Chapter 11
Weighted Voting Systems

For All Practical Purposes: Effective Teaching

- In observing other faculty or TA's, if you discover a teaching technique that you feel was particularly effective, don't hesitate to try it. Effective educators are always willing to lend, borrow, or adapt good teaching techniques. Becoming an effective educator is an evolutionary process, with no limit.
- Although students should have a syllabus with vital course and contact information, remind students throughout the course about your office hours as well as the office hours of the faculty you are conducting the sessions under. Also, if any other resources such as learning labs are available, remind students of this as well.

Chapter Briefing

There are many settings, such as shareholder elections, in which people who are entitled to vote have varying numbers of votes. When this occurs, you will have a *weighted voting system*. In that system you will need to know the weight of each voter along with the quota. The *quota* is the total weight of voters needed to pass a motion.

In such situations, the actual number of votes each can cast may not reflect the voter's *power*, that is, his ability to influence the outcome of the election. Several measures of power have been introduced and two of them are studied in this chapter, the *Banzhaf Power Index* and the *Shapley–Shubik Power Index*.

Certain weighted voting systems, may have a *dictator*, a voter with *veto power*, or *dummy voters*. Being well prepared by knowing how to determine if a system has any of these is crucial. Also, you should be able to determine *permutations of voters*, *winning coalitions*, *blocking coalitions*, and *critical voters*.

In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Weighted Voting Systems**, **The Shapley-Shubik Power Index with 3 or 4 voters**, **The Shapley-Shubik Power Index with 5 or more voters**, **Types of Coalitions and Voters**, **Binary Numbers and Voting Combinations**, **Combinations and Pascal's Triangle**, and **Minimal Winning Coalitions and Equivalent Voting Systems**. Examples that do not appear in the text nor study guide are included. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the six questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Weighted Voting Systems

A **weighted voting system** is one in which each voter has a number of votes, called his or her weight. The number of votes needed to pass a measure is called the quota. If the quota for a system with *n* voters is *q*, and the weights are $w_1, w_2, ..., w_n$, then we use the notation, $[q : w_1, w_2, ..., w_n]$.

Teaching Tip
Try to refer to voters as *A*, *B*, *C*, etc. Referring to them as their weights may cause confusion. Also, notice the initial notation we use is $[q : w_1, w_2, ..., w_n]$, to indicate that there are *n* voters. If there are 4 voters, we will refer to them as $[q : w_A, w_B, w_C, w_D]$.

Teaching Tip
If you are generating examples of weighted voting systems, a requirement of the quota is that it must me no more than the sum of the votes (otherwise nothing will pass) and it must be more than half the sum of the votes.

Example

Consider the weighted voting system $[q : w_A, w_B, w_C] = [23 : 10, 15, 12]$. Determine the number of voters and their weights. State the quota.

Solution
There are 3 voters. Their weights are 10, 15, and 12. The quota is 23.

Special Kinds of Voters

A weighted voting system may or may not have any of the following.

- A **dictator** is a voter whose weight is greater than or equal to the quota. Thus, if the dictator is in favor of a motion, it will pass. Moreover, a motion will fail if the dictator is against it, independent of how the other voters vote.
- A single voter has **veto power** if no issue can pass without his or her vote. Note that this is different from a dictator because the voter with veto power does not need to have a weight of the quota or greater. A separate dictator and a voter with veto power cannot both exist in the same system.
- A **dummy** is one whose vote will never be needed to pass or defeat any measure. Independent of how the dummy voter votes, the outcome will not change.

Teaching Tip
In order to distinguish between the types of voters, treat a voter with veto power as one that is not a dictator but can single-handedly prevent any group of players from passing a motion. Also, whenever there is a dictator, all other voters will have no power and will all be dummies.

Example

Given the following weighted voting systems, classify each voter as a dictator, one with veto power, a dummy, or none of these.

a) $[q : w_A, w_B, w_C] = [45 : 46, 42, 2]$

b) $[q : w_A, w_B, w_C, w_D] = [35 : 26, 16, 10, 4]$

c) $[q : w_A, w_B, w_C] = [23 : 10, 15, 12]$

Solution

a) Voter *A* is a dictator because be has all the votes necessary to pass a motion. Voters *B* and *C* are dummies because their votes will not change the outcome.

b) Voter *A* has veto power. No motion will pass without his or her vote. Voter A does not have enough weight to pass the motion alone. Voters *B* and *C* are none of the classifications because they do have an effect on the outcome. Voter *D* is a dummy.

c) All three voters are none of these. There is no dictator because none of the voters has a weight at least as big as the quota. No single voter has veto power, and all three can have an effect on the outcome.

☝Teaching Tip

With different quotas, the distribution of power can be altered. You may choose to take the previous example and alter the quota and ask students to readdress the same questions. By simply changing the quota you can get very different results.

➾ The Shapley-Shubik Power Index with 3 or 4 voters

A **permutation** of voters is an ordering of <u>all</u> the voters. There is $n!$ (*n* factorial) permutations of *n* voters where $n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$. The first voter in a permutation who, when joined by those coming before him or her, would have enough voting weight to win. This voter is called the **pivotal voter** of that permutation. Each permutation has exactly one pivotal voter. The **Shapley–Shubik power index** of a voter is the fraction of the permutations in which that voter is pivotal.

