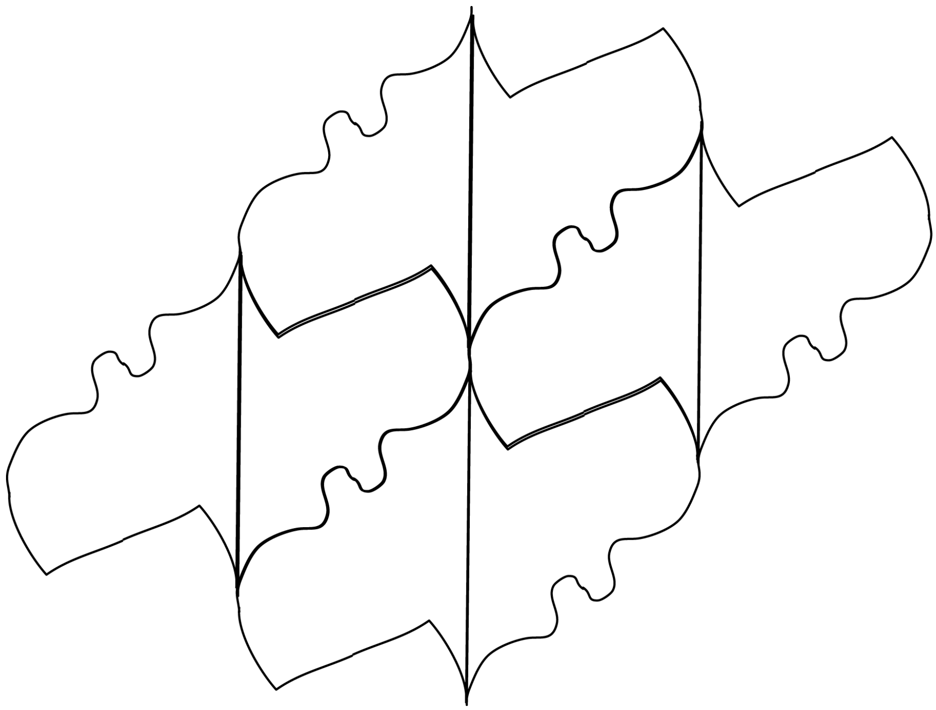
Here is the tiling.



Many fascinating and beautiful examples of these principles are found in the designs of the renowned graphic artist M. C. Escher.

Periodic tilings have a fundamental region that is repeated by translation at regular intervals.

A **fundamental region** consists of a tile, or block of tiles, with which you can tile a plane using translations at regular intervals.

Nonperiodic Tilings

A tiling may be **nonperiodic** because the shape of the tiles varies, or the repetition of the pattern by translation varies. The Penrose tiles are an important example of a set of two tiles which can be used only to tile the plane nonperiodically.

Penrose tilings exhibit the following properties: self-similarity (inflation and deflation), the golden ratio (quasiperiodic repetition in that proportion), and partial five-fold rotational symmetry.

Applying principles of tiling to three dimensional crystals, **Barlow's law** states that a crystal cannot have more than one center of fivefold rotational symmetry.

Solutions to Student Study Guide ✎ Questions

Question 1

What is the measure of each interior and exterior angle of

a) a regular quadrilateral?

b) a regular pentagon?

c) a regular dodecagon (12 sides)?

Solution

The measure of the interior angle of an *n*-gon can be found by evaluating $\frac{n-2}{n}\cdot 180°$. The measure of the exterior angle of an *n*-gon can be found by finding the supplement of an interior angle or evaluating $\frac{360°}{n}$.

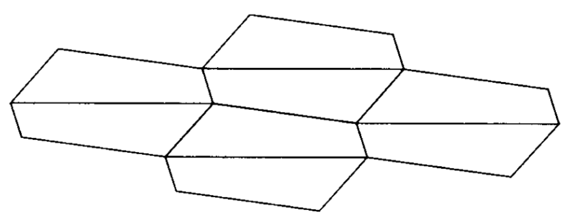
a) A quadrilateral has four sides. Thus the measure of the interior angle of a regular quadrilateral is $\frac{(4-2)180°}{4}=\frac{2\cdot 180°}{4}=\frac{360°}{4}=90°$. Each exterior angle measures $180°-90°=90°$.

b) A pentagon has five sides. Thus the measure of the interior angle of a regular pentagon is $\frac{(5-2)180°}{5}=\frac{3\cdot 180°}{5}=\frac{540°}{5}=108°$. Each exterior angle measures $180°-108°=72°$.

c) A dodecagon has twelve sides. Thus the measure of the interior angle of a regular dodecagon is $\frac{(12-2)180°}{12}=\frac{10\cdot 180°}{12}=\frac{1800°}{12}=150°$. Each exterior angle measures $180°-150°=30°$.

Question 2

Are there two fundamental regions for this tiling?



Solution

Two quadrilaterals which fit together form a hexagon that tiles by translation. The answer is as follows.