Chapter 19
Symmetry and Patterns

For All Practical Purposes: Effective Teaching

- If during your lecture you feel that students are losing focus on your lecture, try asking some true/false or yes/no questions. Don't be concerned about a moment of silence. Inevitably a response (hopefully multiple, even if they differ) should occur. Getting that initial response should allow you to get the attention of the class as a whole and will lead to discussion about your question.
- A technique you can use to keep students involved throughout the course is to ask students to write down a response to a question such as, "What is the most important thing you learned from today's lecture?" If they are used to doing this, it will only take a few minutes and you can scan the responses quickly. The main goal is to keep the students engaged during the class.

Chapter Briefing

In this chapter, you will be mainly examining certain numerical and geometric patterns of growth and structure that can be used to model or describe an amazing variety of phenomena in mathematics and science, art, and nature. The mathematical ideas that the *Fibonacci sequence* leads to, such as the *golden ratio*, spirals, and *selfsimilar* curves, have long been appreciated for their charm and beauty; but no one can really explain why they are echoed so clearly in the world of art and nature. The properties of selfsimilarity, and *reflective* and *rotational symmetry* are ubiquitous in the natural world and are at the core of our ideas of science and art. Plants exhibiting *phyllotaxis* have a number of spiral forms coming from a special sequence of numbers. Certain plants have spirals that are geometrically similar to one another. The spirals are arranged in a regular way, with balance and "proportion". These plants have rotational symmetry.

Being well prepared for class discussion with examples is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Fibonacci Numbers and the Golden Ratio**, **Symmetries Preserve the Pattern**, **Rosette, Strip, and Wallpaper Patterns**, **Notation for Patterns**, **Symmetry Groups**, and **Fractal Patterns and Chaos**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the four questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Fibonacci Numbers and the Golden Ratio

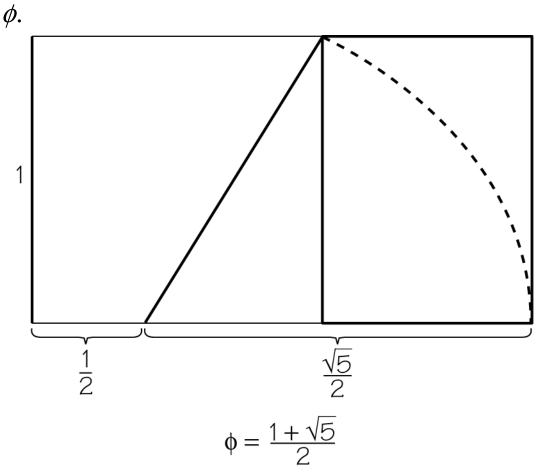
Fibonacci numbers occur in the sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . }. They are generated according to the **recursion** formula that states that each term is the sum of the two terms preceding it. If the n^{th} Fibonacci number is F_n then for $F_1 = F_2 = 1$ and $n \geq 2$, we have the following.

$$F_{n+1} = F_n + F_{n-1}$$

As you go further out in the sequence, the ratio of two consecutive Fibonacci numbers approaches the famous golden ratio $\phi = 1.618034...$. For example, 89/55 = 1.61818... and 377/233 = 1.618025.... .

The number ϕ is also known as the golden mean. The exact value is $\phi = \dfrac{1+\sqrt{5}}{2}$.

A **golden rectangle** (which is considered by many to be visually pleasing) is one such that the ratio of height to width is 1 to ϕ.



The **geometric mean** of two positive numbers, *a* and *b*, is the square root of their product.

$$\sqrt{ab}$$

In general, the geometric mean of *n* numbers is the n^{th} root of their product.

Symmetries Preserve the Pattern

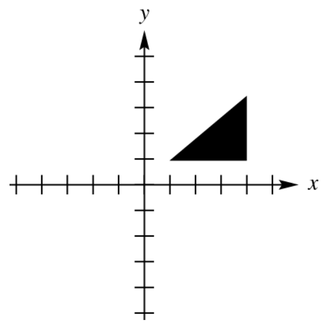
Balance refers to regularity in how repetitions are arranged. Along with **similarity** and **repetition**, balance is a key aspect of symmetry.