☝Teaching Tip

You may choose to point out the following factorial definitions/calculations to students. By showing them how quickly the size grows as n increases, you can reassure them that the expectations are only to deal with all permutations up to a group of 4 voters.

- $0! = 1$
- $1! = 1$
- $2! = 2 \times 1 = 2$
- $3! = 3 \times 2 \times 1 = 6$
- $4! = 4 \times 3 \times 2 \times 1 = 24$
- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
- $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

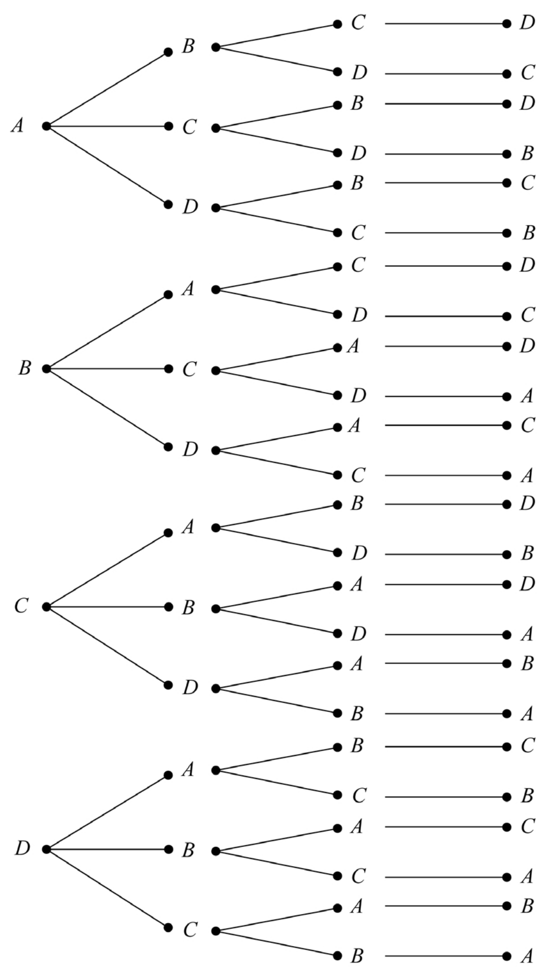
Also, to get a handle on the actual permutations, you should be systematic. You may choose to use tree diagrams.

Example

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q: w_A, w_B, w_C, w_D] = [20: 11, 8, 7, 3]$.

Solution

There are 4 voters. Thus, there are $4! = 4\times3\times2\times1 = 24$ permutations of voters. The following diagram is helpful in determining all 24 permutations in a systematic fashion.



Permutations	Weights			
A B ***C*** *D*	11	19	**26**	29
A B ***D*** *C*	11	19	**22**	29
A C ***B*** *D*	11	18	**26**	29
A C ***D*** *B*	11	18	**21**	29
A D ***B*** *C*	11	14	**22**	29
A D ***C*** *B*	11	14	**21**	29
B A ***C*** *D*	8	19	**26**	29
B A ***D*** *C*	8	19	**22**	29
B C ***A*** *D*	8	15	**26**	29
B C D ***A***	8	15	18	**29**
B D ***A*** *C*	8	11	**22**	29
B D C ***A***	8	11	18	**29**
C A ***B*** *D*	7	18	**26**	29
C A ***D*** *B*	7	18	**21**	29
C B ***A*** *D*	7	15	**26**	29
C B D ***A***	7	15	18	**29**
C D ***A*** *B*	7	10	**21**	29
C D B ***A***	7	10	18	**29**
D A ***B*** *C*	3	14	**22**	29
D A ***C*** *B*	3	14	**21**	29
D B ***A*** *C*	3	11	**22**	29
D B C ***A***	3	11	18	**29**
D C ***A*** *B*	3	10	**21**	29
D C B ***A***	3	10	18	**29**

Since *A* is the pivotal voter 12 times, *B* is pivotal 4 times, *C* is pivotal 4 times, and *D* is pivotal 4 times, the Shapley–Shubik power index for this weighted system is the following.

$$\left(\frac{12}{24}, \frac{4}{24}, \frac{4}{24}, \frac{4}{24}\right) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

The Shapley-Shubik Power Index with 5 or more voters

If there are 5 or more voters, a direct calculation of the Shapley–Shubik index would be difficult. If, however, many of the voters have equal votes, it is possible to compute this index by counting the number of permutations.

Example

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q: w_A, w_B, w_C, w_D, w_E] = [6: 3, 2, 2, 2, 2]$.

Solution

There are 5 voters. Thus, there are $5! = 5\times4\times3\times2\times1 = 120$ permutations of voters. A is pivotal in the following type of permutations.

Permutations					Weights				
X_1	X_2	<u>**A**</u>	X_3	X_4	2	2	<u>**7**</u>	9	11

There are $4! = 4\times3\times2\times1 = 24$ associated permutations for this case. Thus, the Shapley–Shubik power index for A is $\frac{24}{120} = \frac{1}{5}$. The remaining four voters share equally the remaining $1 - \frac{1}{5} = \frac{4}{5}$ of the power. Thus, each of them has an index $\frac{4}{5} \div 4 = \frac{4}{5} \times \frac{1}{4} = \frac{4}{20} = \frac{1}{5}$. The Shapley–Shubik power index for this weighted system is therefore $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$.

