Chapter 9
Social Choice: The Impossible Dream

For All Practical Purposes: Effective Teaching

- Talk to your fellow TA's during the term and find out how their sessions are set up and how they are going. Talking about student problems and how they were handled can help both you and your colleagues in becoming confident educators.
- In this chapter and others involving voting, you will represent candidates by letters such as *A*, *B*, *C*, or *D*. A good habit to get into early on, in general, is to use capital letters in cases such as this or if multiple-choice responses are requested. You can also impart to students that if lower case letters are used (like "a" or "d") and their answers are not clear, then it is possible that their response will be marked as incorrect.

Chapter Briefing

Social choice theory was developed to analyze the various types of voting methods, to discover the potential pitfalls in each, and to attempt to find improved systems of voting.

One difficulty you may encounter imparting to students is the need for various voting methods and their mathematical nature. Discussions can start with examining two candidate races and the simplicity of invoking *majority rule*. When there are three or more candidates, however, different and more complicated voting techniques may be required.

Students may have difficulties grasping the differences between the voting methods and in turn the different criterion. Being well prepared by knowing these methods and criterion is essential for successful classroom discussions. In your academic preparation, you may not have encountered the topic of Social Choice. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Preference List Ballots**, **Voting Methods**, **Voting Criteria**, **Condorcet's voting paradox**, and **Arrow's Impossibility Theorem**. For each of the voting methods, an example with solution that does not appear in the text nor study guide is included in order to demonstrate each method. In order to simplify things, the same preference list ballots are used for all the methods except approval voting. You should feel free to use these examples in class, if needed. A description of each voting criterion, summary of criteria, and guided examples showing voting method violation of criteria are given. Although some violations reference a text example, you can feel free to use the *Teaching Guide*'s (counter) examples in class.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the twelve questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Preference List Ballots

Throughout the chapter, you will be interpreting and manipulating **preference list ballots**. The following is an example of preference list ballots.

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Teaching Tip

This is most likely the first time students are exposed to preference list ballots. Make sure to carefully go through the parts and what the numbers and letters mean (i.e. voters versus candidates). Also, at this time you may choose to have students practice eliminating a candidate or candidates and reorganizing preference list ballots. It may be surprising to you that some students will have difficulties with this, so doing some practice early on before examining the voting methods may be beneficial. For example, have students list four or five makes of cars that they prefer. Then have students put their preferences in order. On the board create preference list ballots making sure to combine like ballots. Break students into smaller groups and ask them to eliminate different makes of cars and reorganize the ballots.

Voting Methods

In the chapter it will be assumed that there are an odd number of voters. This minimizes the number of special cases that could occur with an even number of voters. Naturally, with an even number of voters, ties can occur.

In this chapter there are seven main voting systems introduced. They are the **Condorcet's method**, **plurality voting**, the **Borda count method**, **sequential pairwise** voting, the **Hare system**, the **plurality runoff method**, and the **approval voting method**. A description of each will be followed by a short example.

In **Condorcet's method**, a candidate is declared the winner if he or she can defeat every other candidate in a one-on-one competition using majority rule. That is, the winner of each one-on-one competition will have over 50% of the votes.

Example

Determine if there is a winner using Condorcet's method. If so, who is it?

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

You must determine the outcome of three one-on-one competitions. The candidates not considered in each one-on-one competition can be ignored.

A vs *B*

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*A*
Second	*B*	*A*	*B*

***A*: 19**; *B*: 8

A vs *C*

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*A*	*C*
Second	*C*	*C*	*A*

***A*: 17**; *C*: 10

B vs *C*

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*B*	*B*	*C*
Second	*C*	*C*	*B*

***B*: 17**; *C*: 10

Since *A* can defeat both *B* and *C* in a one-on-one competition, *A* is the winner by the Condorcet method.

In **plurality voting**, the candidate with the most first-place votes on the preference list ballots is the winner. We do not take into account the voters' preferences for the second, third, etc., places.

Example

Determine if there is a winner using plurality voting.

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

A has 9 first-place votes. *B* has 8 first-place votes. *C* has 10 first-place votes. Since *C* has the most first-place votes, *C* is the winner.

☝Teaching Tip

Point out to students that, in general, there is a difference between a plurality and a majority. Discuss under what conditions they would be the same.

☝Teaching Tip

You may choose to point out to students that in an example such as this, one can get different results depending on the method used. Notice with plurality voting, *C* was the winner. However, with Condorcet's method, *A* is the winner.

In the **Borda count** method, points are assigned to each position in the set of preference lists. For example, in a 3-person election, first-place votes may be awarded 2 points each, second-place votes receive 1 point each, and third-place votes are given 0 points each. (Other distributions of points may be used to create similar rank methods.)

