Chapter 8
Probability: The Mathematics of Chance

For All Practical Purposes: Effective Teaching

- It is frustrating when you are going over a problem and an error throws off the final answer. What is even more frustrating is when students spot that error early on and don't say anything. To help facilitate student interaction in such a case, let them know early on in the course that you appreciate it when they (the students) point out errors as they occur.
- Often times when a student does point out an error, they may not do it in the best possible way. Most likely they are not meaning to be confrontational. If they say for example, "That 16 is *supposed to be* a 17." You can pause for a second and look at the board and ask the student for further clarification. By asking them, "Where is the mistake?" you are acknowledging the possibility that there is a mistake and allowing yourself to regroup your thoughts as they answer. If you did make a mistake, accept it and thank the student. If you didn't make a mistake, still encourage the student for following along. You can emphasize that if you do make a mistake at the board it is the students' responsibility to say something.

Chapter Briefing

A *probability model* for a random phenomenon consists of a *sample space* and a way of assigning probabilities to events. In finite sample spaces, counting methods such as *combinatorics* can used to determine how many elements are to be considered. We then would calculate a ratio in order to determine the probability. Also, we can assign probabilities to intervals of numbers as areas under a curve.

The probability of an event must be a number between 0 and 1, and the probabilities of all outcomes must add up to 1. Moreover, if two events *A* and *B* are disjoint, then the probability of the union of events is the sum of the individual probabilities.

For a discrete *probability distribution,* a *probability histogram* gives a visual representation of a probability model. *Continuous probability models* such as the *uniform distribution* or the *normal distribution* assign probabilities as area under *probability density curves*. You will be revisiting the 68–95–99.7 rule as it applies to normal distributions in this chapter.

For a random phenomenon with numerical outcomes, the long run outcome is called its mean, denoted μ. It is a weighted average and the law of large numbers tells us that the mean, $\overline{x}$, of observed outcomes will approach μ as the number of observations increases.

Sampling distributions are important in statistical inference. Random sampling ensures that each sample is equally likely to be chosen. The term *sampling distribution* is applied to the distribution of any statistic. The central limit theorem tells us that the sampling distribution of a statistic is approximately normal if the sample size is large enough.

In order to facilitate your classroom preparation, the **Chapter Topics to the Point** has been broken down into **Probability Models**, **Discrete Probability Models**, **Mean and Standard Deviation of Discrete Probability Models**, **Continuous Probability Models**, **Normal Distributions**, **The Central Limit Theorem**, and **Law of Large Numbers**. Any examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the nine questions included in the Student Study Guide. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Probability Models

A repeatable phenomenon is **random** if any particular outcome is quite unpredictable, while in the long run, after a large number of repeated trials, a regular, predictable pattern emerges (think of tossing a coin or throwing a die). A **probability model** is a mathematical description of a random phenomenon consisting of two parts as follows.

- a **sample space**, S
  The sample space, S, of a random phenomenon is the set of all possible outcomes. An *event* is any outcome (element of sample space) or any set of outcomes (subset of sample space) of a random phenomenon.
- a way of assigning **probabilities** to events
  The probability of any outcome of a random phenomenon is the proportion of times the outcome would occur in a very long series of repetitions. If A and B are events in sample space S and $P(A)$ is the probability of that event, then the following hold true.
  - ✓ $0 \le P(A) \le 1$: Any probability is a number between 0 and 1, inclusively.
  - ✓ $P(S) = 1$: All possible outcomes together must have probability 1.
  - ✓ $P(A \text{ or } B) = P(A) + P(B)$: If two events ($A$ and B) have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities. This is the *addition rule for disjoint events.*
  - ✓ $P(A^c) = 1 - P(A)$: The probability that an event does not occur is 1 minus the probability that the event does occur. A^c is the complement of event A, which is in sample space S.

Teaching Tip

When constructing a sample space, diagrams or a systematic approach (such as a table) in determining all the elements of the sample space are helpful. One of the most common mistakes is forgetting an element of the sample space. Since the cardinality of the sample space is essential, forgetting an element will throw off all of student calculations.

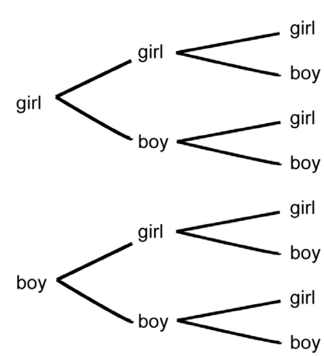
Example

A couple is planning to have three children. What is the sample space?
Solution
The sample space is S = {ggg, ggb, gbg, gbb, bgg, bgb, bbg, bbb}.

Columnar list approach:

Child #1	Child #2	Child #3
girl	girl	girl
girl	girl	boy
girl	boy	girl
girl	boy	boy
boy	girl	girl
boy	girl	boy
boy	boy	girl
boy	boy	boy

Tree diagram approach:

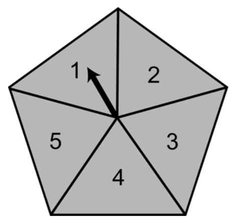


☝Teaching Tip

It's then interesting to note that other experiments that have only two outcomes behave the same way structurally. These binomial probability experiments include tossing coins, answering questions on a true/false test, or recording the sex of each child.

Example

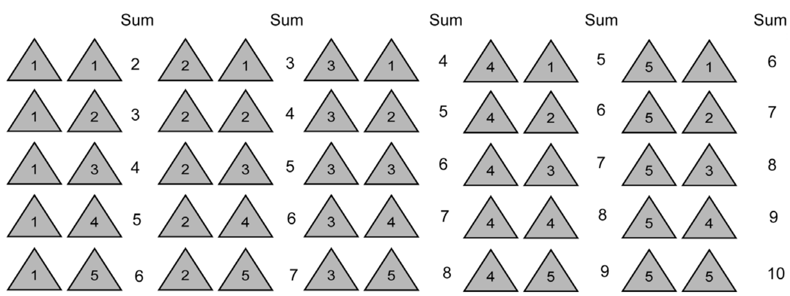
Consider the following spinner.