Example

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q: w_A, w_B, w_C, w_D, w_E, w_F] = [5: 3, 2, 2, 2, 2, 2]$.

Solution

There are 6 voters. Thus, there are $6! = 6\times5\times4\times3\times2\times1 = 720$ permutations of voters. A is pivotal in the following types of permutations.

Permutations						Weights					
X_1	<u>**A**</u>	X_2	X_3	X_4	X_5	2	<u>**5**</u>	7	9	11	13
X_1	X_2	<u>**A**</u>	X_3	X_4	X_5	2	4	<u>**7**</u>	9	11	13

For each of these permutation types, there are $5! = 5\times4\times3\times2\times1 = 120$ associated permutations. Thus, there is a total of $2\times120 = 240$ in which A is pivotal. Thus, the Shapley–Shubik power index for A is $\frac{240}{720} = \frac{1}{3}$. The remaining five voters share equally the remaining $1 - \frac{1}{3} = \frac{2}{3}$ of the power. Thus, each of them has an index $\frac{2}{3} \div 5 = \frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$. The Shapley–Shubik power index for this weighted system is therefore $\left(\frac{1}{3}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}\right)$.

Types of Coalitions and Voters

A **coalition** is a set of voters that vote collectively in favor of or opposed to a motion.

- A **winning coalition** is a combination of voters with enough collective weight to pass a measure. A winning coalition must have a total weight of q or more.
- A **blocking coalition** is a group of voters who have a sufficient number of votes to block a measure from passing. Note that a one-person blocking coalition is said to have veto power. If the combined weight of all voters is n, then a blocking coalition must have a weight more than $n-q$. Since votes are integer values, the blocking coalition must have a weight of at least $n-q+1$.

A voter is **critical** to a winning or blocking coalition if he or she can cause that coalition to lose by single-handedly changing his vote. Some coalitions have several critical voters, while others have none at all. If w is the weight of the winning coalition, then that winning coalition has $w-q$ **extra votes**. Any voter that has a weight that exceeds the number of extra votes is critical to that coalition.

Teaching Tip

The first time you find winning coalitions of voting systems, you may choose to start by calculating the weight of all coalitions. If there are n voters in a system, there will be 2^n-1 coalitions with a combined positive weight (if nobody votes for a motion, then the motion won't pass). The next example is a case with 31 coalitions with a positive weight.

Example

Consider the weighted voting system $[q: w_A, w_B, w_C, w_D, w_E] = [22: 9, 8, 6, 5, 3]$.

a) Find all winning coalitions.

b) Find the critical voters for each winning coalition.

c) List just the coalitions in which the Voter A is critical. Match each with its dual blocking coalition in which Voter A is critical.

Solution

a) If you include the empty set, there are 32 coalitions. Since we want only to consider winning coalitions, we will start by listing all but the empty set.

Coalition	Weight	
$\{A\}$	9	
$\{B\}$	8	
$\{C\}$	6	
$\{D\}$	5	
{E}	3	
$\{A, B\}$	17	
$\{A, C\}$	15	
$\{A, D\}$	14	
$\{A, E\}$	12	
$\{B, C\}$	14	
$\{B, D\}$	13	
$\{B, E\}$	11	
$\{C, D\}$	11	
{C, E}	9	
$\{D, E\}$	8	

Coalition	Weight	
$\{A, B, C\}$	23	winning
$\{A, B, D\}$	22	winning
$\{A, B, E\}$	20	
$\{A, C, D\}$	20	
$\{A, C, E\}$	18	
$\{A, D, E\}$	17	
$\{B, C, D\}$	19	
$\{B, C, E\}$	17	
$\{B, D, E\}$	16	
$\{C, D, E\}$	14	
$\{A, B, C, D\}$	28	winning
$\{A, B, C, E\}$	26	winning
$\{A, B, D, E\}$	25	winning
$\{A, C, D, E\}$	23	winning
$\{B, C, D, E\}$	22	winning
$\{A, B, C, D, E\}$	31	winning

The winning coalitions are $\{A, B, C\}$, $\{A, B, D\}$, $\{A, B, C, D\}$, $\{A, B, C, E\}$, $\{A, B, D, E\}$, $\{A, C, D, E\}$, $\{B, C, D, E\}$, and $\{A, B, C, D, E\}$.

b) There are eight winning coalitions. We will determine the number of extra votes. Any voter that has a weight that exceeds the number of extra votes is critical. A voter is critical in a winning coalition if a "1" appears in its column.

Winning coalition	Weight	Extra votes	Critical votes				
			A	*B*	*C*	*D*	E
{*A*, *B*, *C*}	23	1	1	1	1	0	0
{*A*, *B*, *D*}	22	0	1	1	0	1	0
{*A*, *B*, *C*, *D*}	28	6	1	1	0	0	0
{*A*, *B*, *C*, *E*}	26	4	1	1	1	0	0
{*A*, *B*, *D*, *E*}	25	3	1	1	0	1	0
{*A*, *C*, *D*, *E*}	23	1	1	0	1	1	1
{*B*, *C*, *D*, *E*}	22	0	0	1	1	1	1
{*A*, *B*, *C*, *D*, *E*}	31	9	0	0	0	0	0

c) Voter *A* is critical in 6 winning coalitions. The dual blocking coalition is listed next to the winning coalition.