☝Teaching Tip

After the Borda scores are determined, it is a good idea to point out to students that they can check their calculations by determining what the sum of the Borda scores should be. The product of the sum of the points to be distributed times the number of voters will be the sum of the Borda scores.

☝Teaching Tip

Another method of determining Borda scores outlined by the text is to individually replace the candidates below the one you are determining the score for by a box. You then count up the boxes, being careful to take note of the number of voters in each column. You may choose to point out to students that this method is easiest when there is just one voter per preference ballot. Also, this method will yield a correct Borda score only if the points are distributed according to the method of the text. That is, 0 for last-place increasing to $n-1$ points for first-place, where n is the number of candidates on a ballot.

Example

Who is the winner using Borda count?

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

Preference	1st place votes × 2	2nd place votes × 1	3rd place votes × 0	Borda score
A	9×2	18×1	0×0	**36**
B	8×2	9×1	10×0	25
C	10×2	0×1	17×0	20

The winner is *A*.

The sum is 36 + 25 + 20 = 81. This is the same as the product of 3 (2 + 1 + 0) and 27 (number of voters).

An agenda is the listing (in some order) of the candidates. **Sequential pairwise** voting pits the first candidate against the second in a one-on-one contest. The winner goes on to confront the third candidate on the agenda, while the loser is eliminated. The candidate remaining at the end is the winner. The choice of the agenda can affect the result.

Example

Who is the winner using sequential pairwise voting with the agenda *C*, *A*, *B*?

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

In sequential pairwise voting with the agenda *C*, *A*, *B*, we first pit *C* against *A*. There are 10 voters who prefer *C* to *A* and 17 prefer *A* to *C*. Thus, *A* wins by a score of 17 to 10. *C* is therefore eliminated, and *A* moves on to confront *B*.

There are 19 voters who prefer *A* to *B* and 8 prefer *B* to *A*. Thus, *A* wins by a score of 19 to 8.

Thus, *A* is the winner by sequential pairwise voting with the agenda *C*, *A*, *B*.

☝Teaching Tip

You may choose to ask students to contemplate if *A* is the winner by Condorcet's method, then would they in fact know the result of sequential pairwise voting for any agenda.

In the **Hare system**, the winner is determined by repeatedly deleting candidates that are the least preferred, in the sense of being at the top of the fewest preference lists.

Example

Who is the winner using the Hare system?

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

A has 9 first-place votes. *B* has 8 first-place votes. *C* has 10 first-place votes. Since *B* has the least number of first-place votes, *B* is eliminated. Candidates *A* and *C* move up as indicated to form a new table.

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*A*	*C*
Second	*C*	*C*	*A*

A now has 17 first-place votes. *C* now has 10 first-place votes. Thus, *A* is the winner by the Hare system.

☝Teaching Tip

You may choose to ask students to contemplate in a race with three candidates what occurs in the Hare system if there is a tie for second-place. Will this necessarily be the outcome if there are more than three candidates with a tie for second-place?

Plurality runoff is the voting system in which there is a runoff between the two candidates receiving the most first-place votes. In the case of ties between first or second, three candidates participate in the runoff. This system is basically a one-step process. Determine the candidates that are in the runoff, create new preference list ballots by deleting the candidates not in the runoff, and then determine which candidate has the plurality of the votes.

Example

Who is the winner using the plurality runoff?

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*B*	*C*
Second	*B*	*A*	*A*
Third	*C*	*C*	*B*

Solution

A has 9 first-place votes. *B* has 8 first-place votes. *C* has 10 first-place votes. Since *A* and *C* have the highest first- and second-place votes, *B* is eliminated. Candidates *A* and *C* move up as indicated to form a new table.

	Number of voters (27)		
Rank	**9**	**8**	**10**
First	*A*	*A*	*C*
Second	*C*	*C*	*A*

A now has 17 first-place votes. *C* now has 10 first-place votes. Thus, *A* is the winner by the plurality runoff.

☝Teaching Tip

If there are only two candidates in the runoff, then you will look for the candidate with the majority of the votes. You may choose to ask students to contemplate why this is called plurality runoff, instead of majority runoff.

☝Teaching Tip

In these examples of the Hare system and plurality runoff, the same steps were taken in implementing each method. Ask students if this is true in general or only specific cases, like having three candidates. Also, you may choose to ask students to contemplate what occurs if there is a tie for first- or second-place with each method.