Spin the arrow twice and record the pair of numbers.

a) What is the sample space?
b) If you are to sum the two numbers, what is the sample space? What is the probability model?
c) What is the probability that the sum of the two spins is 5 or 6?
d) What is the probability that the sum is not 6.
e) What is the probability that the sum is at most 9?
f) What is the probability that the sum is at least 10?

Solution



a) The sample space is as follows.

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),$$
$$(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$$

b) The sample space is $S=\{2,3,4,5,6,7,8,9,10\}$

The probability model is as follows.

Outcome	2	3	4	5	6	7	8	9	10
Probability	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{5}{25}=\frac{1}{5}$	$\frac{4}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$\frac{1}{25}$

c) Since the outcomes of a sum of 5 or 6 are separate outcomes, we have the following.

$$P(5 \text{ or } 6)=P(5)+P(6)=\frac{4}{25}+\frac{5}{25}=\frac{9}{25}$$

d) The easiest approach is to use the complement to the event "the sum is not 6." This complementary event would be "the sum is 6." Since $P(A^c)=1-P(A)$, we have $P(A)=1-P(A^c)$. Thus, $P(\text{the sum is not } 6)=1-P(\text{the sum is } 6)=1-\frac{1}{5}=\frac{4}{5}$.

Continued on next page

e) The easiest approach is to use the complement to the event "the sum is at most 9." This complementary event would be "the sum is more than 9" or in this case "the sum is 10." Thus, $P(\text{the sum is at most } 9) = 1 - P(\text{the sum is } 10) = 1 - \frac{1}{25} = \frac{24}{25}$.

f) This is the same as asking "What is the probability sum is 10" since there are no higher sums possible. Thus, $P(\text{sum is at least } 10) = \frac{1}{25}$.

☝Teaching Tip

With *at least* or *at most* and other phrasings, one needs to be careful when stating the complement. If one thinks of symbols and negations, we have the following.

statement	phrase	negation	complementary phrase
$x = a$	is	$x \neq a$	is not
$x < a$	less than	$x \geq a$	at least (no less than)
$x > a$	more than	$x \leq a$	at most (no more than)

Discrete Probability Models

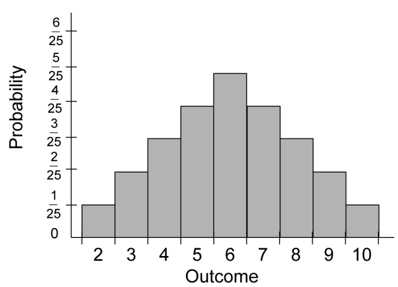
The previous examples have had a finite sample space and are considered **discrete**. In a discrete probability model, you are able to individually list out all events in the sample space and assign probabilities to each event. Notice that in a discrete probability model, an element of the sample space does not need to be a numerical value.

In discrete probability models, **probability histograms** graphically show the likelihood of each outcome. The height of each bar shows the probability of the outcome at its base, and the sum of the heights is 1.

Example

Construct the probability histogram for the sum of spins probability model in the last example.

Solution



☝Teaching Tip

When possible, point out symmetry in the probability histograms. This will help students create the graphs quicker and be able to interpret them easier in exercises.

If a random phenomenon has k possible outcomes, all **equally likely**, then each individual outcome has probability $\frac{1}{k}$. The probability of any event A is $P(A) = \frac{\text{count of outcomes in } A}{\text{count of outcomes in } S} = \frac{\text{count of outcomes in } A}{k}$.

With equally likely outcomes, probability calculations come from **combinatorics**, or the study of counting methods.

- Rule A: Arranging k objects chosen from a set of n possibilities, with **repetitions allowed**, can be done in $n \times n \times \ldots \times n = n^k$ distinct ways.
- Rule B: Arranging k objects chosen from a set of n possibilities, with **no repetitions allowed**, can be done in $n \times (n-1) \times \ldots \times (n-k+1)$. (notice there are k factors here).

☝Teaching Tip

In discussing the last two rules (Rule A and Rule B), students might see the solution easier by using blanks to fill in the number of possible choices as opposed to the formulas. This is shown in the next example.

Example

Suppose a code consists of a 4-digit number. What is the probability that a code chosen at random will have the first two numbers differing by 2 if

a) repetition of digits is allowed?
b) repetition of digits is not allowed?

Solution

a) Since this is a 4-digit number, the first digit cannot be zero. Thus, the possible number of codes is $\underline{9}\times\underline{10}\times\underline{10}\times\underline{10}=9000$. To determine the number of codes in which the first two digits differ by two, we first examine the possible choices. The first two digits could be 13, 24, 35, 46, 57, 68, 79, 86, 97, 75, 64, 53, 42, 31, 20. For each of these, there are $\underline{10}\times\underline{10}=100$ possible last two digits. Thus, there is a total of $15\times100=1500$ possible codes whose first two digits differ by 2. Therefore the probability of choosing such a code would be $\frac{1500}{9000}=\frac{1}{6}$.

b) Since this is a 4-digit number, the first digit cannot be zero. Since repetition of digits is not allowed, the possible number of codes is $\underline{9}\times\underline{9}\times\underline{8}\times\underline{7}=4536$. Like is part a, there are 15 different possible first two digits that satisfy the criteria. For each of these, there are $\underline{8}\times\underline{7}=56$ possible last two digits. Thus, there is a total of $15\times56=840$ possible codes whose first two digits differ by 2. Therefore the probability of choosing such a code would be $\frac{840}{4536}=\frac{5}{27}$.

