# Chapter 13
## Fair Division

## For All Practical Purposes: Effective Teaching

- Even a *seasoned* instructor can get nervous before a lecture. Classroom preparation not only involves knowing the material and the intended structure of your class, but also involves being psychologically ready for the classroom interaction. Try to set aside a period of time to sit and think about what you plan to say and gather your thoughts right before class time.
- Students often get nervous before taking an exam. Often times they try to “cram” material in at the last minute. Also, students tend to talk to oneanother just prior to the exam by discussing what materials or examples they may have studied. This can make a nervous student even more nervous. Recommend to students that they take a period of time just prior to the exam to relax (ten minutes or so). You could recommend that prior to class that they get a soda, use the restroom, or take a quick walk to help them relax.

## Chapter Briefing

In this chapter, you will be examining applications where you find a suitable division of goods. *Fair-division problems* arise in many situations, including divorce, inheritance, or the liquidation of a business. The problem is for the individuals involved, called the *players*, to devise a scheme for dividing an object or a set of objects in such a way that each of the players obtains a share that she considers fair. Such a scheme is called a *fair-division procedure*.

Being well prepared for class discussion with examples is essential. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **The Adjusted Winner Procedure**, **The Knaster Inheritance Procedure**, **Taking Turns**, **Divide-and-Choose**, **Cake-Division Procedures: Proportionality**, and **Cake-Division Procedures: The Problem of Envy**. The material in this chapter of the *Teaching Guide* is presented in the same order as the text. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the three questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

# Chapter Topics to the Point

## The Adjusted Winner Procedure

In the **adjusted winner procedure** for two players, each of the players is given 100 points to distribute over the items that are to be divided. Each party is then initially given those items for which he or she placed more points than the other party. The steps are as follows.

- Each party distributes 100 points over the items in a way that reflects their relative worth to that party.
- Each item is initially given to the party that assigned it more points. Each party then assesses how many of his or her own points he or she has received. The party with the fewest points is now given each item on which both parties placed the same number of points.
- Let *A* denote the party with the higher point total and *B* be the other party. Determine the following for each item.

$$\frac{A\text{'s point value of the item}}{B\text{'s point value of the item}}$$

  Transfer items from *A* to *B*, in order of increasing **point ratio**, until the point totals are equal. (The point at which equality is achieved may involve a fractional transfer of one item.)

### Example

Suppose that Sami and Hanna place the following valuations on the three major assets, which will be divided up:

| Asset | Sami | Hanna |
|---|---|---|
| Artwork | 30 | 40 |
| Business | 20 | 30 |
| House | 50 | 30 |

a) Who gets each of the assets initially?

b) How many points does each of the players get according to this allocation?

c) What further exchange of property is necessary in order to equalize the allocations?

d) After this final reallocation of property, how many points does each of the players receive?

### Solution

a) Hanna gets the artwork and business, while Sami gets the house.

b) Sami gets 50 points for the house, while Hanna gets 70 for the other assets (40 for the artwork and 30 for the business).

c) Some of Hanna's property has to be transferred to Sami, since she has received more points initially. To determine how much, we first compute the fractions.

$$\text{Artwork: } \tfrac{40}{30} \approx 1.33 \quad \text{Business: } \tfrac{30}{20} = 1.50$$

We now transfer part of Hanna's assets to Sami as follows.

- Let *x* equal the fraction of the artwork which Hanna will retain.
- To equalize the number of points, we solve the following equation.

$$30 + 40x = 50 + 30(1 - x) \Rightarrow x = \tfrac{5}{7}$$

- Since Hanna retains $\frac{5}{7}$ of the artwork, she must transfer $1 - \frac{5}{7} = \frac{2}{7}$ of the artwork to Sami.

d) Hanna gets to keep the business, which is worth 30 points to her, she also retains a $\frac{5}{7}$ share in the artwork, which is worth $40\left(\frac{5}{7}\right) = 28\frac{4}{7}$ points, giving her a total of $58\frac{4}{7}$ points. Sami gets the house, worth 50 points to him, and a $\frac{2}{7}$ share in the artwork, worth $30\left(\frac{2}{7}\right) = 8\frac{4}{7}$ points, totaling $58\frac{4}{7}$ points.

### ☝Teaching Tip

Although the total points for both parties should be the same after transferring shares, students should be asked to calculate the total number of points for both parties.

The adjusted winner procedure satisfies three important properties. The allocation must be:

- **equitable:** Both players receive the same number of points.
- **envy-free:** Neither player would be happier with what the other received.
- **Pareto-optimal:** No other allocation, arrived at by any means, can make one party better off without making the other party worse off.

## ➾The Knaster Inheritance Procedure

The adjusted winner procedure applies only when there are two heirs. With three or more, the **Knaster inheritance procedure** can be used.