In **approval voting**, each voter may vote for as many candidates as he or she chooses. The candidate with the highest number of approval votes wins the election. A candidate receives an X if he or she has obtained approval from one or more of the voters. There are two types of tables presented in this section. One is where the voter has an identification number (1, 2, 3, …), and the other indicates how many voters voted for the same combination of nominees.

Example

Suppose the following table is the result of approving four nominees. Treat the table in two ways.

	Number of voters ( )			
Nominee	**1**	**2**	**3**	**4**
A	X			X
B		X		X
C				X
D	X	X	X	

a) Assume there are 4 voters. Who is the winner and how would they be ranked?

b) Assume there are 10 voters (1 + 2 + 3 + 4). Who is the winner and how would they be ranked?

Solution

a)

	Number of voters (4)			
Nominee	**1**	**2**	**3**	**4**
A	X			X
B		X		X
C				X
D	X	X	X	

A has 2 approval votes. *B* has 2 approval votes. *C* has 1 approval vote. *D* has 3 approval votes. Since *D* has the most approval votes, *D* is the winner. Ranking the candidates we have, *D* (3), *A* and *B* (2), and *C* (1).

b)

	Number of voters (10)			
Nominee	**1**	**2**	**3**	**4**
A	X			X
B		X		X
C				X
D	X	X	X	

A has $1+4=5$ approval votes. *B* has $2+4=6$ approval votes. *C* has 4 approval votes. *D* has $1+2+3=6$ approval votes. Since *B* and *D* have the most approval votes, *B* and *D* tie as winners. Ranking the candidates we have, *B* and *D* (6), *A* (5), and *C* (4).

☝Teaching Tip

With all these different voting techniques introduced in this chapter, you might ask students which method might be used and where. For example, approval voting is often used in sports. Louisiana uses the *Cajun primary system* for its state and federal offices. In October's primary election, voters select from potential candidates. If a candidate earns a majority of the votes, then that candidate wins. If not, a runoff is held in November between the top two candidates of any party. You may choose to ask students to discuss the similarities and differences with the plurality runoff method as it is described in the text.

Voting Criteria

Definitions/rules/theorems that appear in the chapter are as follows.

- In a **dictatorship**, all ballots except that of the dictator are ignored.
- In **imposed rule**, candidate *X* wins regardless of who votes for whom.
- In **minority rule**, the candidate with the fewest votes wins.
- When there are only two candidates or alternatives, **May's theorem** states that majority rule is the only voting method that satisfies three desirable properties, given an odd number of voters and no ties.

In this chapter there are four criterion of voting systems introduced. They are the **Condorcet Winner Criterion (CWC)**, **independence of irrelevant alternatives (IIA)**, **monotonicity criterion**, and the **Pareto condition**. A description of each is followed by a table that summaries how the voting methods satisfy or don't satisfy the criterion.

A voting system that satisfies the **Condorcet Winner Criterion (CWC)** either has no Condorcet winner or the voting produces exactly the same winner as does Condorcet's method.

A voting system satisfies **independence of irrelevant alternatives (IIA)** if it is impossible for a candidate *B* to move from nonwinner status to winner status unless at least one voter reverses the order in which he or she had *B* and the winning candidate ranked.

A voting system satisfies **monotonicity criterion** if a candidate that wins the first election and gains support without losing any of his or her original support wins the second election as well.

A voting system satisfies the **Pareto condition** if everyone prefers one candidate, say *A*, to another, say *B*, then *B* cannot be the winner.

☝Teaching Tip

Encourage students to state these conditions/criteria in their own words. You may choose to pair students up and ask them to explain a condition/criteria to their partner.

☝Teaching Tip

With the different voting techniques and criteria, students may easily confuse which voting method satisfies or doesn't satisfy individual criterion. You may choose to display a blank version of the following table and discuss each possibility. You are guided to examples showing where a voting method violates a criterion.

	Condorcet Winner Criterion	Independence of Irrelevant Alternatives (IIA)	Pareto	Monotonicity
Plurality	✗(a)	✗(e)	✓	✓
Borda count	✗(b)	✗(f)	✓	✓
Sequential pairs	✓	✗(g)	✗(j)	✓
Hare system	✗(c)	✗(h)	✓	✗(k)
Plurality runoff	✗(d)	✗(i)	✓	✗(l)

(a) Plurality voting does not satisfy the Condorcet winner criterion. The example given earlier in this section of the manual in which *C* was the winner using plurality voting, but *A* was the Condorcet winner demonstrates this violation.

(b) The Borda count method does not satisfy the Condorcet winner criterion. The solution to Exercise 25 illustrates this fact.

(c) The Hare system does not satisfy the Condorcet winner criterion. This is illustrated as follows.