Winning coalition	Dual Blocking coalition	Weight	Extra votes
{*A*, *B*, *C*}	{*A*, *D*, *E*}	17	7
{*A*, *B*, *D*}	{*A*, *C*, *E*}	18	8
{*A*, *B*, *C*, *D*}	{*A*, *E*}	12	2
{*A*, *B*, *C*, *E*}	{*A*, *D*}	14	4
{*A*, *B*, *D*, *E*}	{*A*, *C*}	15	5
{*A*, *C*, *D*, *E*}	{*A*, *B*}	17	7

Note that if the combined weight of all voters is n, then a blocking coalition must have a weight at least $n-q+1$. For this weighted voting system, a blocking coalition must have a weight of at least $(9+8+6+5+3)-22+1=31-22+1=10$. By looking at the number of extra votes of the blocking coalition (extra weight), we see Voter *A* is critical.

The Banzhaf Power Index with 3 or 4 voters

A voter's **Banzhaf power index** equals the number of distinct winning coalitions in which he is a critical voter. To determine this index for a voting system, perform the following.

- Make a list of the winning and blocking coalitions
- Determine the number of extra votes a coalition has in order to identify the critical voters.
- The number of winning coalitions in which a voter is critical is equal to the number of blocking coalitions in which the same voter is critical. This is known as **winning/blocking duality**. Thus, to calculate a voter's Banzhaf power index, one can double the number of times that voter is critical in a winning coalition.

Example

Consider the weighted voting system $\left[q: w_A, w_B, w_C, w_D, w_E\right] = [22:9,\ 8,\ 6,\ 5,\ 3]$. Find the Banzhaf power index for the system.

Solution

From the last example, we can tally the number of times a voter is critical in a winning coalition.

Winning coalition	Weight	Extra votes	Critical votes A	B	C	D	E
$\{A, B, C\}$	23	1	1	1	1	0	0
$\{A, B, D\}$	22	0	1	1	0	1	0
$\{A, B, C, D\}$	28	6	1	1	0	0	0
$\{A, B, C, E\}$	26	4	1	1	1	0	0
$\{A, B, D, E\}$	25	3	1	1	0	1	0
$\{A, C, D, E\}$	23	1	1	0	1	1	1
$\{B, C, D, E\}$	22	0	0	1	1	1	1
$\{A, B, C, D, E\}$	31	9	0	0	0	0	0
			6	6	4	4	2

Although we can double the number of times a voter is critical in a winning coalition to find its Banzhaf index, we will go ahead and find the blocking coalitions to verify this. A blocking coalition must have a weight of 10 or more. We will only list those blocking coalitions that have 8 or less extra votes (since the highest weight of a single voter is 9) because we are only looking for voters critical in the blocking coalition.

Blocking coalition	Weight	Extra votes	Critical votes A	B	C	D	E
$\{A, B\}$	17	7	1	1	0	0	0
$\{A, C\}$	15	5	1	0	1	0	0
$\{A, D\}$	14	4	1	0	0	1	0
$\{A, E\}$	12	2	1	0	0	0	1
$\{B, C\}$	14	4	0	1	1	0	0
$\{B, D\}$	13	3	0	1	0	1	0
$\{B, E\}$	11	1	0	1	0	0	1
$\{C, D\}$	11	1	0	0	1	1	0
$\{A, C, E\}$	18	8	1	0	0	0	0
$\{A, D, E\}$	17	7	1	0	0	0	0
$\{B, C, E\}$	17	7	0	1	0	0	0
$\{B, D, E\}$	16	6	0	1	0	0	0
$\{C, D, E\}$	14	4	0	0	1	1	0
			6	6	4	4	2

Adding the number of times each voter is critical in either a winning coalition or blocking coalition, the Banzhaf index of this system is $(12, 12, 8, 8, 4)$.

Binary Numbers and Voting Combinations

A sequence made up of 0's and/or 1's (**bits**) can represent a **binary number** or a base-2 number. One should be able to express a number (base 10) in binary form (base 2) and vice versa. Generating powers of 2 is often helpful in such conversions.

n	0	1	2	3	4	5	6	7
2^n	1	2	4	8	16	32	64	128

n	8	9	10	11	12	13	14
2^n	256	512	1024	2048	4096	8192	16,384

Example

a) Express the binary number 10110 in a standard form (base 10).

b) Express 10,110 in binary notation.

