Chapter 15
Game Theory: The Mathematics of Competition

For All Practical Purposes: Effective Teaching

- On the last day of class, students may be as receptive to new information as they were during other days of the semester. You may consider having the last day as a review session or an informal discussion of the how the course objectives were met and the kinds of things you feel are valuable in terms of overall learning outcomes.
- As the end of the term comes, students often ask, "What do I need to get on the final?" Emphasize that they should do their best and prepare well and not focus on a number or a letter outcome. If they truly synthesize the information, they should do well.

Chapter Briefing

In this chapter, you will be mainly examining competitive situations. In such situations, parties in a conflict frequently have to make decisions that will influence the outcome of their competition. Often, the players are aware of the options – called *strategies* – of their opponent(s), and this knowledge will influence their own choice of strategies. Game theory studies the *rational choice* of strategies, how the players select among their options to optimize the outcome. Some two-person games involve *total conflict*, in which what one player wins the other loses. However, there are also games of *partial conflict*, in which cooperation can often benefit the players.

Being well prepared for class discussion with examples is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Two-Person Total-Conflict Games: Pure Strategies**, **Two-Person Total-Conflict Games: Mixed Strategies**, **Partial-Conflict Games**, and **Larger Games**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the three questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Two-Person Total-Conflict Games: Pure Strategies

The simplest games involve two players, each of whom has two strategies. The payoffs to each of the players is best described by a 2×2 **payoff matrix**, in which a positive entry represents a payoff from the column player to the row player, while a negative entry represents a payment from the row player to the column player.

Example
Consider the following payoff matrix.

$$\begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 2 & -4 \\ 3 & 6 \end{bmatrix} \end{array}$$

a) If the row player chooses *D* and the column player chooses *A*, what is the outcome of the game?
b) If the row player chooses *D*, what is the minimum payoff he can obtain?
c) If the row player chooses *C*, what is the minimum payoff he can obtain?
d) If the column player chooses *A*, what is the most she can lose?
e) If the column player chooses *B*, what is the most she can lose?

Solution
a) The payoff associated with this outcome is the entry in Row *D* and Column *A*. The outcome is 3.
b) The minimum payoff is 3
c) An outcome of -4 means that the row player loses 4 to the column player.
d) The outcome is 3. An outcome of 3 means that the column player loses 3 to the row player.
e) The outcome is 6.

We see in these examples that the row player can guarantee himself a payoff of at least 3 by playing *D*, and that the column player can guarantee that she will not lose more than 3 by playing *A*. The entry 3 is the minimum of its row, and it is larger than the minimum of the other row, -4. 3 is thus the **maximin**, and choosing *D* is the row player's **maximin strategy**. Similarly, 3 is the maximum of column *A*, and it is smaller than 6, which is the maximum of column *B*. Hence, 3 is the **minimax** of the columns, and if the column player chooses *A*, then she is playing her **minimax strategy**. When the maximin and minimax coincide, the resulting outcome is called a **saddlepoint**. The saddlepoint is the **value** of the game, because each player can guarantee at least this value by playing his/her maximin and minimax strategies. However, not every game has a saddlepoint. Games which do not will be studied in the next section.

Example

Consider a game in which each of the players (Kevin and Gwen) has a coin, and each chooses to put out either a head or a tail. (Note: The players do not flip the coins.) If the coins match, Gwen (the row player) wins, while if they do not match, Kevin (column player) wins. The payoffs are as follows.

		Kevin Head	Tail
Gwen	Head	3	–6
	Tail	–4	5

a) What is the row player's maximin?

b) What is the column player's minimax?

c) Does this game have a saddlepoint?

Solution

$$\begin{array}{rcc} & & \text{Row Minima} \\ & \begin{bmatrix} 3 & -6 \\ -4 & 5 \end{bmatrix} & \begin{array}{c} -6 \\ \boxed{-4} \end{array} \\ \text{Column Maxima} & \begin{array}{cc} \boxed{3} & 5 \end{array} & \end{array}$$

a) –4

b) 3

c) No. If the maximin is different from the minimax, then there is no saddlepoint.

☝Teaching Tip

Emphasize several times about the terminology of maximin and minimax as it relates to the columns and rows. If students know one of the procedures (such as finding the maximum in the columns) the rest of the procedure should fall into place. You could use a saying like, "The largest value must *fall* down."

Two-Person Total-Conflict Games: Mixed Strategies

When a game fails to have a saddlepoint, the players can benefit from using mixed **strategies**, rather than **pure strategies**.

The notion of **expected value** is necessary in order to calculate the proper mix of the players' strategies.

Example

What is the expected value of a situation in which there are four payoffs, \$2, \$3, $-$\$4, and \$4, which occur with probabilities 0.3, 0.2, 0.4, and 0.1, respectively?

Solution

The expected value is found by multiplying each payoff by its corresponding probability and adding these products. We obtain the following.

$$\$2(0.3)+\$3(0.2)-\$4(0.4)+\$4(0.1)=\$0$$

Example

Let's reconsider the game of matching coins, described by the following payoff matrix.