Mean and Standard Deviation of Discrete Probability Models

The **mean** of a discrete probability model is the sum of the possible outcomes times the probability of each outcome. If there are *k* possible outcomes in the sample space, then there will be *k* terms in the sum. Each term will have a probability associated with it. The sum of all the probabilities will be 1.

The **variance** of a discrete probability model that has numerical outcomes $x_1, x_2, ..., x_k$ in a sample space will have variance as follows.

$$\sigma^2=(x_1-\mu)^2\,p_1+(x_2-\mu)^2\,p_2+...+(x_k-\mu)^2\,p_k$$

Where p_j is the probability of outcome x_j. The **standard deviation** σ is the square root of the variance.

Example

What is the mean and standard deviation for the sum of spins probability model?

Solution

Mean:

$$\tfrac{1}{25}(2)+\tfrac{2}{25}(3)+\tfrac{3}{25}(4)+\tfrac{4}{25}(5)+\tfrac{5}{25}(6)+\tfrac{4}{25}(7)+\tfrac{3}{25}(8)+\tfrac{2}{25}(9)+\tfrac{1}{25}(10)=$$
$$\tfrac{2}{25}+\tfrac{6}{25}+\tfrac{12}{25}+\tfrac{20}{25}+\tfrac{30}{25}+\tfrac{28}{25}+\tfrac{24}{25}+\tfrac{18}{25}+\tfrac{10}{25}=\tfrac{150}{25}=6$$

Due to the symmetry of the probability histogram, this mean should be intuitively obvious.

Continued on next page

Variance:

$$\begin{aligned}\sigma^2 &= (2-6)^2\left(\tfrac{1}{25}\right)+(3-6)^2\left(\tfrac{2}{25}\right)+(4-6)^2\left(\tfrac{3}{25}\right)+(5-6)^2\left(\tfrac{4}{25}\right)+(6-6)^2\left(\tfrac{5}{25}\right)\\ &\quad+(7-6)^2\left(\tfrac{4}{25}\right)+(8-6)^2\left(\tfrac{3}{25}\right)+(9-6)^2\left(\tfrac{2}{25}\right)+(10-6)^2\left(\tfrac{1}{25}\right)\\ &= (-4)^2\left(\tfrac{1}{25}\right)+(-3)^2\left(\tfrac{2}{25}\right)+(-2)^2\left(\tfrac{3}{25}\right)+(-1)^2\left(\tfrac{4}{25}\right)+0^2\left(\tfrac{5}{25}\right)+1^2\left(\tfrac{4}{25}\right)+2^2\left(\tfrac{3}{25}\right)+3^2\left(\tfrac{2}{25}\right)+4^2\left(\tfrac{1}{25}\right)\\ &= 16\left(\tfrac{1}{25}\right)+9\left(\tfrac{2}{25}\right)+4\left(\tfrac{3}{25}\right)+1\left(\tfrac{4}{25}\right)+0\left(\tfrac{5}{25}\right)+1\left(\tfrac{4}{25}\right)+4\left(\tfrac{3}{25}\right)+9\left(\tfrac{2}{25}\right)+16\left(\tfrac{1}{25}\right)\\ &= \tfrac{16}{25}+\tfrac{18}{25}+\tfrac{12}{25}+\tfrac{4}{25}+\tfrac{0}{25}+\tfrac{4}{25}+\tfrac{12}{25}+\tfrac{18}{25}+\tfrac{16}{25}=\tfrac{100}{25}=4\end{aligned}$$

Thus, the standard deviation is $\sigma = \sqrt{4} = 2$.

Continuous Probability Models

In a **continuous probability model**, there are infinitely many possible events that could occur. In order to assign probabilities to events, we look at area under a *density curve*. The total area under the curve bounded by a horizontal axis must equal 1. The probability of a single value occurring is 0. In the case of continuous probability models, you will be looking at the probability of a range of values (an interval) occurring.

Teaching Tip

For the types of density curves that will be investigated in this chapter, students should be familiar with the geometric formulas for finding the area of a rectangle and the area of a triangle. You may choose to ask students to determine the heights of rectangle and a triangle given a base and the total area of 1 $\left(h = \frac{1}{b} \text{ and } h = \frac{2}{b}, \text{ respectively}\right)$.

The **uniform probability model** has a density curve that creates a rectangle along a horizontal axis. The area of the rectangle will always be 1 for the continuous uniform model. Please note that figures are not drawn to scale in the examples.

Example

Suppose you specify that the range random number generators is to be all numbers between 0 and 8. The density curve for the outcome has constant height between 0 and 8, and height 0 elsewhere.

a) Draw a graph of the density curve.

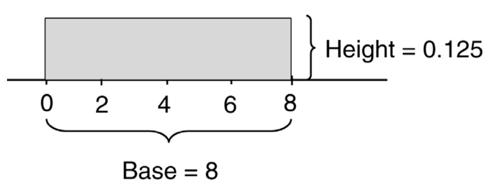
b) Suppose the generator produces a number X. Find $P(X \geq 3)$.

c) Find $P(2.5 \leq X \leq 7.5)$.

d) Find $P(X = 7.5)$.

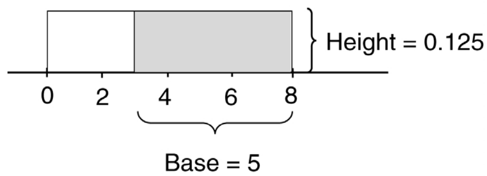
Solution

a) Since the width of the base is 8, the height would be $\frac{1}{8} = 0.125$.

