Chapter 10
The Manipulability of Voting Systems

For All Practical Purposes: Effective Teaching

- As a teaching assistant, you most likely will administer and proctor many exams. Although it is tempting to work on one's own graduate studies or grade papers, to be effective during the testing process always be attentive to the class. Besides not allowing cheating by your aggressive proctoring, you will also help maintain the teaching assistant-student relation during the term.
- When proctoring an exam, it is a good idea to take attendance before the exam begins. If you visually recognize all students then go ahead and start the exam. If you don't recognize a student, ask for identification. When students turn in an exam, make eye contact. This will reinforce that you are alert and aware of who they are as individuals.

Chapter Briefing

It is possible to manipulate the outcome of an election by having a voter or a group of voters choose to cast their vote in such a way that it does fully represent their actual overall preferences, but it does yield a favorable or more-preferred outcome.

One difficulty you may encounter in this chapter is students retaining the different voting techniques discussed in Chapter 9. Knowing these different methods is crucial for discussion of how they can or cannot be manipulated. Discussions can start by comparing two or three different voting methods in order to make sure students are quickly recalling the following voting methods.

- **Condorcet's method**
- **majority and plurality voting**
- **Borda count method**
- **sequential pairwise, with agenda**
- **Hare system**
- **plurality runoff method**

Being well prepared by knowing these methods, the conditions for manipulation (or non-manipulation), and how to manipulate the outcome of an election is essential for successful classroom discussions. In your academic preparation, you may not have encountered the topic of manipulation of voting systems. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Voting Manipulation**, **Majority Rule Non-Manipulation**, **Condorcet's Method Non-Manipulation**, **Borda Count Non-Manipulation and Manipulation**, **PluralityVoting Non-Manipulation and Manipulation**, **Plurality Runoff Rule and Hare system Manipulation**, **Sequential Pairwise Voting With Agenda Manipulation**, **Impossibility**, and **Chair's Paradox**. Examples of manipulation of voting systems with solutions that do not appear in the text nor study guide are included. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the three questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Voting Manipulation

Voting in a strategic manner is called **manipulation**. This occurs when a voter casts a ballot, which does not represent their actual overall preferences. The outcome of this manipulated election is either the preferred or a *more-preferred* candidate of the voter that is doing the manipulation. This type of ballot is known as an **insincere** or **disingenuous ballot**, and the term **unilateral change** is used when one voter changes his or her ballot.

Definition of Manipulability: A voting system is said to be **manipulable** if there exist two sequences of preference list ballots and a voter (call the voter *j*) such that

- Neither election results in a tie. (Ties in an election present a problem in determining sincere preference.)
- The only ballot change is by voter *j* (This is a unilateral change)
- Voter *j* prefers the outcome (overall winner) of the second election even though the first election showed his or her true (overall order) preferences.

Teaching Tip

You may choose to convey to students that when a voter casts a disingenuous ballot, it may not be possible for him or her to obtain their first choice in candidates. So, in manipulating an election, the effect of the unilateral change by this voter may be to obtain a "better" or "more-preferred" result. The result of Election 2, when compared to Election 1, has the winner in a higher ranking in the preference list ballot of the voter that is doing the manipulation. It is generally assumed that the preference list ballots represent the voters' true preferences.

Majority Rule Non-Manipulation

For this voting system, it is assumed that the number of voters is odd and we are only considering the two-candidate case. Three desirable properties of majority rule are as follows.

- All voters are treated equally
- Both candidates are treated equally.
- If there is a unilateral change from the loser in Election 1 to the winner, then this has no effect on the outcome in Election 2. This statement implies that the system is monotone.

May's theorem for manipulability states that given the initial conditions (odd number of voters and only two candidates), majority rule is the *only* voting method that satisfies the above properties. Where the *manipulation* part of the theorem comes in is that the third condition is equivalent to stating that majority rule is **non-manipulable**.

Teaching Tip

You may choose to take the above properties and examine them with an example like the following. This will reinforce the terms that are used throughout the chapter and the idea that Election 2 is a result of knowing the preferences of the voters in Election 1. Election 2, in the end, is the election that will prevail. It is not a "do over" so to speak. Also, in an example as follows one sees that the voter that gave the disingenuous ballot in Election 2 could in no way get a preferred or more-preferred outcome (non-manipulation).

Example

Consider the following election. Who is the winner if majority rule is used? What would happen if the left-most voter changed his or her ballot?