		Kevin Head	Tail	
Gwen	Head	3	–6	q
	Tail	–4	5	$1-q$
		p	$1-p$	

a) Suppose the row player, Gwen, mixes her strategy by choosing head with probability q and tails with probability $1-q$. If the column player always chooses heads, what is the row player's expected value?

b) Suppose the row player, Gwen, mixes her strategy by choosing head with probability q and tails with probability $1-q$. If the column player always chooses tails, what is the row player's expected value?

c) Find the best value of q, that is, the one which guarantees row player the best possible return. What is the (*mixed-strategy*) *value* in this case?

d) Suppose the column player, Kevin, mixes his strategy by choosing head with probability p and tails with probability $1-p$. If the row player always chooses heads, what is the column player's expected value?

e) Suppose the column player, Kevin, mixes his strategy by choosing head with probability p and tails with probability $1-p$. If the row player always chooses tails, what is the column player's expected value?

f) Find the best value of p, that is, the one which guarantees column player the best possible return.

g) Is this game fair?

Solution

a) The expected value is $E_{Head} = 3q + (-4)(1-q) = 3q - 4 + 4q = -4 + 7q$.

b) The expected value is $E_{Tail} = -6q + (5)(1-q) = -6q + 5 - 5q = 5 - 11q$.

c) The optimal value of q can be found in this case by setting E_{Head} equal to E_{Tail}, and solving for q

$$E_{Head} = E_{Tail} \Rightarrow -4 + 7q = 5 - 11q \Rightarrow 18q = 9$$
$$q = \tfrac{9}{18} = \tfrac{1}{2} \Rightarrow 1 - q = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Gwen's optimal mixed strategy is $(q, 1-q) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$.

To find the value, substitute the q into E_{Head} or E_{Tail}.

The value is $E_{Head} = E_{Tail} = E = 5 - 11\left(\tfrac{1}{2}\right) = \tfrac{10}{2} - \tfrac{11}{2} = -\tfrac{1}{2}$.

d) The expected value is $E_{Head} = 3p + (-6)(1-p) = 3p - 6 + 6p = -6 + 9p$.

e) The expected value is $E_{Tail} = -4p + (5)(1-p) = -4p + 5 - 5p = 5 - 9p$.

f) $-6 + 9p = 5 - 9p \Rightarrow p = \tfrac{11}{18}$

g) Since the value of the game is negative for Gwen, it is unfair to her.

A game in which the payoff to one player is the negative of the payoff to the other player is called a **zero-sum game**. A zero-sum game can be **non-symmetrical** and yet fair.

The **minimax theorem** guarantees that there is a unique game value and an optimal strategy for each player. If this value is positive, then the row player can realize at least this value provided he plays his optimal strategy. Similarly, the column player can assure herself that she will not lose more than this value by playing her optimal strategy. If either one deviates from his or her optimal strategy, then the opponent may obtain a payoff greater than the guaranteed value.

Partial-Conflict Games

In a game of total conflict, the sum of the payoffs of each outcome is 0, since one player's gain is the other's loss. **Variable-sum games,** on the other hand, are those in which the sum of the payoffs at the different outcomes varies. These are games of partial conflict, because, through cooperation, the players can often achieve outcomes that are more favorable than would be obtained by being pure adversaries.

In many games of partial conflict, it is difficult to assign precise numerical payoffs to the outcomes. However, the preferences of the parties for the various outcomes may be clear. In such a case, the payoffs are **ordinal**, with 4 representing the best outcome, 3 the second best, 2 next, and 1 worst. The payoff matrix now consists of pairs of numbers, the first number representing the row player's payoff, with the second number of the pair being the column player's payoff. Now, both like high numbers.

When neither player can benefit by departing unilaterally from a strategy associated with an outcome, the outcome constitutes a **Nash equilibrium**.

Example

Consider the following matrix

	A	*B*
C	$(2,2)$	$(1,5)$
D	$(1,1)$	$(3,3)$

a) If the row player chooses *C* and the column player chooses *B*, what will the payoffs be to the players?
b) Does either player have a dominant strategy?
c) Is there a Nash equilibrium in this matrix?

Solution

a) The first entry in the outcome $(1,5)$ represents the payoff to the row player, and the second entry, the payoff to the column player. The payoffs will be 1 to the row player and 5 to the column player.
b) The column player gets a better payoff in both cases by choosing strategy *B* (5 to 2 if the row player selects strategy *A*, and 3 to 1 if the row player selects strategy *D*).
The row player gets a more desirable payoff by switching from *D* to *C* when the column player selects strategy *A;* however, she gets a less desirable payoff by making the same switch when the column player selects strategy *B.* Thus, *B* is a dominant strategy for the column player. The row player does not have a dominant strategy.
c) If Column player chooses *A* and row player chooses *C*, column player would benefit by changing her strategy.
If Column player chooses *A* and row player chooses *D*, both players would benefit by changing their strategies.
If Column player chooses *B* and row player chooses *C*, row player would benefit by changing his strategy.
If Column player chooses *B* and row player chooses *D*, neither player would benefit by changing their strategy.
Thus there is a Nash equilibrium, namely $(3,3)$.