Rigid motions are **translations**, **rotations**, **reflections**, and **glide reflections**.

Preservation of the pattern occurs when the pattern looks exactly the same, with all parts appearing in the same places, after a particular motion is applied.

Example

Consider the following drawing.

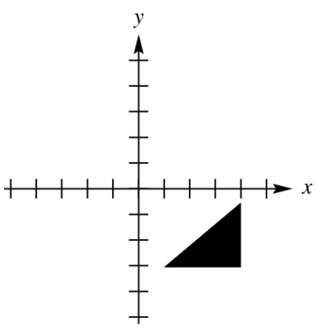


Render a drawing that involves a

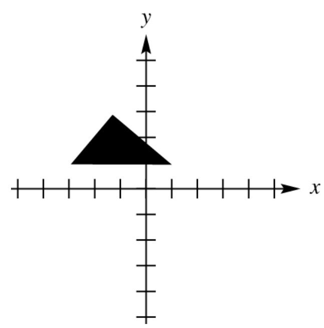
a) translation.
b) rotation.
c) reflection.
d) glide reflection.

Solution

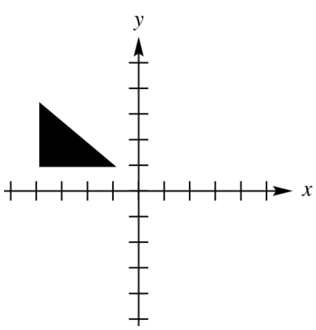
a)



b)



c)



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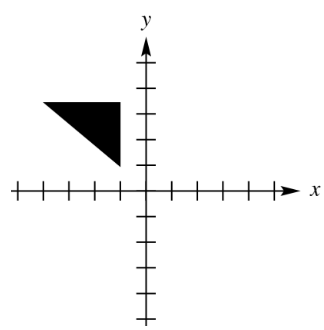


☝Teaching Tip

Try having students investigate rigid motions on graph paper in a rectangular plane. Each motion can be described by translating points in the plane. Rotations may be a little tricky, so try using $90°$.

Rosette, Strip, and Wallpaper Patterns

Patterns are analyzed by determining which rigid motions preserve the pattern. These rigid motions are called **symmetries of the pattern**.

- **Rosette patterns** contain only rotations and reflections.
- **Wallpaper patterns** repeat a design element in more than one direction.
- **Strip patterns** repeat a design element along a line, so all of them have translation symmetry along the direction of the strip and may also contain glide reflections.

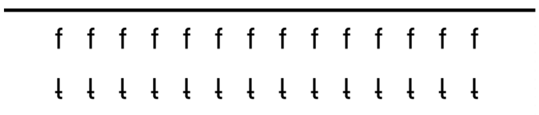
Other possible symmetries for a strip pattern are horizontal or vertical reflection, rotation by 180°, or glide reflection.

Notation for Patterns

We can classify strip patterns according to their symmetry types; there are exactly seven different classes, designated by four symbols *p* * * *. The first symbol is always the *p*. The second symbol is either *m* or 1 indicating the presence or absence of a vertical line of reflection. The third symbol is *m* if there is a horizontal line of reflection, *a* if there is a glide reflection but no horizontal reflection, or 1 if there is horizontal or glide reflection. The fourth symbol is a 2 if there is half-turn rotational symmetry; otherwise it is a 1.

Example

What is the symbolic notation for the following pattern?



Solution

Refering to the flow chart (Figure 19.12) on page 729 of the text, we see that there is no vertical line reflection, there is a horizontal line of reflection. It is a *p*1*m*1.

It is useful to have a standard notation for patterns, for purposes of communication. Crystallographer's notation is the one most commonly used.

In applying notation to patterns, it must be taken into account that patterns may not be perfectly rendered, especially if they are on a rounded surface.

Teaching Tip

Impart to students that a goal to keep in mind is that they should be able to look at a wallpaper pattern, and identify which symmetries preserve the pattern. They should also be able to locate centers of rotations and axes of reflections.