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*B*
Second	*B*	*B*	*A*	*A*	*A*

Solution

Since *B* has the majority (over half) of the first-place votes, he or she would be the winner. Suppose though that the left-most voter changes his or her ballot. We would have the following.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*A*	*B*	*B*	*B*
Second	*A*	*B*	*A*	*A*	*A*

B still has the majority of the first-place votes and is still the winner.

Condorcet's Method Non-Manipulation

For this voting system, it is assumed that the number of voters is odd. Condorcet's method is based on majority rule in which one candidate can beat all others in a one-on-one contest. Condorcet's method is non-manipulable in the sense that a voter cannot change the outcome from one candidate to another that he or she prefers. It is possible however for a voter to change the outcome from one candidate to having no winner.

Example

Consider the following election with four candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*B*	*A*	*A*
Second	*A*	*B*	*A*	*C*	*B*
Third	*C*	*A*	*D*	*D*	*D*
Fourth	*B*	*D*	*C*	*B*	*C*

Show that if Condorcet's method is being used, the voter on the left can change the outcome so that there is no winner.

Solution

There are 6 one-on-one contests as summarized below.

A vs *B*	*A:*	3	*B:*	2
A vs *C*	*A:*	4	*C:*	1
A vs *D*	*A:*	4	*D:*	1
B vs *C*	*B:*	2	*C:*	3
B vs *D*	*B:*	3	*D:*	2
C vs *D*	*C:*	2	*D:*	3

Since *A* can beat the other candidates in a one-on-one contest, *A* is declared the winner by Condorcet's method.

Continued on next page

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*B*	*A*	*A*
Second	*C*	*B*	*A*	*C*	*B*
Third	*B*	*A*	*D*	*D*	*D*
Fourth	*A*	*D*	*C*	*B*	*C*

A vs *B*	*A:*	2	*B:*	3
A vs *C*	*A:*	3	*C:*	2
A vs *D*	*A:*	4	*D:*	1
B vs *C*	*B:*	2	*C:*	3
B vs *D*	*B:*	3	*D:*	2
C vs *D*	*C:*	2	*D:*	3

Since no candidate can beat all other candidates in a one-on-one contest, there is no winner by Condorcet's method.

Borda Count Non-Manipulation and Manipulation

For this voting system, the assumption for an odd number of voters is dropped.

- The Borda count method is **non-manipulable for three candidates**. We do assume that the preference list ballots in the first election represent the sincere preferences of the voters and any change in a voter's preference ballot is an attempt to obtain a preferred or more-preferred outcome (which cannot happen). The argument for this case of non-manipulation is outlined in Section 10.2.
- The Borda count method is **manipulable for four or more candidates** (and at least two voters). We do assume that the preference list ballots in the first election represent the sincere preferences of the voters and any change in a voter's preference ballot is an attempt to obtain a preferred or more-preferred outcome. The argument for this is outlined in the text by showing that you can handle the case of an even number of voters by starting with the four candidate/two-voter example (Example 1) and then adding any even number of voters that would essentially cancel out the effect of the additional voters. This of course extends for any number of candidates. In order to handle the case of an odd number of voters, Exercise 9 provides an example with four candidates and three voters. This example can be extended to show the manipulation of any number of candidates and any odd number of voters.

☝Teaching Tip

In dealing with the manipulation of the Borda count, there are two types of exercises that students should be able to master. One is **extending an example** of vote manipulation for a specified number of candidates and voters. To extend to an even number of voters, the initial example must have an even number of voters. Naturally to extend to an odd number of voters, the initial example must have an odd number of voters. The additional voters are thought of in groups of 2, canceling out the influence of each other. Question 2 from the *Student Study Guide* asks to demonstrate an extension of an even number of voters. The solution appears at the end of this chapter of your guide. Exercise 9 of the text asks to demonstrate the extension of an odd number of voters. The solution appears in both the *Student Solution Manual* and the *Instructor Solution Manual*.

The other type of exercise students should master is finding an example of a **unilateral change that results in a manipulation** of an election. The following example and solution demonstrates such an exercise.

Example

Consider the following election with four candidates and four voters.

Election 1

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*B*	*A*	*C*	*D*
Second	*D*	*C*	*A*	*B*
Third	*C*	*D*	*B*	*C*
Fourth	*A*	*B*	*D*	*A*

Show that if the Borda count is being used, the voter on the left can manipulate the outcome (assuming the above ballot represents his true preferences).