Prisoners' Dilemma is a game with four possible outcomes. Here, *A* stands for "arm," and *D* for "disarm."
There are four possible outcomes:

(D,D): Red and Blue disarm, which is *next best* for both because, while advantageous to each, it also entails certain risks.

(A,A): Red and Blue arm, which is *next worst* for both, because they spend needlessly on arms and are comparatively no better off than at (*D*, *D*).

(A,D): Red arms and Blue disarms, which is *best for Red* and *worst for Blue*, because Red gains a big edge over Blue.

(D,A): Red disarms and Blue arms, which is *worst for Red* and *best for Blue*, because Blue gains a big edge over Red.

		Blue	
		A	*D*
Red	*A*	(A,A)	(A,D)
	D	(D,A)	(D,D)

This matrix is also used to model other situations.

Chicken is a game with a payoff matrix such as the following.

	Swerve	*Don't swerve*
Swerve	(c,y)	(b,z)
Don't swerve	(d,x)	(a,w)

☝Teaching Tip

For examples to be realistic, $a<b<c<d$ and $w<x<y<z$. In this case there will be two Nash equilibria, namely (b,z) and (d,x). To elicit class discussion, ask students why this would be the case.

☝Teaching Tip

At the time of printing this *Guide*, the Website www.gametheory.net shows interactive applets. One such applet is for the Repeated Prisoner's Dilemma. You might find this Website helpful.

Larger Games

If one of three players has a dominant strategy in a $3\times3\times3$ game, we assume this player will choose it and the game can then be reduced to a 3×3 game between the other two players. (If no player has a dominant strategy in a three-person game, it cannot be reduced to a two-person game.)

The 3×3 game is not one of total conflict, so the minimax theorem, guaranteeing players the value in a two-person zero-sum game, is not applicable. Even if the game were zero-sum, the fact that we assume the players can only rank outcomes, but not assign numerical values to them, prevents their calculating optimal mixed strategies in it.

The problem in finding a solution to the 3×3 game is not a lack of Nash equilibria. So the question becomes which, if any, are likely to be selected by the players. Is one more appealing than the others?

Yes, but it requires extending the idea of dominance to its successive application in different stages of play.

In a small group voting situation (such as a committee of three), **sophisticated voting** can lead to Nash equilibria with surprising results. An example is the status quo paradox. In this situation, supporting the apparently favored outcome actually hurts.

The analysis of a "**truel**" (three-person duel) is very different when the players move sequentially, rather than simultaneously.

Sequential truels may be analyzed through the use of a **game tree**, examining it from the bottom up through **backward induction**.

The **theory of moves** (**TOM**) introduces a dynamic element into the analysis of game strategy. It is assumed that play begins in an initial state, from which the players, thinking ahead, may make subsequent moves and countermoves. Backward induction is the essential reasoning tool the players should use to find optimal strategies.

Solutions to Student Study Guide Questions

Question 1

Consider the following payoff matrix.

$$\begin{bmatrix} 3 & 7 \\ 2 & 6 \\ 6 & 9 \end{bmatrix}$$

a) What is the row player's maximin?
b) What is the column player's minimax?
c) Does this game have a saddlepoint?

Solution

$$\begin{array}{rcc} & & \text{Row Minima} \\ & \begin{bmatrix} 3 & 7 \\ 2 & 6 \\ 6 & 9 \end{bmatrix} & \begin{array}{c} 3 \\ 2 \\ \boxed{6} \end{array} \\ \text{Column Maxima} & \begin{array}{cc} \boxed{6} & 9 \end{array} & \end{array}$$

a) 6
b) 6
c) Since the maximin and the minimax are the same, yes there is a saddlepoint, namely 6.

Question 2

What is the expected value of a situation in which there are four payoffs, \$2, $-\$4$, \$4, and \$9, which occur with probabilities 0.25, 0.15, 0.45, and 0.15, respectively?

Solution

$\$2(0.25) - \$4(0.15) + \$4(0.45) + \$9(0.15) = \$3.05$

Question 3

Let's reconsider the game of matching coins, described by the following payoff matrix.

		John Head	Tail	
Jane	Head	2	–4	q
	Tail	–3	5	$1-q$
		p	$1-p$	

a) Suppose the column player, John, mixes his strategy by choosing head with probability p and tails with probability $1-p$. If the row player always chooses heads, what is the column player's expected value?
b) Suppose the column player, John, mixes his strategy by choosing head with probability p and tails with probability $1-p$. If the row player always chooses tails, what is the column player's expected value?
c) Find the best value of p, that is, the one which guarantees column player the best possible return.

Solution

a) The expected value is $E_{Head} = 2p + (-4)(1-p) = 2p - 4 + 4p = -4 + 6p$.
b) The expected value is $E_{Tail} = -3p + (5)(1-p) = -3p + 5 - 5p = 5 - 8p$.
c) $E_{Head} = E_{Tail} \Rightarrow -4 + 6p = 5 - 8p \Rightarrow 14p = 9 \Rightarrow p = \frac{9}{14}$