Solution

a) $10110 = 2^4 + 2^2 + 2^1 = 16 + 4 + 2 = 22.$

b) Since 2^{13} represents the largest power of 2 that doesn't exceed 10110, we start there.

$$10{,}110 - 8192 = 10{,}110 - 2^{13} = 1918$$
$$1918 - 1024 = 1918 - 2^{10} = 894$$
$$894 - 512 = 894 - 2^9 = 382$$
$$382 - 256 = 382 - 2^8 = 126$$
$$126 - 64 = 126 - 2^6 = 62$$
$$62 - 32 = 62 - 2^5 = 30$$
$$30 - 16 = 30 - 2^4 = 14$$
$$14 - 8 = 14 - 2^3 = 6$$
$$6 - 4 = 6 - 2^2 = 2$$
$$2 - 2 = 2 - 2^1 = 0$$

Thus, the nonzero bits are b_{13}, b_{10}, b_9, b_8, b_6, b_5, b_4, b_3,b_2, and b_1. Thus, we have the following.

$$(10{,}110)_2 = 10011101111110$$

A **voting combination** is a record of how the voters cast their votes for or against a given proposition. Since there are only two outcomes, in favor or against, n voters can have 2^n voting combinations. If a vote against (no) a proposition is cast, that voter can be represented by a 0. If a vote for (yes) a proposition is cast, that voter can be represented by a 1. Thus, a voting combination can be sequence of 0's and/or 1's.

Teaching Tip

To emphasize that a voting combination can be recorded, take a vote or a poll on a particular issue. For any row or column of students you can demonstrate their voting record by 0's and 1's, where a 1 is indicated by a raised hand.

Combinations and Pascal's Triangle

The number of voting combinations with n voters and exactly k "yes" votes can be determined by finding C_k^n. This value can be found by using the **combination formula**, $C_k^n = \frac{n!}{k!(n-k)!}$. Since there is only one way to obtain all "no" votes or all "yes" votes, it should be noted that $C_0^n = C_n^n = 1$. This does agree with the combination formula. The **duality formula for combinations** is $C_k^n = C_{n-k}^n$ and the **addition formula** is $C_{k-1}^n + C_k^n = C_k^{n+1}$.

Teaching Tip

You may choose to demonstrate the duality and addition formulas with numerical examples. The duality formula is convenient if the dual combination has already been calculated. The addition formula motivates Pascal's triangle.

Using the addition formula, the pattern known as **Pascal's Triangle** can be justified. The first 11 rows are displayed below.

$$\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1\\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1\\
1\quad 9\quad 36\quad 84\quad 126\quad 126\quad 84\quad 36\quad 9\quad 1\\
1\quad 10\quad 45\quad 120\quad 210\quad 252\quad 210\quad 120\quad 45\quad 10\quad 1
\end{array}$$

Pascal's triangle is useful in finding values of C_k^n, where n is relatively small.

Example

Use the combination formula as well as Pascal's triangle to find C_7^{10}.

Solution

Since $C_k^n = \frac{n!}{k!(n-k)!}$, we have $C_7^{10} = \frac{10!}{7!(10-7)!} = \frac{10!}{7!3!} = \frac{10\times 9\times 8}{3\times 2\times 1} = 10\times 3\times 4 = 120.$

With the very top 1 being the starting place (0th row), go down to row 10 and to entry 7 (with 1st entry labeled as 0th entry).

$$\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1\\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1\\
1\quad 9\quad 36\quad 84\quad 126\quad 126\quad 84\quad 36\quad 9\quad 1\\
1\quad 10\quad 45\quad 120\quad 210\quad 252\quad 210\quad 120\quad 45\quad 10\quad 1
\end{array}$$

Thus, $C_7^{10} = 120$.

The Banzhaf Power Index with 5 or more voters

When there are many voters, the number of winning coalitions can be very large, and calculating the Banzhaf index will then be cumbersome. However, there are settings in which most, but not all, of the voters have equal weights. In such situations, we can compute the Banzhaf index by means of combinations, using the numbers C_k^n.

Example

Consider the weighted voting system $[q : w_A, w_B, w_C, w_D, w_E, w_F] = [5 : 3, 1, 1, 1, 1, 1]$. Find the Banzhaf power index for the system.

Solution

A is a critical voter in three types of winning coalitions. One in which there are no extra votes, 1 extra vote, or 2 extra votes. X_i indicates a weight-one voter, where i =1, 2, 3, 4, or 5.

Winning coalition	Weight	Extra votes
$\{A, X_1, X_2\}$	5	0
$\{A, X_1, X_2, X_3\}$	6	1
$\{A, X_1, X_2, X_3, X_4\}$	7	2

In the first situation, the two additional voters are drawn from the five other voters, and there are $C_2^5 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5\times4}{2\times1} = 5\times2 = 10$ ways of choosing these two voters. Similarly, in the second case, there are $C_3^5 = 10$ ways of choosing three voters from among five. In the third case there are $C_4^5 = 5$ ways of choosing four voters from among five. Hence, *A* is critical in 25 winning coalitions, and there are 25 blocking coalitions in which he is also critical, so that his Banzhaf index is 50. Now let us consider one of the voters with just one vote, say *B*, who is also critical in two types of winning coalitions.

Winning coalition	Weight	Extra votes
$\{A, B, X_1\}$	5	0
$\{B, X_1, X_2, X_3, X_4\}$	5	0

In the first situation, the additional voter is drawn from four other voters. and there are $C_1^4 = 4$ ways of choosing this voter. Similarly, in the second case, there is $C_4^4 = 1$ way of choosing four voters from among four. This, together with an equal number of blocking coalitions in which he or she is critical, yields a Banzhaf index of 10.

Thus, the Banzhaf power index for the system is $(50,10,10,10,10,10)$.