Solution

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	1×2	0×1	2×0	5
B	1×3	1×2	1×1	1×0	6
C	1×3	1×2	2×1	0×0	7
D	1×3	1×2	1×1	1×0	6

With the given ballots, the winner using the Borda count is *C*. However, if the left-most voter changes his or her preference ballot, we have the following.

Election 2

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*D*	*A*	*C*	*D*
Second	*B*	*C*	*A*	*B*
Third	*A*	*D*	*B*	*C*
Fourth	*C*	*B*	*D*	*A*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	1×2	1×1	1×0	6
B	0×3	2×2	1×1	1×0	5
C	1×3	1×2	1×1	1×0	6
D	2×3	0×2	1×1	1×0	7

With the new ballots, the winner using the Borda count is *C*. Although *B* was the left-most voters top choice, having *D* declared winner is a *more-preferred* outcome because that voter prefers *D* to *C*.

☝Teaching Tip

In trying to determine what unilateral change will cause a manipulation to an election, you may choose to point out to students that they must always first determine the outcome using the original preference list ballots. These are generally assumed to be the true preferences of all voters of an election. If they are given a particular voter which to do the manipulation, students should determine which candidate is preferred (ranked first) or which candidates are more-preferred (ranked *above* the winner of Election 1 on the potential manipulator's preference list ballot). These should be the candidates that students should be focusing on to be the winner of Election 2.

Plurality Voting Non-Manipulation and Manipulation

For this voting system, the assumption for an odd number of voters is dropped.

Plurality voting is not manipulable by a single voter. This type of non-manipulation is in the same spirit of Condorcet's method. Like Condorcet's method, it is possible to have a winner in Election 1 and then no winner in Election 2. In the case of plurality voting, there could be a tie and thus in Election 2, there is no *single* winner.

Plurality voting can be, however, **group-manipulable**. Group-manipulable is when group of voters can change the outcome of an election (as a group) to something they all prefer.

Teaching Tip

In discussing group-manipulation as it relates to plurality voting, you may choose to make the point to students that it is not implied that every election can be manipulated. You may choose to consider the following two examples. The first example shows that group-manipulation is possible. The second example shows that group-manipulation is not possible for that particular election because the winner of the election has 50% or more of the first-place votes.

Example

Consider the following election with four candidates and 12 voters.

Election 1

	Number of voters (12)		
Rank	**3**	**5**	**4**
First	*C*	*A*	*D*
Second	*B*	*C*	*A*
Third	*D*	*B*	*C*
Fourth	*A*	*D*	*B*

Show that if plurality voting is used, the group of voters on the left can secure a **more-preferred** outcome.

Solution

Since Candidate *A* has the most first-place votes, *A* is declared the winner.

Election 2

	Number of voters (12)		
Rank	**3**	**5**	**4**
First	*D*	*A*	*D*
Second	*C*	*C*	*A*
Third	*B*	*B*	*C*
Fourth	*A*	*D*	*B*

Since the group on the left changes their ballots, *D* now has the most votes and is declared the winner. Having *D* win the election was **more-preferred** by the left most group of voters, rather than having *A* win the election.

Teaching Tip

In discussing group-manipulation as it relates to plurality voting, you may choose to make the point to students that when a group changes its preference list ballot, they are always looking for a more-preferred outcome. This is because ultimately we are only looking at first-place votes in this method. The group of voters that are casting the disingenuous ballot(s) cannot possibly change their top choice to win the election.

Example

Consider the following two elections with four candidates and 14 voters.

Election A

	Number of voters (14)		
Rank	**3**	**7**	**4**
First	*C*	*A*	*D*
Second	*B*	*C*	*C*
Third	*D*	*B*	*B*
Fourth	*A*	*D*	*A*

Election B

	Number of voters (14)		
Rank	**2**	**9**	**3**
First	*C*	*A*	*D*
Second	*B*	*C*	*C*
Third	*D*	*B*	*B*
Fourth	*A*	*D*	*A*

Is it possible for either of the two losing groups in either election (the two elections are independent of each other) to manipulate the outcome of their respective elections?

Solution

In both Election A and B, the winner of the election is Candidate *A*. It is not possible to have the vote manipulated by either the left-most or right-most groups. In Election A, if either non-winning group throws their support to the other, it is not enough to create a winner because a tie will result.