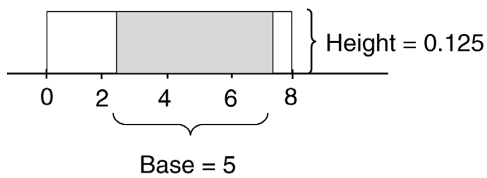


Continued on next page

b) We need to determine the length of the base. It will be $8-3=5.$ Since the area of a rectangle base$\times$height, the probability will be $(5)(0.125)=0.625.$



c) We need to determine the length of the base. It will be $7.5-2.5=5.$ Thus, the probability will be Area $=$ base$\times$height $=(5)(0.125)=0.625.$



d) The probability of a single value occurring is 0. This can be visualized by a region that has a height of 0.125 and a base of 0.

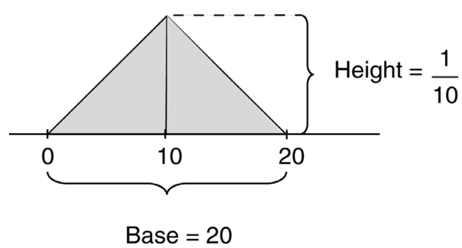
Example

Generate two random numbers between 0 and 10 and take their sum. The sum can take any value between 0 and 20. The density curve is the triangle.

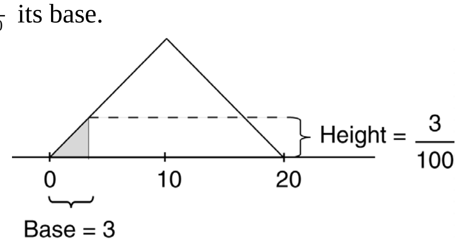
a) What is the probability that the sum is less than 3?
b) What is the probability that the sum is less than 17?

Solution

Since the base is 20, the height is $h=\frac{2}{b}=\frac{2}{20}=\frac{1}{10}$.



a) By looking at the midpoint of the base, 10, we can see in general that the height of the triangle (from 0 to 10) will be $\frac{1}{100}$ its base.



Thus, the height is $\frac{1}{100}(3)=\frac{3}{100}$. Thus, the desired probability is indicated by the area of the triangle, or $\frac{1}{2}\times 3\times\frac{3}{100}=\frac{9}{200}$

b) We can use the fact that the area of the triangle is 1 and the probability that the sum is more than 17 is the same as the probability that the sum is less than 3. Thus, the desired probability is $1-\frac{9}{200}=\frac{191}{200}$.

☝Teaching Tip

You may note to the students that since the probability of a single value occurring is 0, there is some flexibility in how the complement of an event is stated. For example, the complement to the event "the sum is less than 17" in a discrete model should be stated as "the sum is at least 17" or "the sum is no less than 17", implying 17 must be included. In the continuous model, one could say the complementary event is "the sum is more than 17."

Normal Distributions

Normal distributions are continuous probability models. The 68–95–99.7 rule applies to a normal distribution, and we can use it for determining probabilities. It is useful in determining the proportion of a population with values falling in certain ranges. The **sample proportion,** $\hat{p}$, will vary from sample to sample according to a normal distribution with mean, p, and standard deviation $\sqrt{\frac{p(1-p)}{n}}$, where n is the number in the sample. (p is the population proportion.)

☝Teaching Tip

When determining ranges by applying the 68–95–99.7 rule, the end-values of the intervals may differ slightly. Although there is a possibility of accumulated error that makes these end-values approximations anyway, discuss with student what happens with rounding the lower bound down and the upper bound is rounded up.

Example

Suppose 24% of all drivers over the age of 55 prefer not to drive at night. You choose 10,134 drivers over the age of 55 to sample at random. What are the mean and standard deviation of the proportion of adults that prefer not to drive at night. The probability is 0.997 that $\hat{p}$ lies between what two values?

Solution

The population proportion of drivers over 55 that prefer not to drive at night is $p = 0.24$. The sample proportion, $\hat{p}$, of drivers over 55 that prefer not to drive at night in a random sample of n = 10,134 has mean $p = 0.24$ and standard deviation as follows.

$$\sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.24(1-0.76)}{10{,}134}}\sqrt{\frac{0.24(0.76)}{10{,}134}} = \sqrt{\frac{0.1824}{10{,}134}} \approx 0.004$$

Using the 68–95–99.7 rule, the probability is 0.997 that $\hat{p}$ (the sample proportion) lies between the following.

$$0.24-3(0.004)=0.24-0.012=0.228 \text{ to } 0.24-3(0.004)=0.24+0.012=0.252$$

or

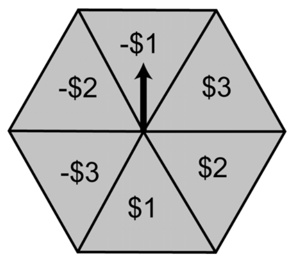
22.8% to 25.2%

The Central Limit Theorem

The **central limit theorem** says that the distribution of any random phenomenon tends to be normal if we average it over a large number of independent repetitions. This theorem applies to both discrete and continuous probability models. It also says that a sample distribution will have the same mean, μ, as the original phenomenon. It will also have a standard deviation equal to $\frac{\sigma}{\sqrt{n}}$, where σ is the standard deviation of a single trial and n is the number of trials. For example, the sampling distribution of sample means is the distribution of sample means when one draws samples of the same size from the same population. This distribution will tend to be normal as this sampling is repeated, regardless of the distribution of the population.

Example

Consider the following spinner game. It costs \$1 to play. If you spin a negative value, you lose your dollar as well as the additional amount indicated (\$1, \$2, or \$3). If you spin a positive value, you keep your dollar and you receive the additional amount indicated (\$1, \$2 or \$3).



a) What is the mean of a single game?
b) What is the standard deviation?
c) What is the mean and the standard deviation of the average win/loss of the game if it was played 100 times in one day?
d) Apply the 99.7 part of the 68-95-99.7 rule to determine a range of average win/loss of playing this game 100 times per day for 1000 days.
e) Repeat Parts a – d with the cost of the game being \$2. If you spin a negative value, you lose the \$2. If you spin a positive value, you keep your \$2 and receive the additional amount. Comment on the results.
f) What should happen with this game if there was no cost to play?