Minimal Winning Coalitions and Equivalent Voting Systems

A **minimal winning coalition** is one in which each voter is critical to the passage of a measure; that is, if anyone defects, then the coalition is turned into a losing one.

Example

Consider the weighted voting system $[q: w_A, w_B, w_C, w_D] = [7: 5, 3, 2, 2]$. Find the minimal winning coalitions.

Solution

Since there are only $2^4 - 1 = 16 - 1 = 15$ coalitions with a positive weight, we will go ahead and list them all out.

Coalition	Weight	
$\{A\}$	5	
$\{B\}$	3	
$\{C\}$	2	
$\{D\}$	2	
$\{A, B\}$	8	winning
$\{A, C\}$	7	winning
$\{A, D\}$	7	winning
$\{B, C\}$	5	
$\{B, D\}$	5	
$\{C, D\}$	4	
$\{A, B, C\}$	10	winning
$\{A, B, D\}$	10	winning
$\{A, C, D\}$	9	winning
$\{B, C, D\}$	7	winning
$\{A, B, C, D\}$	12	winning

We now look at the eight winning coalitions and determine which have all members as critical.

Winning coalition	Weight	Extra votes	Critical votes			
			A	B	C	D
$\{A, B\}$	8	1	1	1	0	0
$\{A, C\}$	7	0	1	0	1	0
$\{A, D\}$	7	0	1	0	0	1
$\{A, B, C\}$	10	3	1	0	0	0
$\{A, B, D\}$	10	3	1	0	0	0
$\{A, C, D\}$	9	2	1	0	0	0
$\{B, C, D\}$	7	0	0	1	1	1
$\{A, B, C, D\}$	12	5	0	0	0	0

Thus, the minimal winning coalitions are $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$.

A voting system can be described completely by stating its minimal winning coalitions. There are three requirements of this list of winning coalitions.

- You must have at least one coalition on the list; otherwise, a motion can't pass.
- A minimal coalition cannot be contained in another minimal one. This would contradict the idea that this is a minimal winning coalition.
- Every pair of minimal winning coalitions should overlap; otherwise, two opposing motions can pass.

Example

Consider the weighted voting system in the last example. Verify that the minimal winning coalitions $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$ satisfy the three conditions stated above.

Solution

a) You must have at least one coalition on the list; otherwise, a motion can't pass.

There are four minimal coalitions. Thus, this condition is satisfied.

b) A minimal coalition cannot be contained in another minimal one.

Comparing all minimal coalitions with the others, we have the following.

- $\{A, B\}$ is not contained in $\{A, C\}$, $\{A, D\}$, nor $\{B, C, D\}$.
- $\{A, C\}$ is not contained in $\{A, B\}$, $\{A, D\}$, nor $\{B, C, D\}$.
- $\{A, D\}$ is not contained in $\{A, B\}$, $\{A, C\}$, nor $\{B, C, D\}$.
- $\{B, C, D\}$ is not contained in $\{A, B\}$, $\{A, C\}$, nor $\{A, D\}$.

Thus, this condition is satisfied.

c) Every pair of minimal winning coalitions should overlap.

Comparing all minimal coalitions with the others, we have the following.

Overlap	$\{A, B\}$	$\{A, C\}$	$\{A, D\}$	$\{B, C, D\}$
$\{A, B\}$		$\{A\}$	$\{A\}$	$\{B\}$
$\{A, C\}$			$\{A\}$	$\{C\}$
$\{A, D\}$				$\{D\}$
$\{B, C, D\}$				

Each pair of minimal coalitions has a non-empty intersection. Thus, this condition is satisfied.

Two voting systems are **equivalent** if there is a way to exchange all voters from the first system with voters of the second while maintaining the same winning coalitions.

Example

A committee has 7 members. *A* and *B* are co-chairs, and *C*, *D*, *E*, *F* and *G* are the ordinary members. The minimal winning coalitions are *A* and *B*, *A* and any three of the ordinary members, and *B* and any three of the ordinary members. Express this situation as an equivalent weighted voting system.

Solution

There are many answers possible. Consider assigning a weight of 1 to each regular committee member. Then assign weights to the two co-chairs both based on the voting requirements. One possible answer is therefore $[q : w_A, w_B, w_C, w_D, w_E, w_F, w_G] = [6 : 3, 3, 1, 1, 1, 1, 1]$. Another possibility would be to assign a weight of 2 to each regular committee member. We could therefore have the following $[q : w_A, w_B, w_C, w_D, w_E, w_F, w_G] = [11 : 6, 5, 2, 2, 2, 2, 2]$.

Solutions to Student Study Guide ✎ Questions

Question 1

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q : w_A, w_B, w_C, w_D] = [9 : 4, 3, 3, 1]$.

Solution

There are 4 voters. Thus, there are $4! = 4\times3\times2\times1 = 24$ permutations of voters.