In Election B, if either non-winning group throws their support to the other, it is not enough to create a winner because the number of first-place votes will still be less than that of Candidate *A*.

Plurality Runoff Rule And Hare System Manipulation

For these voting systems, the assumption for an odd number of voters is dropped. Both plurality runoff and the Hare system are manipulable runoff voting methods.

Teaching Tip

In order to continually emphasize the different voting methods, you may choose to ask students to describe what they believe the term “runoff” means. In doing so, they will be able to distinguish the voting methods and visualize what needs to occur in order for manipulation to take place. Recall in the plurality runoff method, two (or three if there is a tie for second place) candidates go forward for one more election that eliminates all but these candidates. In the Hare System, the candidate (or candidates if a tie) with the least number of first-place votes is (are) eliminated. Depending on the number of candidates, it could take several steps to determine the winner.

Teaching Tip

In discussing manipulation of the runoff systems, you may wish to again make the point to students that a system is manipulable if there is at least one scenario in which manipulation occurs. Because of the possibility of ties with an even number of voters, students may come to the conclusion that a system is not manipulable because they may not be able to manipulate certain examples.

Example

Consider the following election with four candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*C*	*C*	*D*	*B*	*B*
Second	*A*	*D*	*B*	*A*	*D*
Third	*D*	*A*	*C*	*C*	*A*
Fourth	*B*	*B*	*A*	*D*	*C*

Show how the left-most voter can secure a **more-preferred** outcome by a unilateral change of ballot using the plurality runoff rule.

Solution

Since *B* and *C* have the most number of first-place votes, *A* and *D* are eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*C*	*C*	*B*	*B*	*B*
Second	*B*	*B*	*C*	*C*	*C*

Since *B* has the most number of first-place votes, the winner using the plurality runoff rule is *B*. But the winner becomes *D* if the leftmost voter changes his or her ballot as the following shows.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*D*	*B*	*B*
Second	*A*	*D*	*B*	*A*	*D*
Third	*C*	*A*	*C*	*C*	*A*
Fourth	*B*	*B*	*A*	*D*	*C*

Since *B* and *D* have the most number of first-place votes, *A* and *C* are eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*D*	*D*	*B*	*B*
Second	*B*	*B*	*B*	*D*	*D*

Since *D* has the most number of first-place votes, the winner using the plurality runoff rule is *D*. For the first voter, having *D* win the election was **more-preferred** than having *B* win the election.

Example

Consider the following election with four candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*C*	*C*	*D*	*B*	*B*
Second	*A*	*D*	*B*	*A*	*D*
Third	*D*	*A*	*C*	*C*	*A*
Fourth	*B*	*B*	*A*	*D*	*C*

Show how the left-most voter can secure a **more-preferred** outcome by a unilateral change of ballot using the Hare system.

Solution

A has the fewest first-place votes and is thus eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*C*	*C*	*D*	*B*	*B*
Second	*D*	*D*	*B*	*C*	*D*
Third	*B*	*B*	*C*	*D*	*C*

D now has the fewest first-place votes and is eliminated

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*C*	*C*	*B*	*B*	*B*
Second	*B*	*B*	*C*	*C*	*C*

C now has the fewest first-place votes and is eliminated, leaving *B* as the winner.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*D*	*B*	*B*
Second	*A*	*D*	*B*	*A*	*D*
Third	*C*	*A*	*C*	*C*	*A*
Fourth	*B*	*B*	*A*	*D*	*C*

A has the fewest first-place votes and is eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*D*	*B*	*B*
Second	*C*	*D*	*B*	*C*	*D*
Third	*B*	*B*	*C*	*D*	*C*

C now has the fewest first-place votes and is eliminated

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*D*	*D*	*B*	*B*
Second	*B*	*B*	*B*	*D*	*D*

B now has the fewest first-place votes and is eliminated, leaving *D* as the winner. For the first voter, having *D* win the election was **more-preferred** than having *B* win the election.

Sequential Pairwise Voting With Agenda Manipulation

In this voting system, the assumption for an odd number of voters is dropped. Sequential pairwise voting itself can be manipulated by a single voter. In this case, the agenda is fixed. The other way manipulation occurs is by not altering the preference list ballots, but by altering the agenda.

Teaching Tip

In order to determine how a change of agenda can alter the outcome of an election, you may choose to advise student to initially determine the winners of all one-on-one contests. Doing this is advance will save students time in the long run.