	Number of voters (17)				
Rank	**5**	**4**	**3**	**3**	**2**
First	*E*	*A*	*C*	*D*	*B*
Second	*B*	*B*	*B*	*B*	*C*
Third	*C*	*C*	*D*	*C*	*D*
Fourth	*D*	*D*	*A*	*A*	*A*
Fifth	*A*	*E*	*E*	*E*	*E*

Since *B* defeats *A* (13 to 4), *C* (14 to 3), *D* (14 to 3), and *E* (12 to 5), *B* is the Condorcet winner. Applying the Hare system, *B* is the first eliminated.

	Number of voters (17)				
Rank	**5**	**4**	**3**	**3**	**2**
First	*E*	*A*	*C*	*D*	*C*
Second	*C*	*C*	*D*	*C*	*D*
Third	*D*	*D*	*A*	*A*	*A*
Fourth	*A*	*E*	*E*	*E*	*E*

D is eliminated next.

	Number of voters (17)				
Rank	**5**	**4**	**3**	**3**	**2**
First	*E*	*A*	*C*	*C*	*C*
Second	*C*	*C*	*A*	*A*	*A*
Third	*A*	*E*	*E*	*E*	*E*

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(c) continued
A is eliminated next.

	Number of voters (17)				
Rank	**5**	**4**	**3**	**3**	**2**
First	*E*	*C*	*C*	*C*	*C*
Second	*C*	*E*	*E*	*E*	*E*

Finally, since *C* has the most first-place votes, *C* is declared the winner. However, *B* was the Condorcet winner.

(d) The Plurality runoff method does not satisfy the Condorcet winner criterion. Using the same table as part c, we know that *B* is the Condorcet winner. Since *E* and *A* have the most first-place votes, they will compete in the runoff. Adjusting the table to eliminate *B*, *C*, and *D*, we have the following.

	Number of voters (17)				
Rank	**5**	**4**	**3**	**3**	**2**
First	*E*	*A*	*A*	*A*	*A*
Second	*A*	*E*	*E*	*E*	*E*

Since *A* has the most first-place votes, *A* is declared the winner. However, *B* was the Condorcet winner.

(e) Plurality voting does not satisfy independence of irrelevant alternatives. This is illustrated as follows.

	Number of voters (13)		
Rank	**6**	**4**	**3**
First	*A*	*C*	*B*
Second	*B*	*A*	*C*
Third	*C*	*B*	*A*

Since *A* has the most first-place votes, *A* is declared the winner using plurality voting. Now consider a second preference list ballots.

	Number of voters (13)		
Rank	**6**	**4**	**3**
First	*A*	*C*	*C*
Second	*B*	*A*	*B*
Third	*C*	*B*	*A*

Now *C* is the winner using plurality voting even though no one changed his or her mind about whether *C* is preferred to *A*, or vice versa.

(f) The Borda count method does not satisfy independence of irrelevant alternatives. This is illustrated in Section 9.2 of the text.

(g) The sequential pairs method does not satisfy independence of irrelevant alternatives. This is illustrated as follows.

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	*C*	*C*	*B*	*D*	*C*
Third	*B*	*C*	*D*	*A*	*D*	*B*	*B*
Fourth	*C*	*D*	*A*	*D*	*A*	*A*	*A*

In the first election, if sequential pairs are used (assume agenda *A*, *B*, *C*, *D*), then *B* defeats *A* (5 to 4), then *B* defeats *C* (6 to 3), and then *D* defeats *B* (5 to 4). *D* wins the election.

Now consider the following table. The changes have been bolded.

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	*C*	*C*	*B*	*D*	***A***
Third	*B*	*C*	*D*	*A*	*D*	*B*	***C***
Fourth	*C*	*D*	*A*	*D*	*A*	*A*	***B***

In the second election, if sequential pairs are used (assume agenda *A*, *B*, *C*, *D*), then *A* defeats *B* (5 to 4), then *A* defeats *C* (5 to 4), and then *A* defeats *D* (5 to 4). *A* wins the election.

Thus, *A* has gone from non-winner status to winner status even though no voter reversed the order in which he or she had ranked *A* and the winning candidate from the previous election (i.e. *D*).

(h) The Hare system does not satisfy independence of irrelevant alternatives. This is illustrated as follows.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	*A*	*B*	*C*
Second	*B*	*B*	*C*	*B*
Third	*C*	*C*	*A*	*A*

In the first election, *B* has the least first-place-votes and is eliminated.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	*A*	*C*	*C*
Second	*C*	*C*	*A*	*A*

C wins with five first-place votes.