Permutations	Weights
A B **<u>*C*</u>** *D*	4 7 **<u>10</u>** 11
A B D **<u>*C*</u>**	4 7 8 **<u>11</u>**
A C **<u>*B*</u>** *D*	4 7 **<u>10</u>** 11
A C D **<u>*B*</u>**	4 7 8 **<u>11</u>**
A D B **<u>*C*</u>**	4 5 8 **<u>11</u>**
A D C **<u>*B*</u>**	4 5 8 **<u>11</u>**

Permutations	Weights
B A **<u>*C*</u>** *D*	3 7 **<u>10</u>** 11
B A D **<u>*C*</u>**	3 7 8 **<u>11</u>**
B C **<u>*A*</u>** *D*	3 6 **<u>10</u>** 11
B C D **<u>*A*</u>**	3 6 7 **<u>11</u>**
B D A **<u>*C*</u>**	3 4 8 **<u>11</u>**
B D C **<u>*A*</u>**	3 4 7 **<u>11</u>**

Permutations	Weights
C A **<u>*B*</u>** *D*	3 7 **<u>10</u>** 11
C A D **<u>*B*</u>**	3 7 8 **<u>11</u>**
C B **<u>*A*</u>** *D*	3 6 **<u>10</u>** 11
C B D **<u>*A*</u>**	3 6 7 **<u>11</u>**
C D A **<u>*B*</u>**	3 4 8 **<u>11</u>**
C D B **<u>*A*</u>**	3 4 7 **<u>11</u>**

Permutations	Weights
D A B **<u>*C*</u>**	1 5 8 **<u>11</u>**
D A C **<u>*B*</u>**	1 5 8 **<u>11</u>**
D B A **<u>*C*</u>**	1 4 8 **<u>11</u>**
D B C **<u>*A*</u>**	1 4 7 **<u>11</u>**
D C A **<u>*B*</u>**	1 4 8 **<u>11</u>**
D C B **<u>*A*</u>**	1 4 7 **<u>11</u>**

Since *A* is the pivotal voter 8 times, *B* is pivotal 8 times, *C* is pivotal 8 times, and *D* is pivotal 0 times the Shapley–Shubik power index for this weighted system is $\left(\frac{8}{24},\frac{8}{24},\frac{8}{24},\frac{0}{24}\right)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$.

You may wish to note that Voter *D* is a dummy.

Question 2

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q : w_A, w_B, w_C, w_D] = [8 : 4, 3, 3, 1]$.

Solution

There are 4 voters. Thus, there are $4! = 4\times3\times2\times1 = 24$ permutations of voters.

Permutations	Weights
A B **$\underline{C}$** D	4 7 **<u>10</u>** 11
A B **$\underline{D}$** C	4 7 **<u>8</u>** 11
A C **$\underline{B}$** D	4 7 **<u>10</u>** 11
A C **$\underline{D}$** B	4 7 **<u>8</u>** 11
A D **$\underline{B}$** C	4 5 **<u>8</u>** 11
A D **$\underline{C}$** B	4 5 **<u>8</u>** 11

Permutations	Weights
B A **$\underline{C}$** D	3 7 **<u>10</u>** 11
B A **$\underline{D}$** C	3 7 **<u>8</u>** 11
B C **$\underline{A}$** D	3 6 **<u>10</u>** 11
B C D **$\underline{A}$**	3 6 7 **<u>11</u>**
B D **$\underline{A}$** C	3 4 **<u>8</u>** 11
B D C **$\underline{A}$**	3 4 7 **<u>11</u>**

Permutations	Weights
C A **$\underline{B}$** D	3 7 **<u>10</u>** 11
C A **$\underline{D}$** B	3 7 **<u>8</u>** 11
C B **$\underline{A}$** D	3 6 **<u>10</u>** 11
C B D **$\underline{A}$**	3 6 7 **<u>11</u>**
C D **$\underline{A}$** B	3 4 **<u>8</u>** 11
C D B **$\underline{A}$**	3 4 7 **<u>11</u>**

Permutations	Weights
D A **$\underline{B}$** C	1 5 **<u>8</u>** 11
D A **$\underline{C}$** B	1 5 **<u>8</u>** 11
D B **$\underline{A}$** C	1 4 **<u>8</u>** 11
D B C **$\underline{A}$**	1 4 7 **<u>11</u>**
D C **$\underline{A}$** B	1 4 **<u>8</u>** 11
D C B **$\underline{A}$**	1 4 7 **<u>11</u>**

Since A is the pivotal voter 12 times, B is pivotal 4 times, C is pivotal 4 times, and D is pivotal 4 times the Shapley–Shubik power index for this weighted system is $\left(\frac{12}{24}, \frac{4}{24}, \frac{4}{24}, \frac{4}{24}\right) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$.

You may wish to note the drastic change in the Shapley–Shubik power index by lowering the quota by 1 from Question 1.

Question 3

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q: w_A, w_B, w_C, w_D] = [6: 3, 2, 2, 2]$, without determining all possible permutations.

Solution

There are 4 voters. Thus, there are $4! = 4\times3\times2\times1 = 24$ permutations of voters. A is pivotal in the following type of permutations.

Permutations				Weights			
X_1	X_2	$\underline{\mathbf{A}}$	X_3	2	4	$\underline{\mathbf{7}}$	9

For this permutation type, there is $3! = 3\times2\times1 = 6$ associated permutations. Thus, there is a total of $1\times6 = 6$ in which A is pivotal. Thus, the Shapley–Shubik power index for A is $\frac{6}{24} = \frac{1}{4}$. The remaining three voters share equally the remaining $1 - \frac{1}{4} = \frac{3}{4}$ of the power. Thus, each of them has an index $\frac{3}{4} \div 3 = \frac{3}{4} \times \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$. The Shapley–Shubik power index for this weighted system is therefore $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

Question 4

Calculate the Shapley–Shubik power index for each of the voters in the weighted voting system $[q: w_A, w_B, w_C, w_D, w_E, w_F] = [10: 4, 4, 1, 1, 1, 1]$, without determining all possible permutations.