Solution

The probability model would be as follows.

Outcome	–\$2	–\$3	–\$4	\$1	\$2	\$3
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

a) The mean of a single game would be as follows.

$$\mu = (-2)\left(\tfrac{1}{6}\right)+(-3)\left(\tfrac{1}{6}\right)+(-4)\left(\tfrac{1}{6}\right)+(1)\left(\tfrac{1}{6}\right)+(2)\left(\tfrac{1}{6}\right)+(3)\left(\tfrac{1}{6}\right)$$
$$= \tfrac{-2}{6}+\tfrac{-3}{6}+\tfrac{-4}{6}+\tfrac{1}{6}+\tfrac{2}{6}+\tfrac{3}{6} = \tfrac{-3}{6} = -\tfrac{1}{2} = -\$0.50$$

Continued on next page

b) The variance of a single game would be as follows.

$$\sigma^2 = \left[-2-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)+\left[-3-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)+\left[-4-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)+\left[1-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)$$
$$+\left[2-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)+\left[3-\left(-\tfrac{1}{2}\right)\right]^2\left(\tfrac{1}{6}\right)$$
$$=\left(-\tfrac{3}{2}\right)^2\left(\tfrac{1}{6}\right)+\left(-\tfrac{5}{2}\right)^2\left(\tfrac{1}{6}\right)+\left(-\tfrac{7}{2}\right)^2\left(\tfrac{1}{6}\right)+\left(\tfrac{3}{2}\right)^2\left(\tfrac{1}{6}\right)+\left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{6}\right)+\left(\tfrac{7}{2}\right)^2\left(\tfrac{1}{6}\right)$$
$$=\left(\tfrac{9}{4}\right)\left(\tfrac{1}{6}\right)+\left(\tfrac{25}{4}\right)\left(\tfrac{1}{6}\right)+\left(\tfrac{49}{4}\right)\left(\tfrac{1}{6}\right)+\left(\tfrac{9}{4}\right)\left(\tfrac{1}{6}\right)+\left(\tfrac{25}{4}\right)\left(\tfrac{1}{6}\right)+\left(\tfrac{49}{4}\right)\left(\tfrac{1}{6}\right)$$
$$=\tfrac{9}{24}+\tfrac{25}{24}+\tfrac{49}{24}+\tfrac{9}{24}+\tfrac{25}{24}+\tfrac{49}{24}=\tfrac{166}{24}=\tfrac{83}{12}$$

Thus, the standard deviation would be $\sqrt{\tfrac{83}{12}} \approx 2.6300.$

c) From the central limit theorem, the mean would be approximately $-\$0.50$. The standard deviation would be $\frac{2.6300}{\sqrt{100}} \approx 0.2630.$

d) Almost all, 99.7%, of all daily win/losses would fall within three standard deviations from the mean. Thus, the total win/losses after playing 100 times will fall between $-0.50-3(0.2630)=-0.50-0.7890=-1.2890$ and $-0.50+3(0.2630)=-0.50+0.7890=0.2890.$

With rounding, we would say between $-\$1.29$ and $\$0.29$.

e) The probability model would be as follows.

Outcome	–\$3	–\$4	–\$5	\$1	\$2	\$3
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

a) The mean of a single game would be as follows.

$$\mu=(-3)\left(\tfrac{1}{6}\right)+(-4)\left(\tfrac{1}{6}\right)+(-5)\left(\tfrac{1}{6}\right)+(1)\left(\tfrac{1}{6}\right)+(2)\left(\tfrac{1}{6}\right)+(3)\left(\tfrac{1}{6}\right)$$
$$=\tfrac{-3}{6}+\tfrac{-4}{6}+\tfrac{-5}{6}+\tfrac{1}{6}+\tfrac{2}{6}+\tfrac{3}{6}=\tfrac{-6}{6}=-1=-\$1.00$$

b) The variance of a single game would be as follows.

$$\sigma^2=\left[-3-(-1)\right]^2\left(\tfrac{1}{6}\right)+\left[-4-(-1)\right]^2\left(\tfrac{1}{6}\right)+\left[-5-(-1)\right]^2\left(\tfrac{1}{6}\right)+\left[1-(-1)\right]^2\left(\tfrac{1}{6}\right)$$
$$+\left[2-(-1)\right]^2\left(\tfrac{1}{6}\right)+\left[3-(-1)\right]^2\left(\tfrac{1}{6}\right)$$
$$=(-2)^2\left(\tfrac{1}{6}\right)+(-3)^2\left(\tfrac{1}{6}\right)+(-4)^2\left(\tfrac{1}{6}\right)+2^2\left(\tfrac{1}{6}\right)+3^2\left(\tfrac{1}{6}\right)+4^2\left(\tfrac{1}{6}\right)$$
$$=4\left(\tfrac{1}{6}\right)+9\left(\tfrac{1}{6}\right)+16\left(\tfrac{1}{6}\right)+4\left(\tfrac{1}{6}\right)+9\left(\tfrac{1}{6}\right)+16\left(\tfrac{1}{6}\right)$$
$$=\tfrac{4}{6}+\tfrac{9}{6}+\tfrac{16}{6}+\tfrac{4}{6}+\tfrac{9}{6}+\tfrac{16}{6}=\tfrac{58}{6}=\tfrac{29}{3}$$

Thus, the standard deviation would be $\sqrt{\tfrac{29}{3}} \approx 3.1091.$

c) From the central limit theorem, the mean would be approximately $-\$1.00$. The standard deviation would be $\frac{3.1091}{\sqrt{100}} \approx 0.3109.$

d) Almost all, 99.7% of all average daily win/losses would fall within three standard deviations of the mean. Thus, the total win/losses after playing 100 times will fall between $-1.00-3(0.3109)=-1.00-0.9327=-1.9327$ and $-1.00+3(0.3109)=-1.00+0.9327=-0.0673.$

With rounding, we would say between $-\$1.93$ and $-\$0.07$. With the increase cost of play, it seems correct that this interval would be in favor of the "house" instead of the player. The player receives no benefit from the increased cost of play.