Now consider the following table. The changes have been bolded.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	***B***	*B*	*C*
Second	*B*	***A***	*C*	*B*
Third	*C*	*C*	*A*	*A*

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(h) continued

In this second election, *A* has the least first-place-votes and is eliminated.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*B*	*B*	*B*	*C*
Second	*C*	*C*	*C*	*B*

B wins with six first-place votes.

Thus, *B* has gone from non-winner status to winner status even though no voter reversed the order in which he or she had ranked *B* and the winning candidate from the previous election (i.e. *C*).

(i) Plurality voting does not satisfy independence of irrelevant alternatives. This is illustrated as follows.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	*A*	*B*	*C*
Second	*B*	*B*	*C*	*B*
Third	*C*	*C*	*A*	*A*

In the first election, if plurality runoff voting is used, then the runoff would be between *A* and *C*.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	*A*	*C*	*C*
Second	*C*	*C*	*A*	*A*

C wins with five first-place votes.

Now consider the following table. The changes have been bolded.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*A*	***B***	*B*	*C*
Second	*B*	***A***	*C*	*B*
Third	*C*	*C*	*A*	*A*

In this second election, if plurality runoff voting is used, then the runoff would be between *B* and *C*.

	Number of voters (9)			
Rank	**2**	**2**	**2**	**3**
First	*B*	*B*	*B*	*C*
Second	*C*	*C*	*C*	*B*

B wins with six first-place votes.

Thus, *B* has gone from non-winner status to winner status even though no voter reversed the order in which he or she had ranked *B* and the winning candidate from the previous election (i.e. *C*).

(j) The sequential pairs method does not satisfy the Pareto condition. This is illustrated in Section 9.2 of the text.

(k) The Hare system does not satisfy the monotonicity criterion. This is illustrated in Section 9.2 of the text.

(l) The plurality runoff method does not satisfy the monotonicity criterion. The solution to Exercise 23 illustrates this fact.

Condorcet's voting paradox

Condorcet's voting paradox can occur with three or more candidates in an election where Condorcet's method yields no winners. For example, in a three-candidate race, two-thirds of voters could favor *A* over *B*, two-thirds of voters could favor *B* over *C*, and two-thirds of voters could favor *C* over *A*.

Teaching Tip

You may choose to convey to students that the voting paradox discovered by Marquis de Condorcet is paradoxical because majority choices are in conflict with each other. Conflicting majorities are made up of different groups. The example that is given in the text is very effective.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Carefully go through the example to clearly demonstrate Condorcet's voting paradox and ask a student to describe it in his or her own words.

Teaching Tip

You may choose to note that Condorcet was a late 18th century French mathematician, philosopher, and an early political scientist that advocated for equal rights for women as well as all races. Giving some historical background to the name that appears throughout the chapter may increase student interest in the topics.

Example

Create preference list ballots with five candidates (*A*, *B*, *C*, *D*, and *E*) such that 4/5 prefer *A* over *B*, 4/5 prefer *B* over *C*, 4/5 prefer *C* over *D*, 4/5 prefer *D* over *E*, and 4/5 prefer *E* over *A*.

Solution

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*C*	*D*	*E*	*A*
Third	*C*	*D*	*E*	*A*	*B*
Fourth	*D*	*E*	*A*	*B*	*C*
Fifth	*E*	*A*	*B*	*C*	*D*

☝Teaching Tip

You may choose to ask students after viewing this past example to work collaboratively to create preference list ballots with 7, 9, or more candidates in which the Condorcet paradox occurs. Ask students to determine in advance what the desired outcome should be, (i.e. how many prefer which candidate to which).

➾Arrow's Impossibility Theorem

Arrow's impossibility theorem states that with three or more candidates and any number of voters, there does not exist (and will never exist) a voting system that produces a winner satisfying Pareto and independence of irrelevant alternatives (IIA), and is not a dictatorship.

☝Teaching Tip

You may choose to read Arrow's impossibility theorem to the class and review the defininition of a dictatorship, which appears in Section 9.1. Also, you may choose emphasize that this statement is a deep result, as the text indicates. Hence, we will be looking at a weaker version of this theorem.

A weak version of Arrow's impossibility theorem states that with three or more candidates and an odd number of voters, there does not exist (and will never exist) a voting system that satisfies both the Condorcet winner criterion (CWC) and independence of irrelevant alternatives (IIA), and that always produces at least one winner every election.

☝Teaching Tip

You may choose to have students carefully go through each part of this theorem. Have students work in pairs to explain each part of the theorem in their own language. Trying to understand it in its entirety may be difficult for students.