Solution

We will start by determining the voting power of C, a weight-1 voter. C is pivotal when the voters before him or her in the permutation have a combined weight of exactly 9. This condition can be met if the two weight-4 voters and one weight-1 voter precede C. There are 6 ways that the two weight-4 voters can fill two of the first three places. Once these are in place, there are $3! = 3\times2\times1 = 6$ ways to put in the remaining weight-1 voters. Thus, the number of permutations is $6\times6 = 36$. The Shapley–Shubik index for A is therefore $\frac{36}{6!} = \frac{36}{6\times5\times4\times3\times2\times1} = \frac{36}{6\times5\times4\times(3\times2)\times1} = \frac{1}{5\times4} = \frac{1}{20}$. The other weight-1 voters have the same Shapley–Shubik index. Since there are 4 weight-1 voters, they share $4\times\frac{1}{20} = \frac{1}{5}$ of the total power. Thus, the two weight-4 voters share $1 - \frac{1}{5} = \frac{4}{5}$ of the power. Each weight-4 voter therefore has $\frac{4}{5} \div 2 = \frac{4}{5}\cdot\frac{1}{2} = \frac{4}{10} = \frac{2}{5}$ of the power. The Shapley–Shubik power index for this weighted system is therefore $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$.

Question 5

Consider the weighted voting system $[q: w_A, w_B, w_C, w_D] = [11: 7, 5, 3, 2]$. Find the Banzhaf power index for the system. Is there a dummy in this system?

Solution

Because of winning/blocking duality, we can double the number of times that voter is critical in a winning coalition to find that voter's Banzhaf power index.

The winning coalitions are those whose weights sum to 11 or more.

Winning coalition	Weight	Extra votes	Critical votes			
			A	*B*	*C*	*D*
{*A*, *B*}	12	1	1	1	0	0
{*A*, *B*, *C*}	15	4	1	1	0	0
{*A*, *B*, *D*}	14	3	1	1	0	0
{*A*, *C*, *D*}	12	1	1	0	1	1
{*A*, *B*, *C*, *D*}	17	6	1	0	0	0
			5	3	1	1

Doubling the number of times each voter is critical in either a winning coalition, the Banzhaf index of this system is $(10, 6, 2, 2)$. There are no dummy voters.

Question 6

Consider the weighted voting system $[q : w_A, w_B, w_C, w_D, w_E, w_F] = [5 : 2, 2, 1, 1, 1, 1]$. Find the Banzhaf power index for the system.

Solution

Following along with Example 11.17 from the text, this problem can be solved using combinations. Start with a weight-1 voter by calling him or her *M*. He or she will be a critical voter in a winning coalition if the votes of the other members in the coalition add up to 4. There are two ways to achieve this total.

- The two weight-2, and no other weight-1 voters, could join with *M*. There is exactly one such coalition.
- One of the two weight-2 voters and two of three other weight-1 voters join *M*. Since $C_k^n = \dfrac{n!}{k!(n-k)!}$, we have $C_1^2 \times C_2^3 = \dfrac{2!}{1!(2-1)!} \times \dfrac{3!}{2!(3-2)!} = \dfrac{2!}{1!1!} \times \dfrac{3!}{2!1!} = 2 \times 3 = 6.$

Thus, *M* is critical voter in 7 coalitions; doubling this, we find that his or her Banzhaf power index is 14.

We must now determine the Banzhaf power index of a weight-2 voter by calling him or her *A*. He or she will be a critical voter in a winning coalition in which the other members have a combined total of 3 or 4.

- If he or she is joined by the other weight-two voter and one of the four weight-1 voters, then there will be a combined weight of 3, not counting *A*. This can be done in C_1^4 ways.
- If he or she is joined by three of the four weight-1 voters, then there will be a combined weight of 3, not counting *A*. This can be done in C_3^4 ways.
- If he or she is joined by the other weight-two voter and two of the four weight-1 voters, then there will be a combined weight of 4, not counting *A*. This can be done in C_2^4 ways.
- If he or she is joined by the other four weight-1 voters, then there will be a combined weight of 4, not counting *A*. This can be done in 1 way.

A is a critical voter in $C_1^4 + C_3^4 + C_2^4 + 1$ coalitions. Performing this calculation, we have the following.

$$C_1^4 + C_3^4 + C_2^4 + 1 = \frac{4!}{1!(4-1)!} + \frac{4!}{3!(4-3)!} + \frac{4!}{2!(4-2)!} + 1 = \frac{4!}{1!3!} + \frac{4!}{3!1!} + \frac{4!}{2!2!} + 1 = 4 + 4 + 6 + 1 = 15$$

Thus, *A* is a critical voter in 15 coalitions; doubling this, we find that his or her Banzhaf power index is 30.

Finally, the Banzhaf index of this system is $(30, 30, 14, 14, 14, 14)$.