Example

Consider the following election with four candidates and five voters.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*C*	*D*
Second	*B*	*A*	*D*	*B*	*B*
Third	*D*	*C*	*A*	*D*	*A*
Fourth	*C*	*D*	*C*	*A*	*C*

Determine which candidate is the winner using sequential pairwise voting, with agenda *A*, *B*, *C*, *D*. Is it possible for any voter to manipulate the outcome by choosing a different agenda so that a preferred candidate wins the election?

Solution

Looking at the 6 one-on-one contests, we can more readily see the solution.

A vs *B*	*A:*	1	*B:*	4
A vs *C*	*A:*	4	*C:*	1
A vs *D*	*A:*	2	*D:*	3
B vs *C*	*B:*	4	*C:*	1
B vs *D*	*B:*	4	*D:*	1
C vs *D*	*C:*	2	*D:*	3

B can beat all other candidates in one-on-one contests (*B* is the winner by Condorcet's Method). *B* will win the election, independent of the agenda.

Example

Consider the following election with four candidates and five voters.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*D*
Second	*C*	*D*	*B*	*B*	*C*
Third	*B*	*A*	*D*	*A*	*B*
Fourth	*D*	*C*	*A*	*C*	*A*

Determine which candidate is the winner using sequential pairwise voting, with agenda *A*, *B*, *C*, *D*. Is it possible for any voter to manipulate the outcome by choosing a different agenda so that a preferred candidate wins the election?

Continued on next page

Solution

Looking at the 6 one-on-one contests, we can more readily see the solution.

A vs *B*	*A:*	1	*B:*	4
A vs *C*	*A:*	3	*C:*	2
A vs *D*	*A:*	1	*D:*	4
B vs *C*	*B:*	2	*C:*	3
B vs *D*	*B:*	3	*D:*	2
C vs *D*	*C:*	2	*D:*	3

In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, we first pit *A* against *B*. Thus, *B* wins by a score of 4 to 1. *B* moves on to confront *C*. *C* wins by a score of 3 to 2. *C* moves on to confront *D*. *D* wins by a score of 3 to 2. Thus, *D* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*.

Noticing that *A* wins over *C* by a score of 3 to 2, we can make the **winner of the election *A*** by changing the agenda to *B*, *D*, *C*, *A*. Notice that we needed to first pit *B* against *D*, so that *B* could win over *D* to then face *C*.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*D*
Second	*C*	*D*	*B*	*B*	*C*
Third	*B*	*A*	*D*	*A*	*B*
Fourth	*D*	*C*	*A*	*C*	*A*

This would be the preferred outcome by the shaded voter.

Noticing that *B* loses to only *C* by a score of 3 to 2, we can make the **winner of the election *B*** by changing the agenda to *A*, *C*, *B*, *D*. Notice that we needed to first pit *A* against *C*, so that *A* could win over *C* to then face *B*.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*D*
Second	*C*	*D*	*B*	*B*	*C*
Third	*B*	*A*	*D*	*A*	*B*
Fourth	*D*	*C*	*A*	*C*	*A*

This would be the preferred outcome by the shaded voters.

Noticing that *C* wins over *B* by a score of 3 to 2, we can make the **winner of the election *C*** by changing the agenda to *B*, *D*, *A*, *C*. Notice that we needed to first pit *B* against *D*, so that *B* could win over *D* to then face *A*.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*D*
Second	*C*	*D*	*B*	*B*	*C*
Third	*B*	*A*	*D*	*A*	*B*
Fourth	*D*	*C*	*A*	*C*	*A*

This would be the preferred outcome by the shaded voters.

Impossibility

An important theorem in social choice is the **Gibbard-Satterthwaite Theorem** (“GS theorem” for short). It says that with three or more candidates and any number of voters, there does not exist (and never will exist) a voting system that always has all of the following features.

- a winner
- no ties
- satisfies the Pareto condition
- non-manipulable
- not a dictatorship.

A *weak* version of the Gibbard-Satterthwaite Theorem refers to any voting system for three candidates that agrees with Condorcet’s method whenever there is a Condorcet winner. This voting system must also produce a unique winner when confronted by the ballots in the Condorcet voting paradox. Given these conditions, this voting system is manipulable.

Teaching Tip

In discussing these theorems, you may choose to review *Arrow’s impossibility theorem* and a *weak version of Arrow’s impossibility theorem* from Chapter 9 to see how these new theorems compare in terms of the conditions and their meaning to those of Chapter 9.