### Example

Dan, Ed, and Bill inherit a house. Their respective evaluations of the house are \$120,000, \$105,000, and \$81,000. Describe a fair division.

### Solution

The highest bidder gets the asset. Since Dan values the house at \$120,000, his share is $\frac{\$120{,}000}{3} = \$40{,}000$ and pays \$80,000 into a **kitty**. Ed and Bill view their fair shares as $\frac{\$105{,}000}{3} = \$35{,}000$ and $\frac{\$81{,}000}{3} = \$27{,}000$, respectively. After they take these amounts from the kitty, \$18,000 remains. This sum is then split equally among the three. Thus, Dan gets the house, and pays \$41,000 to Ed and \$33,000 to Bill.

### Example

Dan, Ed, and Bill are heirs to an estate, which consists of a house, an antique car, and a yacht. Their evaluations of the items in the estate follow.

| Asset | Dan | Ed | Bill |
|---|---|---|---|
| House | \$132,000 | \$138,000 | \$99,000 |
| Car | \$20,000 | \$32,000 | \$21,000 |
| Yacht | \$76,000 | \$52,000 | \$60,000 |

Describe a fair division.

### Solution

It is possible to solve this problem by doing a calculation similar to the previous one for each of the three items, and then adding up the totals. However, an alternative method is to look at the entire estate. For example, Dan's bids indicate that he places a total value of \$228,000 on the estate, which means that his share is \$76,000. Similarly, Ed's estimate is \$222,000, with his share being \$74,000, while Bill's estimate is \$180,000, entitling him to \$60,000. Now Dan gets the yacht having a total value (in his estimate) of \$76,000, which is his fair share. Hence, he places \$0 into a kitty. In the same way, Ed gets the house and car and places \$96,000 in the kitty. Bill then removes his share of \$60,000, which leaves \$36,000 in the kitty. This amount is then divided equally among the three heirs.

Dan gets the yacht and receives \$12,000 from Ed. Ed receives the house and the yacht and pays Dan \$12,000 and also pays \$72,000 to Bill. Bill receives no items but receives \$72,000 from Ed.

## Taking Turns

Two people often split a collection of assets between them using the simplest and most natural of fair-division schemes: **taking turns**. If they each know the other's preferences among the assets, their best strategies may not be to choose their own most highly preferred asset first.

### Example

John and Jane Luecke attend an auction for artwork. They are bidding on 4 pieces of art, *A*, *B*, *C*, and *D*. They will take turns to split the artwork between them, with John choosing first. Here are their preferences:

| | John's Ranking | Jane's Ranking |
|---|---|---|
| Best | *A* | *C* |
| Second best | *C* | *B* |
| Third best | *D* | *A* |
| Worst | *B* | *D* |

If John chooses *A* (his favorite) first, then Jane may choose her favorite, *C*; John loses his second best choice. Is there a way for John to choose strategically to end up with both of his top two?

### Solution

John should choose *C* first. Jane must now resort to choosing *B* (her second favorite). That leaves *A* available to John in the second round.

There is a general procedure for rational players, called the "bottom-up-strategy," for optimizing individual asset allocations when taking turns. It is based on two principles:

- You should never choose your least-preferred available option.
- You should never waste a choice on an option that will come to you automatically in a later round.

### Example

What would the allocation of artwork be in the last example if John and Jane use the bottom-up strategy? Assume that Jane goes first.

| | John's Ranking | Jane's Ranking |
|---|---|---|
| Best | *A* | *C* |
| Second best | *C* | *B* |
| Third best | *D* | *A* |
| Worst | *B* | *D* |

### Solution

| | | | | |
|---|---|---|---|---|
| Jane: | C | | B | |
| John: | | A | | D |

## Divide-and-Choose

A fair-division procedure known as **divide-and-choose** can be used if two people want to divide an object such as a cake or a piece of property. One of the people divides the object into two pieces, and the second person chooses either of the two pieces.

## Cake-Division Procedures: Proportionality

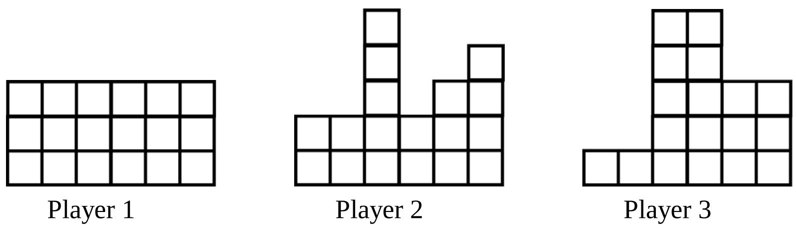
The divide-and-choose method cannot be used if there are more than two players. A **cake-division procedure** is a scheme that *n* players can use to divide a cake among themselves in a way which satisfies each player.