Continued on next page

f) The mean would be zero and the standard deviation should be smaller.
The probability model would be as follows.

Outcome	–\$1	–\$2	–\$3	\$1	\$2	\$3
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The mean of a single game would be as follows.

$$\mu=(-1)\left(\tfrac{1}{6}\right)+(-2)\left(\tfrac{1}{6}\right)+(-3)\left(\tfrac{1}{6}\right)+1\left(\tfrac{1}{6}\right)+2\left(\tfrac{1}{6}\right)+3\left(\tfrac{1}{6}\right)=\$0$$

The variance of a single game would be as follows.

$$\begin{aligned}\sigma^2&=[-1-0]^2\left(\tfrac{1}{6}\right)+[-2-0]^2\left(\tfrac{1}{6}\right)+[-3-0]^2\left(\tfrac{1}{6}\right)+[1-0]^2\left(\tfrac{1}{6}\right)\\&\qquad+[2-0]^2\left(\tfrac{1}{6}\right)+[3-0]^2\left(\tfrac{1}{6}\right)\\&=(-1)^2\left(\tfrac{1}{6}\right)+(-2)^2\left(\tfrac{1}{6}\right)+(-3)^2\left(\tfrac{1}{6}\right)+1^2\left(\tfrac{1}{6}\right)+2^2\left(\tfrac{1}{6}\right)+3^2\left(\tfrac{1}{6}\right)\\&=1\left(\tfrac{1}{6}\right)+4\left(\tfrac{1}{6}\right)+9\left(\tfrac{1}{6}\right)+1\left(\tfrac{1}{6}\right)+4\left(\tfrac{1}{6}\right)+9\left(\tfrac{1}{6}\right)\\&=\tfrac{1}{6}+\tfrac{4}{6}+\tfrac{9}{6}+\tfrac{1}{6}+\tfrac{4}{6}+\tfrac{9}{6}=\tfrac{28}{6}\end{aligned}$$

Thus, the standard deviation would be $\sqrt{\frac{28}{6}}\approx 2.1602.$ This is smaller than the other two scenarios.

Law of Large Numbers

The **law of large numbers** states that as a random phenomenon is repeated a large number of times, the mean of the trials, $\bar{x}$, gets closer and closer to the mean of the probability model, μ.

Example

Consider you tossed a coin 10 times and got all heads. According to the law of large numbers, what do you expect should happen if you keep tossing that coin, say 10,000 times?

Solution

You would expect that the number of observed heads (or tails) should be *fairly* close to 5000.

Teaching Tip

Students may confuse the different symbols used for means and standard deviations, as well as proportions.

- $\bar{x}$ is the average of n observations. This is a discrete **observed** occurrence.
- μ is the **population mean** of the probability model. As noted in the text, think of μ as the theoretical mean or the expected value.
- σ is the **population standard deviation** of the probability model.
- Sometimes $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}$ are used when addressing problems involving the central limit theorem $\left(\mu_{\bar{x}}=\mu \text{ and } \sigma_{\bar{x}}=\frac{\sigma}{\sqrt{n}}\right)$. Also, samples of size of at least than 30 $(n\geq 30)$ is a common rule implied.
- p is the probability of an event or the **population** proportion.
- $\hat{p}$ (sometimes called "p hat") is the **sample** proportion.

Solutions to Student Study Guide ✎ Questions

Question 1
Consider tossing a die and flipping a coin.

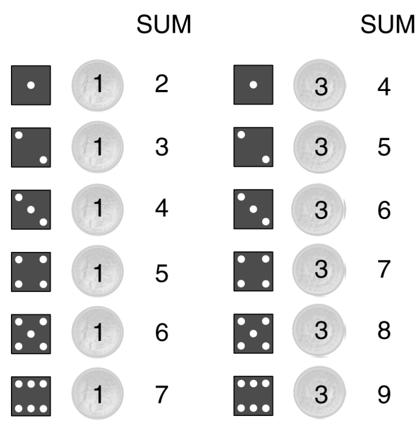
a) Assume a value of "1" was assigned to heads and "3" for tails. Sum together the value on the die with the assigned value on the coin. What is the probability model?

b) What is the probability that the sum is an even number?

c) What is the complement to the event: sum is 2?

d) What is the probability that the sum is not 4?

Solution
a) Consider the following.



The probability model is as follows.

Outcome	2	3	4	5	6	7	8	9
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

b) The probability that the sum is an even number would be as follows.

$$P(2)+P(4)+P(6)+P(8)=\tfrac{1}{12}+\tfrac{2}{12}+\tfrac{2}{12}+\tfrac{1}{12}=\tfrac{6}{12}=\tfrac{1}{2}$$

c) the sum is not 2

d) The best approach would be to state that $P(4)=\frac{1}{6}$. Thus, the desired probability is $1-\frac{1}{6}=\frac{5}{6}$.

Question 2
Consider the situation in Question 1.

a) What is the probability that the sum is more than 3?

b) What is the probability that the sum is less than 3?

c) Should the answers to Parts a and b sum to be 1? Explain.

Solution
a) The complement to the event of the sum is more than 3 is the sum is 3 or less. Thus, the desired probability is $1-\left[P(2)+P(3)\right]=1-\left(\frac{1}{12}+\frac{1}{12}\right)=1-\frac{2}{12}=1-\frac{1}{6}=\frac{5}{6}$. This could also have been found by summing together the probability that the sum is 4, 5, 6, 7, 8, or 9.

b) The probability that the sum is less than 3 is the same as the probability that the sum is 2, or $\frac{1}{12}$.

c) No; a sum of more than 3 and a sum of less than 3 do not consider the possibility of a sum of exact three. The sum of the probabilities of a sum of more than 3, less than 3, and equal to 3 would be 1.