Chair’s Paradox

Some terms that appear in this section are as follows.

- **Strategy**: A (single) choice of which candidate to vote for will be called a *strategy.*
- **Rational**: If a voter is *rational*, he or she will not vote for their least-preferred candidate.
- **Tie-breaking power**: If a candidate gets two or three votes, he or she wins. If each candidate gets one vote (three-way tie), then the chair has *tie-breaking power* and his or her candidate is the winner.
- **Weakly dominates**: The strategy of choosing a candidate, say *X*, *weakly dominates* another choice, say *Y*, if the choice of *X* yields outcomes that are either the same or better than the choice of *Y*.

In examining this paradox, preference list ballots are examined even though a vote for a single candidate will be cast. The interesting outcome involved in the chair’s paradox is that although the chair has tie-breaking power, he or she would be in a better strategic advantage by handing this power off to another voter (i.e. not being the chair in this election). The text (and the *Student Study Guide*) outlines what happens with three voters and three candidates.

Teaching Tip

In order to have students see what is meant by weakly dominates and the paradox involved, you may choose to run through the argument with 5 students and yourself. Make yourself the chair and two of the students on your election committee (voters). With the three other students being the candidates, create the preference list ballots and demonstrate the paradox. Have the three voters and the three candidates sit across from each other as the rest of the class observes the argument being demonstrated. The preference list ballots should be clearly known to the class.

Solutions to Student Study Guide ✐ Questions

Question 1

Consider Example 2 from the text. Is it possible to use the preference list ballots below to create an example of manipulating the Borda count with five candidates and six voters? Justify your yes/no response.

Election 1

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*C*	*B*
Third	*B*	*A*
Fourth	*D*	*D*

Election 2

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*D*	*B*
Third	*B*	*A*
Fourth	*C*	*D*

Solution

Yes.

One way to get an example of manipulation of the Borda count with five candidates and six voters is to alter the elections in Example 2 of the text by adding *E* to the bottom of each of the two ballots in both elections, and then adding the four rightmost columns as shown. The last four voters contribute exactly 8 to the Borda score of each candidate, and so, taken together have no effect on who the winner of the election is.

Election 1

	Number of voters (6)					
Rank	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*A*	*E*	*A*	*E*
Second	*C*	*B*	*B*	*D*	*B*	*D*
Third	*B*	*A*	*C*	*C*	*C*	*C*
Fourth	*D*	*D*	*D*	*B*	*D*	*B*
Fifth	*E*	*E*	*E*	*A*	*E*	*A*

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	3×4	0×3	1×2	0×1	2×0	14
B	0×4	3×3	1×2	2×1	0×0	13
C	1×4	1×3	4×2	0×1	0×0	15
D	0×4	2×3	0×2	4×1	0×0	10
E	2×4	0×3	0×2	0×1	4×0	8

Thus, *C* has the highest Borda score and is declared the winner. This was the expected result.

Continued on next page

Election 2

	Number of voters (6)					
Rank	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*A*	*E*	*A*	*E*
Second	*D*	*B*	*B*	*D*	*B*	*D*
Third	*B*	*A*	*C*	*C*	*C*	*C*
Fourth	*C*	*D*	*D*	*B*	*D*	*B*
Fifth	*E*	*E*	*E*	*A*	*E*	*A*

Preference	1^{st} place votes $\times$ 4	2^{nd} place votes $\times$ 3	3^{rd} place votes $\times$ 2	4^{th} place votes $\times$ 1	5^{th} place votes $\times$ 0	Borda score
A	3×4	0×3	1×2	0×1	2×0	14
B	0×4	3×3	1×2	2×1	0×0	13
C	1×4	0×3	4×2	1×1	0×0	13
D	0×4	3×3	0×2	3×1	0×0	12
E	2×4	0×3	0×2	0×1	4×0	8

Thus, *A* has the highest Borda score and is declared the winner.

One could also add four ballots canceling each other out first, and then add *E* to the bottom of all six ballots in each election. By doing this, the last four voters contribute exactly 8 to the Borda score of each of the top candidates, and so, taken together have no effect on who the winner of the election is. Because *E* holds the fifth place on all ballots, it has no effect on the candidates above them.