A cake-division scheme is said to be proportional if each player's strategy guarantees him a piece of size at least $\frac{1}{n}$ in his own estimation. It is envy-free if each player feels that no other player's piece is bigger than the one he has received.

When there are three players, the **lone-divider method** guarantees proportional shares.

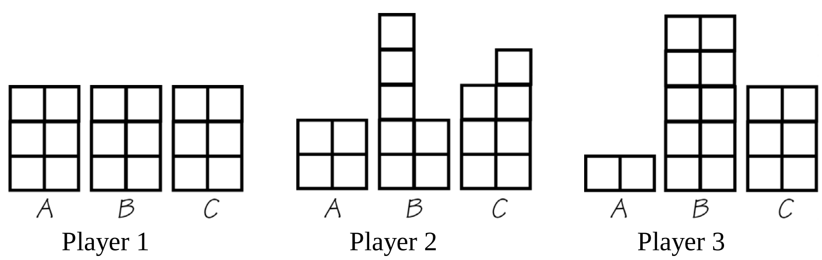
### Example

Suppose that players 1, 2, and 3 view a cake as follows.




If player 1 cuts the cake into what she perceives as three equal pieces, draw three diagrams to show how each player will view the division. Can all players be satisfied by this division?

### Solution



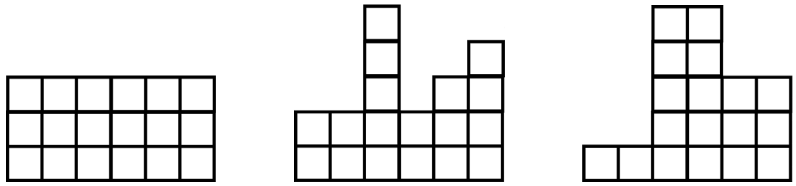

Each player needs to have a piece that they see as having at least 6 square units. Player 1 must choose piece *A* in order for the other players to be satisfied. Both players 2 and 3 would be satisfied with either of the remaining pieces. Since player 2 views both *B* and *C* the same, it would seem reasonable that player 3 would prefer piece *B* to piece *C*, but would be satisfied with either. Thus, all players can be satisfied.

## Teaching Tip

In creating examples of cake cutting division problems, if you have three players you may choose cakes that are made up of 12, 18, or 24 square units (divisible by 3). Also, make note that the base of each players' view needs to be the same.

### Example

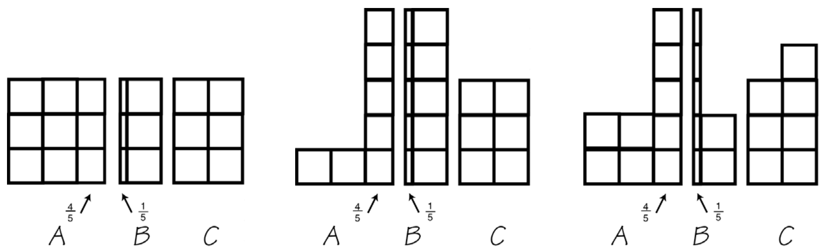
Suppose that players 1, 2, and 3 view a cake as follows.



Player 1 Player 2 Player 3

If player 2 cuts the cake into what she perceives as three equal pieces, draw three diagrams to show how each player will view the division. Can all players be satisfied by this division?

### Solution




Player 1 Player 2 Player 3

Player 1 can be satisfied with piece *A* or *C*. If player 1 chooses *A*, then player 3 would be satisfied with piece *C* and player 2 can take piece *B*.

### ☝Teaching Tip

Students may find using graph paper for these types of problems will help in organizing their work.

The **last-diminisher method** is a more complicated procedure, which guarantees proportional shares with any number of players. Like the lone-divider method, it is proportional but not envy-free.

## Cake-Division Procedures: The Problem of Envy

Envy-free cake-division schemes are still more complicated. The **Selfridge-Conway procedure** solves this problem for the case of three players. For more than three players, a scheme which involves a **trimming procedure** has been developed. In it, proportions of the cake are successively allocated in an envy-free fashion, with the remaining portions diminishing in size. Eventually, the remainder of the cake is so small that it will not affect the perception of each player that he or she has obtained the largest piece. If players cannot find a way to make an item fairly divisible, there may be no alternative but to sell it and share the proceeds equally. The procedure is as follows.

- Step 1 Player 1 cuts the cake into three pieces he considers to be the same size. He hands the three pieces to Player 2.
- Step 2 Player 2 trims at most one of the three pieces so as to create at least a two-way tie for largest. Setting the trimmings aside, Player 2 hands the three pieces (one of which may have been trimmed) to Player 3.
- Step 3 Player 3 now chooses, from among the three pieces, one that he considers to be at least tied for largest.
- Step 4 Player 2 next chooses, from the two remaining pieces, one that she considers to be at least tied for largest, with the proviso that if she trimmed a piece in Step 2, and Player 3 did not choose this piece, then she must now choose it.
- Step 5 Player 1 receives the remaining piece.