Question 3

When rolling two dice, what is the probability of obtaining a sum of 10?

Solution

There are three combinations of two dice that will add to 10: $(4,6),(5,5)$, and $(6,4)$. Thus, the probability is $\frac{3}{36}=\frac{1}{12}$.

Question 4

Consider an identification code, which made up of three letters of the alphabet followed by three digits. What is the probability that a randomly chosen identification code is ABC123 given that

a) repetition is allowed?
b) repetition is not allowed?

Solution

a) There is only one code that is the desired code of ABC123. Thus, the probability will be 1 divided by the number of possible codes. The number of possible codes would be as follows.

$$26\times 26\times 26\times 10\times 10\times 10 = 17,576,000$$

Thus, the desired probability is $\frac{1}{17,576,000}$.

b) There is only one code that is the desired code of ABC123. Thus, the probability will be 1 divided by the number of possible codes. The number of possible codes would be as follows.

$$26\times 25\times 24\times 10\times 9\times 8 = 11,232,000$$

Thus, the desired probability is $\frac{1}{11,232,000}$.

Question 5

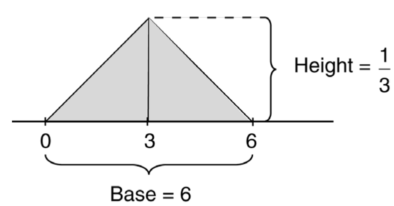
Generate two random numbers between 0 and 3 and take their sum. The sum can take any value between 0 and 6. The density curve is the triangle.

a) What is the probability that the sum is less than 2?
b) What is the probability that the sum is between 1.5 and 4?

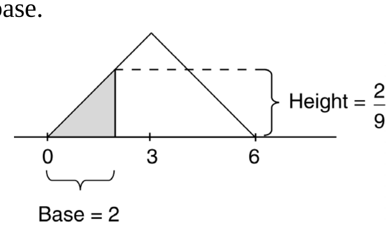
Solution

In drawing the density curve, we realize that the total area must be 1. Since the base is 6, the height would be found by solving the following for *h*.

$$1=\tfrac{1}{2}\times 6\times h \Rightarrow 1=3\times h \Rightarrow h=\tfrac{1}{3}$$



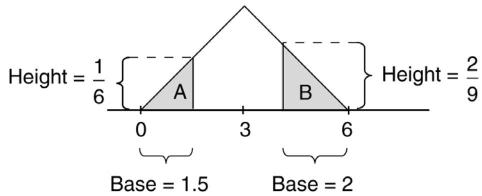
a) By looking at the midpoint of the base, 3, we can see in general that the height of the triangle (from 0 to 3) will be $\frac{1}{9}$ its base.



Thus, the height is $\frac{1}{9}(2)=\frac{2}{9}$. Thus, the desired probability is indicated by the area of the triangle, or $\frac{1}{2}\times 2\times \frac{2}{9}=\frac{2}{9}$

Continued on next page

b) The desired area in this case is the non-shaded part. The easiest way to determine this area is to find the two shaded areas and subtract from the total, 1.



Labeling the left triangle A and the right B, we determine the heights are $\frac{1}{9}\left(\frac{3}{2}\right)=\frac{1}{6}$ and $\frac{1}{9}(2)=\frac{2}{9}$, respectively.

- Area of triangle A: $\frac{1}{2}\times\frac{3}{2}\times\frac{1}{6}=\frac{1}{8}$
- Area of triangle B: $\frac{1}{2}\times 2\times\frac{2}{9}=\frac{2}{9}$
- Sum of the two triangular areas: $\frac{1}{8}+\frac{2}{9}=\frac{9}{72}+\frac{16}{72}=\frac{25}{72}$
- Desired area (probability): $1-\frac{25}{72}=\frac{47}{72}$

Question 6

Suppose 12% of all adults over the age of 25 watch a certain TV show, say the *Captain Buckaroo Show*. You choose 171 adults over the age of 25 to sample at random. By applying the 68–95–99.7 rule, determine 95% of the time the sample proportion will be in what interval? Round your answer to the nearest tenth of a percent.

Solution

The population proportion all adults over the age of 25 who watch the *Captain Buckaroo Show* is $p=0.12$. The sample proportion, $\hat{p}$, of children under the age of 6 who watch the *Captain Buckaroo Show* in a random sample of n = 171 has mean $p=0.12$ and standard deviation as follows.

$$\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.12(1-0.12)}{171}}\sqrt{\frac{0.12(0.88)}{171}}=\sqrt{\frac{0.1056}{171}}\approx 0.025$$

Thus, by applying the 68–95–99.7 rule, 95% of the time the sample proportion will be in the following interval.

$$0.12-2(0.025)=0.12-0.05=0.07 \text{ to } 0.12+2(0.025)=0.12+0.05=0.17$$

or 7.0% to 17.0%

Question 7

Consider the situation in Question 1. What is the mean sum?

Solution

The probability model is as follows.

Outcome	2	3	4	5	6	7	8	9
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

The mean is as follows.

$$\mu=(2)\left(\tfrac{1}{12}\right)+(3)\left(\tfrac{1}{12}\right)+(4)\left(\tfrac{2}{12}\right)+(5)\left(\tfrac{2}{12}\right)+(6)\left(\tfrac{2}{12}\right)+(7)\left(\tfrac{2}{12}\right)+(8)\left(\tfrac{1}{12}\right)+(9)\left(\tfrac{1}{12}\right)$$
$$=\tfrac{2}{12}+\tfrac{3}{12}+\tfrac{8}{12}+\tfrac{10}{12}+\tfrac{12}{12}+\tfrac{14}{12}+\tfrac{8}{12}+\tfrac{9}{12}=\tfrac{66}{12}=5.5$$

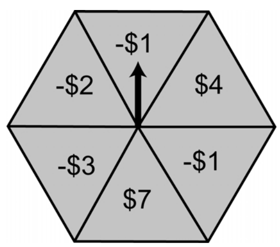
Question 8
Consider the situation in Question 1. What is the standard deviation of the sum?