Election 1

	Number of voters (6)					
Rank	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*A*	*D*	*A*	*D*
Second	*C*	*B*	*B*	*C*	*B*	*C*
Third	*B*	*A*	*C*	*B*	*C*	*B*
Fourth	*D*	*D*	*D*	*A*	*D*	*A*
Fifth	*E*	*E*	*E*	*E*	*E*	*E*

Preference	1^{st} place votes $\times$ 4	2^{nd} place votes $\times$ 3	3^{rd} place votes $\times$ 2	4^{th} place votes $\times$ 1	5^{th} place votes $\times$ 0	Borda score
A	3×4	0×3	1×2	2×1	0×0	16
B	0×4	3×3	3×2	0×1	0×0	15
C	1×4	3×3	2×2	0×1	0×0	17
D	2×4	0×3	0×2	4×1	0×0	12
E	0×4	0×3	0×2	0×1	6×0	0

Thus, *C* has the highest Borda score and is declared the winner. This was the expected result.

Election 2

	Number of voters (6)					
Rank	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*A*	*D*	*A*	*D*
Second	*D*	*B*	*B*	*C*	*B*	*C*
Third	*B*	*A*	*C*	*B*	*C*	*B*
Fourth	*C*	*D*	*D*	*A*	*D*	*A*
Fifth	*E*	*E*	*E*	*E*	*E*	*E*

Continued on next page

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	3×4	0×3	1×2	2×1	0×0	16
B	0×4	3×3	3×2	0×1	0×0	15
C	1×4	2×3	2×2	1×1	0×0	15
D	2×4	1×3	0×2	3×1	0×0	14
E	0×4	0×3	0×2	0×1	6×0	0

Thus, *A* has the highest Borda score and is declared the winner.

Question 2

Consider the following election with four candidates and 3 voters.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*D*
Second	*B*	*C*	*A*
Third	*C*	*D*	*B*
Fourth	*D*	*C*	*C*

If sequential pairwise voting, with agenda is used, is it possible to make all candidates winners (i.e. four separate manipulations/agendas) by different agendas? Explain your yes/no answer.

Solution

Yes.

Looking at the 6 one-on-one contests, we can more readily see the solution.

A vs *B*	*A:*	2	*B:*	1
A vs *C*	*A:*	2	*C:*	1
A vs *D*	*A:*	1	*D:*	2
B vs *C*	*B:*	3	*C:*	0
B vs *D*	*B:*	2	*D:*	1
C vs *D*	*C:*	2	*D:*	1

- For *A* to win, we use agenda *B*, *D*, *A*, *C* (or *D*, *B*, *A*, *C*).

  We first pit *B* against *D*. Thus, *B* wins by a score of 2 to 1. *B* moves on to confront *A*. *A* wins by a score of 2 to 1. *A* moves on to confront *C*. *A* wins by a score of 2 to 1. Thus, *A* is the winner.

- For *B* to win, we use agenda *A*, *D*, *B*, *C* (or *D*, *A*, *B*, *C*).

  We first pit *A* against *D*. Thus, *D* wins by a score of 2 to 1. *D* moves on to confront *B*. *B* wins by a score of 2 to 1. *B* moves on to confront *C*. *B* wins by a score of 3 to 0. Thus, *B* is the winner.

- For *C* to win, we use agenda *A*, *B*, *D*, *C* (or *B*, *A*, *D*, *C*).

  We first pit *A* against *B*. Thus, *A* wins by a score of 2 to 1. *A* moves on to confront *D*. *D* wins by a score of 2 to 1. *D* moves on to confront *C*. *C* wins by a score of 2 to 1. Thus, *C* is the winner.

- For *D* to win, we use agenda *A*, *B*, *C*, *D* (or *B*, *A*, *C*, *D*).

  We first pit *A* against *B*. Thus, *A* wins by a score of 2 to 1. *A* moves on to confront *C*. *A* wins by a score of 2 to 1. *A* moves on to confront *D*. *D* wins by a score of 2 to 1. Thus, *D* is the winner.

Question 3

Consider the following election with four candidates and 11 voters.

	Number of voters (11)		
Rank	**2**	**5**	**4**
First	*B*	*C*	*D*
Second	*C*	*B*	*A*
Third	*A*	*D*	*C*
Fourth	*D*	*A*	*B*

If plurality voting is used, can the group of voters on the left secure a **more preferred** outcome? Explain your yes/no answer.

Solution

No. The winner by plurality voting is candidate *C*. The only candidate they prefer more than *C* is *B* and they already have that candidate as their first choice.