☝Teaching Tip

As the chapter comes to a close, remind students of all the resources they have available to them in preparation for an examination. There are Skills Check exercises (with answers) in the text, Practice Quiz (with answers) in the *Student Study Guide*, flashcards of Review Vocabulary in the *Student Study Guide* as well as web versions for students that have Internet access. Students should be comfortable with the different voting techniques and criteria. If review sessions or other materials are made available, write this information on the board and refer to it several times before the examination date.

Solutions to Student Study Guide ✐ Questions

Question 1

What are three properties satisfied by majority rule?

Solution

The properties are stated in Section 9.1 of the text.

The three properties are:

1. All voters are treated equally.
2. Both candidates are treated equally.
3. If a single voter who voted for the loser, *B*, changes his mind and votes for the winner, *A*, then *A* is still the winner. This is what is called **monotone.**

Question 2

Determine if there is a winner using Condorcet's method. If so, who is it?

	Number of voters (15)			
Rank	**5**	**2**	**7**	**1**
First	*A*	*C*	*B*	*B*
Second	*B*	*B*	*C*	*A*
Third	*C*	*A*	*A*	*C*

Solution

You must determine the outcome of three one-on-one competitions. The candidates not considered in each one-on-one competition can be ignored.

A vs *B*

	Number of voters (15)			
Rank	**5**	**2**	**7**	**1**
First	*A*	*B*	*B*	*B*
Second	*B*	*A*	*A*	*A*

A: 5; ***B*: 2 + 7 + 1 = 10**

A vs *C*

	Number of voters (15)			
Rank	**5**	**2**	**7**	**1**
First	*A*	*C*	*C*	*A*
Second	*C*	*A*	*A*	*C*

A: 5 + 1 = 6; ***C*: 2 + 7 = 9**

B vs *C*

	Number of voters (15)			
Rank	**5**	**2**	**7**	**1**
First	*B*	*C*	*B*	*B*
Second	*C*	*B*	*C*	*C*

***B*: 5 + 7 + 1 = 13**; *C*: 2

Since *B* can defeat both *A* and *C* in a one-on-one competition, *B* is the winner by the Condorcet method.

Question 3

In the following table, is there a Condorcet winner? If so, who is it?

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*A*	*B*	*C*	*D*
Second	*B*	*C*	*B*	*C*
Third	*C*	*A*	*A*	*B*
Fourth	*D*	*D*	*D*	*A*

Solution

You must determine the outcome of six one-on-one competitions. The candidates not considered in each one-on-one competition can be ignored.

A vs *B*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*A*	*B*	*B*	*B*
Second	*B*	*A*	*A*	*A*

A: 3; ***B*: 8 + 7 + 5 = 20**

A vs *C*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*A*	*C*	*C*	*C*
Second	*C*	*A*	*A*	*A*

A: 3; ***C*: 8 + 7 + 5 = 20**

A vs *D*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*A*	*A*	*A*	*D*
Second	*D*	*D*	*D*	*A*

***A*: 3 + 8 + 7 = 18**; *D*: 5

B vs *C*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*B*	*B*	*C*	*C*
Second	*C*	*C*	*B*	*B*

B: 3 + 8 = 11; ***C*: 7 + 5 = 12**

B vs *D*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*B*	*B*	*B*	*D*
Second	*D*	*D*	*D*	*B*

***B*: 3 + 8 + 7 = 18**; *D*: 5

C vs *D*

	Number of voters (23)			
Rank	**3**	**8**	**7**	**5**
First	*C*	*C*	*C*	*D*
Second	*D*	*D*	*D*	*C*

***C*: 3 + 8 + 7 = 18**; *D*: 5

C can defeat *A*, *B*, and *D*. Thus, *C* is the winner.

Question 4

Does Condorcet's voting paradox occur in the following table?

	Number of voters (23)			
Rank	**9**	**4**	**2**	**8**
First	*A*	*B*	*B*	*C*
Second	*B*	*C*	*A*	*A*
Third	*C*	*A*	*C*	*B*

Solution

The answer is yes. Voters prefer *A* over *B* (17 to 6). Voters prefer *B* over *C* (15 to 8). However, voters prefer *C* over *A* (12 to 11).

Question 5

In the following table, who is the winner by plurality voting? Is there a winner by Condorcet's method? Is there a violation of CWC?

	Number of voters (27)				
Rank	**11**	**2**	**8**	**2**	**4**
First	*A*	*A*	*B*	*C*	*C*
Second	*B*	*C*	*C*	*A*	*B*
Third	*C*	*B*	*A*	*B*	*A*

Solution

A has 13 first-place votes. *B* has 8 first-place votes. *C* has 6 first-place votes. Since *A* has the highest number of first-place votes, *A* is the winner by plurality voting.