Symmetry Groups

A group is made up of a set of elements and an operation that has the following properties.

- Closure: The result of an operation on any two elements of the set yields an element of the set.
- Identity: The set has a special element I such that if the operation is performed with any element of the set, say A, the result of $A \circ I = I \circ A$, which is A.
- Inverse: For any element A, there exists an element of the set A^{-1} such that $A \circ A^{-1} = A^{-1} \circ A = I$.
- Associatively: For any three elements of the set, A, B, and C, we have the following.

$$A \circ B \circ C = A \circ (B \circ C) = (A \circ B) \circ C$$

The full list of symmetries of any pattern forms a **symmetry group**. Symmetry of a pattern has the following properties.

- The combination of two symmetries A and B is written $A \circ B$. This combination is another symmetry.
- The "null" symmetry doesn't move anything. It is considered the identity.
- Every symmetry has an inverse or an opposite that "undoes" the effect of the original symmetry.
- In applying a number of symmetries one after another, we may combine consecutive ones without affecting the result.

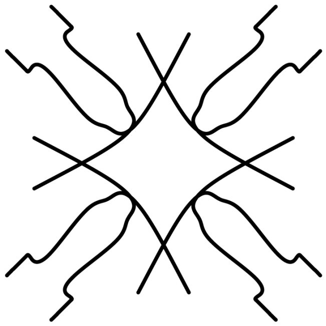
The symmetries of a rectangle are as follows.

- *I*: leaves every point unchanged in location
- *R*: A $180°$ half-turn through the center)
- *V*: A reflection in a vertical line through the center
- *H*: A reflection in a horizontal line through the center

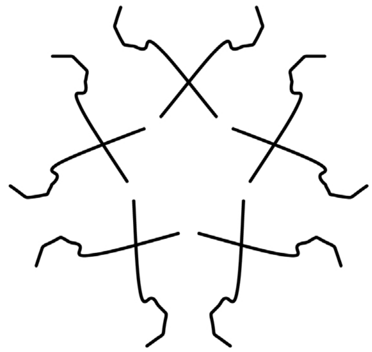
Symmetry groups of rosette patterns contain only rotations and reflections.

Example

What are the symmetries of these rosettes?



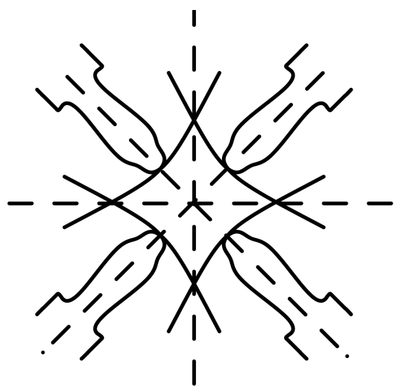
a)



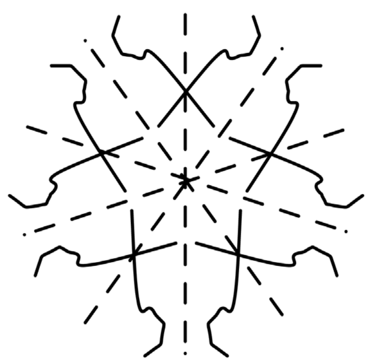
b)

Solution

a) Four rotations $\{I, R, R^2, R^3\}$, where R is a rotation $90°$, and four reflections across the axes shown below.



b) Five rotations $\{I, R, R^2, R^3, R^4\}$, where R is a rotation $72°$, and five reflections across the axes shown below.