# Solutions to Student Study Guide Questions

## Question 1
Suppose that Adam and Nadia place the following valuations on the three major assets, which will be divided up:

| Asset | Adam | Nadia |
|---|---|---|
| House | 30 | 40 |
| Car | 60 | 20 |
| Boat | 10 | 40 |

After this final reallocation of property, how many points does each of the players receive?

## Solution
Nadia gets the house and boat, while Adam gets the car. Nadia gets 80 points (40 for the car and 40 for the boat), while Adam gets 60 for the house. Some of Nadia's property has to be transferred to Adam, since she has received more points initially. To determine how much, we first compute the fractions.

$$\text{House: } \tfrac{40}{30} \approx 1.33 \quad \text{Boat: } \tfrac{40}{10} = 4.00$$

We now transfer part of Nadia's assets to Adam as follows.

- Let $x$ equal the fraction of the house which Nadia will retain.
- To equalize the number of points, we solve the following equation.

$$40 + 40x = 60 + 30(1 - x)$$
$$40 + 40x = 60 + 30 - 30x$$
$$40 + 40x = 90 - 30x$$
$$70x = 50 \Rightarrow x = \tfrac{5}{7}$$

- Since Nadia retains $\frac{5}{7}$ of the house, she must transfer $1 - \frac{5}{7} = \frac{2}{7}$ of the house to Adam.

Nadia gets to keep the boat, which is worth 40 points to her, she also retains a $\frac{5}{7}$ share in the house, which is worth $40\left(\frac{5}{7}\right) = 28\frac{4}{7}$ points, giving her a total of $68\frac{4}{7}$ points. Adam gets the car, worth 60 points to him, and a $\frac{2}{7}$ share in the house, worth $30\left(\frac{2}{7}\right) = 8\frac{4}{7}$ points, totaling $68\frac{4}{7}$ points.

## Example F (from the Student Study Guide)
What would the allocation of cars be in the last example if Rachel and Kari use the bottom-up strategy? Assume that Rachel goes first.

| | Rachel's Ranking | Kari's Ranking |
|---|---|---|
| Best | C | P |
| Second best | P | H |
| Third best | H | C |
| Worst | A | A |

## Question 2 (refers to Example F)
Who would get the worst choice if Kari went first?

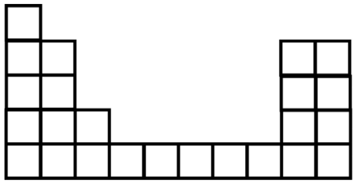
## Solution

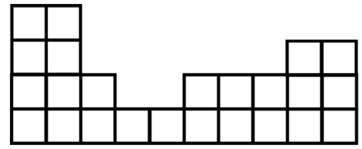
| | | | | |
|---|---|---|---|---|
| Rachel: | | <u>C</u> | | <u>A</u> |
| Kari: | <u>P</u> | | <u>H</u> | |

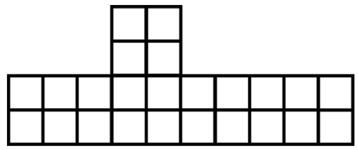
Rachel would get the worst choice, A.

**Example J (from the Student Study Guide)**

Suppose that Players 1, 2, and 3 view a cake as follows.


Player 1

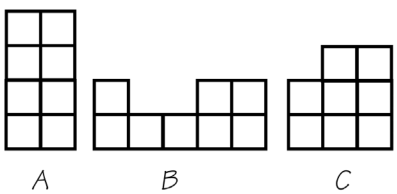

Player 2


Player 3

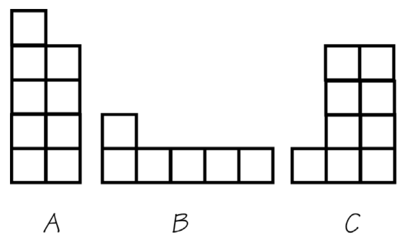
**Question 3 (refers to Example J)**

If Player 2 did the cutting, how many pieces would be acceptable to Players 1 and 3?

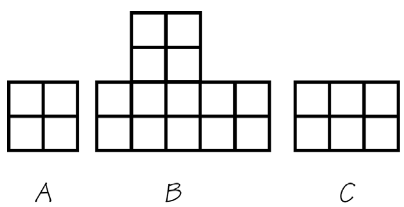
**Solution**



Player 2



Player 1



Player 3

Since an acceptable piece would be at least of size $\frac{24}{3}=8$, Player 1 would find pieces *A* and *B* acceptable. Player 3 also would find piece *B* acceptable.