To determine the if there is a winner by Condorcet's method, you must determine the outcome of three one-on-one competitions. The candidates not considered in each one-on-one competition can be ignored.

A vs *B*

	Number of voters (27)				
Rank	**11**	**2**	**8**	**2**	**4**
First	*A*	*A*	*B*	*A*	*B*
Second	*B*	*B*	*A*	*B*	*A*

***A*: 11 + 2 + 2 = 15**; *B*: 8 + 4 = 12

A vs *C*

	Number of voters (27)				
Rank	**11**	**2**	**8**	**2**	**4**
First	*A*	*A*	*C*	*C*	*C*
Second	*C*	*C*	*A*	*A*	*A*

A: 11 + 2 = 13; ***C*: 8 + 2 + 4 = 14**

B vs *C*

	Number of voters (27)				
Rank	**11**	**2**	**8**	**2**	**4**
First	*B*	*C*	*B*	*C*	*C*
Second	*C*	*B*	*C*	*B*	*B*

***B*: 11 + 8 = 19**; *C*: 2 + 2 + 4 = 8

Since no one candidate can beat all other candidates in one-on-one competitions, there is no winner. There is no violation since having no Condorcet winner is a possibility in the Condorcet Winner Criterion (CWC).

Question 6

In plurality voting, must there always be a Condorcet winner?

Solution

The answer is no. The solution to Question 5 demonstrates this.

Question 7

In the following table, who is the winner by Borda count? What is the sum of all the Borda scores?

	Number of voters (37)					
Rank	**8**	**8**	**2**	**12**	**5**	**2**
First	*A*	*A*	*B*	*C*	*C*	*D*
Second	*B*	*D*	*C*	*B*	*D*	*A*
Third	*C*	*B*	*D*	*A*	*A*	*B*
Fourth	*D*	*C*	*A*	*D*	*B*	*C*

Solution

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	16×3	2×2	17×1	2×0	**69**
B	2×3	20×2	10×1	5×0	56
C	17×3	2×2	8×1	10×0	63
D	2×3	13×2	2×1	20×0	34

The winner is *A*.

The sum is 69 + 56 + 63 + 34 = 222. This is the same as the product of 6 (3 + 2 + 1 + 0) and 37 (number of voters).

Question 8

Who is the winner with sequential pairwise voting with the agenda *A*, *B*, *C*, *D*? with agenda *B*, *D*, *C*, *A*?

	Number of voters (19)				
Rank	**5**	**3**	**2**	**1**	**8**
First	*A*	*A*	*B*	*C*	*D*
Second	*B*	*D*	*C*	*B*	*B*
Third	*D*	*B*	*A*	*A*	*A*
Fourth	*C*	*C*	*D*	*D*	*C*

Solution

In sequential pairwise voting with the agenda ***A***, ***B***, ***C***, ***D***, we first pit *A* against *B*. There are 8 voters who prefer *A* to *B* and 11 prefer *B* to *A*. Thus, *B* wins by a score of 11 to 8. *A* is therefore eliminated, and *B* moves on to confront *C*.

There are 18 voters who prefer *B* to *C* and 1 prefers *C* to *B*. Thus, *B* wins by a score of 18 to 1. *C* is therefore eliminated, and *B* moves on to confront *D*.

There are 8 voters who prefer *B* to *D* and 11 prefer *D* to *B*. Thus, *D* wins by a score of 11 to 8.

Thus, *D* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*.

In sequential pairwise voting with the agenda ***B***, ***D***, ***C***, ***A***, we first pit *B* against *D*. There are 8 voters who prefer *B* to *D* and 11 prefer *D* to *B*. Thus, *D* wins by a score of 11 to 8. *B* is therefore eliminated, and *D* moves on to confront *C*.

There are 16 voters who prefer *D* to *C* and 3 prefer *C* to *D*. Thus, *D* wins by a score of 16 to 3. *C* is therefore eliminated, and *D* moves on to confront *A*.

There are 8 voters who prefer *D* to *A* and 11 prefer *A* to *D*. Thus, *A* wins by a score of 11 to 8.

Thus, *A* is the winner by sequential pairwise voting with the agenda *B*, *D*, *C*, *A*.

Question 9

Who is the winner when the Hare system is applied?