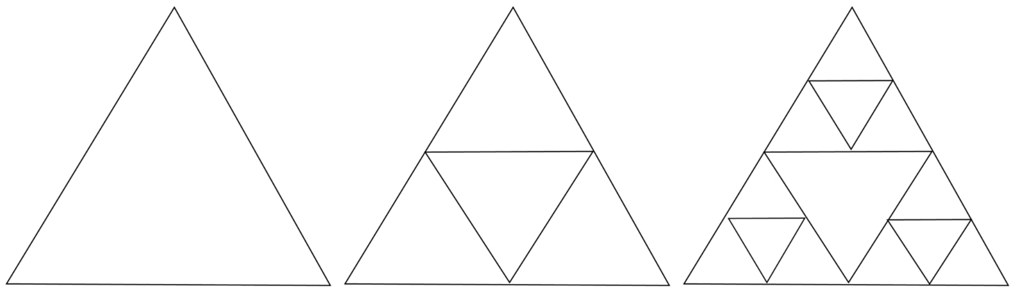
☝Teaching Tip

At the time of printing this *Guide*, an interactive Website that allows one to create wallpaper and rosette groups was available. You may find it very interesting to have students investigate the website http://www.scienceu.com/geometry/handson/kali/kali.html. This program is also available for download at http://geometrygames.org/Kali/index.html.

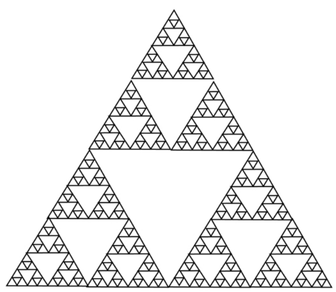
Fractal Patterns and Chaos

A **fractal** is a kind of pattern that exhibits similarity at ever finer scales. When you "zoom in", the resulting figure looks like the original figure.

Fractals can be created with a replication rule called an **iterative function system** (**IFS**). One particular geometric pattern is called **Sierpinski's triangle**. It starts with a triangle, and the "middle triangle" is removed. This process is repeated for the resulting triangles.



After a few more iterations, the figure looks like the following.



This process continues.

Solutions to Student Study Guide Questions

Question 1

Suppose a sequence begins with numbers 2 and 4, and continues by adding the previous two numbers to get the next number in sequence. What would the sum of the 10^{th} and 11^{th} terms in this sequence be?

Solution

The sequence would be 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, 288, 466,…, 466 is the sum of the 10^{th} and 11^{th} terms, which is also the 12^{th} term.

Question 2

a) What is the geometric mean of 5 and 7? (round to three decimal places)

b) What is the geometric mean of 5, 6, and 7? (round to three decimal place)

Solution

a) The geometric mean of 5 and 7 is $\sqrt{5 \cdot 7} = \sqrt{35} \approx 5.916$.

b) The geometric mean of 5, 6, and 7 is $\sqrt[3]{5 \cdot 6 \cdot 7} = \sqrt[3]{210} \approx 5.944$.

Question 3

Which of the following letters has (have) a shape that is preserved by reflection or rotation?

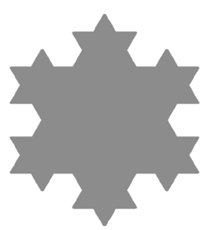
A E I S

Solution

A, I, and E have a shape preserved by reflection. I and S have a shape preserved by rotation. Thus, all four letters have a shaped preserved by reflection or rotation.

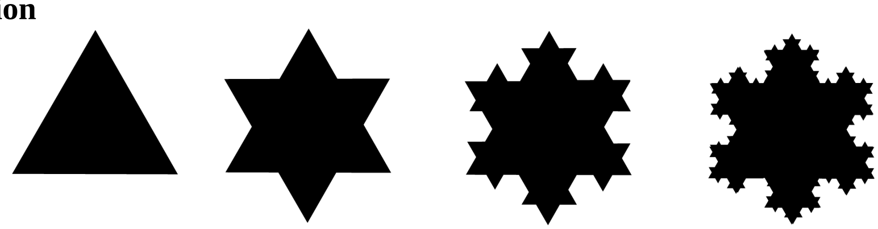
Question 4

A fractal known as the Koch snowflake starts with an equilateral triangle. Each side of the triangle is cut into three parts with the middle section removed. The removed segment is replaced by two line segments of the same length as that was removed. If the following is a picture of what occurs after doing this process twice, draw the first two pictures and the one that follows (the first is the equilateral triangle).



Is the area in the interior of the figure increasing as you do each iteration?

Solution



The area is increasing, but it is bounded.