	Number of voters (17)				
Rank	**5**	**2**	**6**	**3**	**1**
First	*A*	*B*	*C*	*D*	*B*
Second	*B*	*A*	*A*	*C*	*D*
Third	*C*	*C*	*B*	*B*	*C*
Fourth	*D*	*D*	*D*	*A*	*A*

Solution

A has 5 first-place votes. *B* has 3 first-place votes. *C* has 6 first-place votes. *D* has 3 first-place votes. Since *B* and *D* both have the least number of first-place votes, *B* and *D* are both eliminated.

Candidates *A* and *C* move up as indicated to form a new table.

	Number of voters (17)				
Rank	**5**	**2**	**6**	**3**	**1**
First	*A*	*A*	*C*	*C*	*C*
Second	*C*	*C*	*A*	*A*	*A*

A now has 7 first-place votes. *C* now has 10 first-place votes. Thus, *C* is the winner by the Hare system.

Question 10

Who is the winner when the Hare system is applied?

	Number of voters (21)				
Rank	**1**	**8**	**7**	**3**	**2**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*E*	*A*	*A*	*A*
Third	*C*	*C*	*B*	*C*	*C*
Fourth	*D*	*D*	*D*	*B*	*B*
Fifth	*E*	*A*	*E*	*E*	*D*

Solution

A has 1 first-place vote. *B* has 8 first-place votes. *C* has 7 first-place votes. *D* has 3 first-place votes. *E* has 2 first-place votes. Since *A* has the least number of first-place votes, *A* is eliminated.

Candidates *B*, *C*, *D*, and *E* move up as indicated to form a new table.

	Number of voters (21)				
Rank	**1**	**8**	**7**	**3**	**2**
First	*B*	*B*	*C*	*D*	*E*
Second	*C*	*E*	*B*	*C*	*C*
Third	*D*	*C*	*D*	*B*	*B*
Fourth	*E*	*D*	*E*	*E*	*D*

B has 9 first-place votes. *C* has 7 first-place votes. *D* has 3 first-place votes. *E* has 2 first-place votes. Since *E* has the least number of first-place votes, *E* is eliminated.

Candidates *B*, *C*, and *D* move up as indicated to form a new table.

	Number of voters (21)				
Rank	**1**	**8**	**7**	**3**	**2**
First	*B*	*B*	*C*	*D*	*C*
Second	*C*	*C*	*B*	*C*	*B*
Third	*D*	*D*	*D*	*B*	*D*

B has 9 first-place votes. *C* has 9 first-place votes. *D* has 3 first-place votes. Since *D* has the least number of first-place votes, *D* is eliminated.

Candidates *B* and *C* move up as indicated to form a new table.

	Number of voters (21)				
Rank	**1**	**8**	**7**	**3**	**2**
First	*B*	*B*	*C*	*C*	*C*
Second	*C*	*C*	*B*	*B*	*B*

B now has 9 first-place votes. *C* now has 12 first-place votes. Thus, *C* is the winner by the Hare system.

Question 11

Who is the winner when the plurality runoff method is applied?

	Number of voters (29)				
Rank	**9**	**6**	**7**	**2**	**5**
First	*A*	*B*	*C*	*D*	*D*
Second	*C*	*D*	*B*	*C*	*B*
Third	*B*	*A*	*A*	*A*	*C*
Fourth	*D*	*C*	*D*	*B*	*A*

Solution

A has 9 first-place votes. *B* has 6 first-place votes. *C* has 7 first-place votes. *D* has 7 first-place votes. So, *A* has the most first-place votes and both *C* and *D* have the second-highest total. Thus, the runoff is between candidates *A*, *C*, and *D*.

	Number of voters (29)				
Rank	**9**	**6**	**7**	**2**	**5**
First	*A*	*D*	*C*	*D*	*D*
Second	*C*	*A*	*A*	*C*	*C*
Third	*D*	*C*	*D*	*A*	*A*

A has 9 first-place votes. *C* has 7 first-place votes. *D* has 13 first-place votes. Thus, *D* is the winner using the plurality runoff method.

Question 12

There are 71 voters in a committee. Who is the winner in the following table, where X indicates that the voter approves of that particular candidate? How would they be ranked?

	Number of voters (71)							
Nominee	**8**	**9**	**2**	**12**	**10**	**14**	**12**	**4**
A	X			X	X			X
B		X		X		X		X
C						X		X
D		X		X		X		X
E	X	X	X		X	X		X

Solution

A has $8+12+10+4=34$ approval votes. *B* has $9+12+14+4=39$ approval votes.
C has $14+4=18$ approval votes. *D* has $9+12+14+4=39$ approval votes. *E* has $8+9+2+10+14+4=47$ approval votes. Since *E* has the most approval votes, *E* is the winner.
Ranking the candidates, we have *E* (47), *B* and *D* (39), *A* (34), and *C* (18).