Solution

$$\begin{aligned}\sigma^2 &= (2-5.5)^2\left(\tfrac{1}{12}\right)+(3-5.5)^2\left(\tfrac{1}{12}\right)+(4-5.5)^2\left(\tfrac{2}{12}\right)+(5-5.5)^2\left(\tfrac{2}{12}\right)\\ &\quad+(6-5.5)^2\left(\tfrac{2}{12}\right)+(7-5.5)^2\left(\tfrac{2}{12}\right)+(8-5.5)^2\left(\tfrac{1}{12}\right)+(9-5.5)^2\left(\tfrac{1}{12}\right)\\ &=(-3.5)^2\left(\tfrac{1}{12}\right)+(-2.5)^2\left(\tfrac{1}{12}\right)+(-1.5)^2\left(\tfrac{2}{12}\right)+(-0.5)^2\left(\tfrac{2}{12}\right)\\ &\quad+(0.5)^2\left(\tfrac{2}{12}\right)+(1.5)^2\left(\tfrac{2}{12}\right)+(2.5)^2\left(\tfrac{1}{12}\right)+(3.5)^2\left(\tfrac{1}{12}\right)\\ &=(12.25)\left(\tfrac{1}{12}\right)+(6.25)\left(\tfrac{1}{12}\right)+(2.25)\left(\tfrac{2}{12}\right)+(0.25)\left(\tfrac{2}{12}\right)\\ &\quad+(0.25)\left(\tfrac{2}{12}\right)+(2.25)\left(\tfrac{2}{12}\right)+(6.25)\left(\tfrac{1}{12}\right)+(12.25)\left(\tfrac{1}{12}\right)\\ &=\tfrac{12.25}{12}+\tfrac{6.25}{12}+\tfrac{4.5}{12}+\tfrac{0.5}{12}+\tfrac{0.5}{12}+\tfrac{4.5}{12}+\tfrac{6.25}{12}+\tfrac{12.25}{12}=\tfrac{47}{12}\end{aligned}$$

Thus, the standard deviation is $\sigma=\sqrt{\tfrac{47}{12}}\approx 1.9791$.

Question 9
Consider the following spinner game. It costs \$1 to play. If you spin a negative value, you lose your dollar as well as the additional amount indicated (\$1, \$2, or \$3). If you spin a positive value, you keep your dollar and you receive the additional amount indicated (\$4 or \$7).



a) What is the mean and standard deviation of a single game?

b) What is the mean and the standard deviation of the average win/loss of the game if it was played 1000 times in one day?

c) Apply the 99.7 part of the 68-95-99.7 rule to determine a range of average win/loss of playing this game 1000 times per day for 1000 days.

Solution
The probability model would be as follows.

Outcome	–\$2	–\$3	–\$4	\$7	–\$2	\$4
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Continued on next page

a) The mean of a single game would be as follows.

$$\mu = (-2)\left(\tfrac{1}{6}\right) + (-3)\left(\tfrac{1}{6}\right) + (-4)\left(\tfrac{1}{6}\right) + (7)\left(\tfrac{1}{6}\right) + (-2)\left(\tfrac{1}{6}\right) + (4)\left(\tfrac{1}{6}\right)$$
$$= \tfrac{-2}{6} + \tfrac{-3}{6} + \tfrac{-4}{6} + \tfrac{7}{6} + \tfrac{-2}{6} + \tfrac{4}{6} = \tfrac{0}{6} = \$0$$

The variance of a single game would be as follows.

$$\sigma^2 = (-2-0)^2\left(\tfrac{1}{6}\right) + (-3-0)^2\left(\tfrac{1}{6}\right) + (-4-0)^2\left(\tfrac{1}{6}\right) + (7-0)^2\left(\tfrac{1}{6}\right) + (-2-0)^2\left(\tfrac{1}{6}\right) + (4-0)^2\left(\tfrac{1}{6}\right)$$
$$= (-2)^2\left(\tfrac{1}{6}\right) + (-3)^2\left(\tfrac{1}{6}\right) + (-4)^2\left(\tfrac{1}{6}\right) + 7^2\left(\tfrac{1}{6}\right) + (-2)^2\left(\tfrac{1}{6}\right) + 4^2\left(\tfrac{1}{6}\right)$$
$$= 4\left(\tfrac{1}{6}\right) + 9\left(\tfrac{1}{6}\right) + 16\left(\tfrac{1}{6}\right) + 49\left(\tfrac{1}{6}\right) + 4\left(\tfrac{1}{6}\right) + 16\left(\tfrac{1}{6}\right)$$
$$= \tfrac{4}{6} + \tfrac{9}{6} + \tfrac{16}{6} + \tfrac{49}{6} + \tfrac{4}{6} + \tfrac{16}{6} = \tfrac{98}{6} = \tfrac{49}{3}$$

Thus, the standard deviation would be $\sqrt{\tfrac{49}{3}} \approx 4.0415$.

b) From the central limit theorem, the mean would be \$0. The standard deviation would be $\tfrac{4.0415}{\sqrt{1000}} \approx 0.1278$.

c) Almost all, 99.7% of all average daily win/losses would fall within three standard deviations from the mean \$0. Thus, the total win/losses after playing 1000 times will fall between $0-3(0.1278) = 0-0.3834 = -0.3834$ and $0+3(0.1278) = 0+0.3834 = 0.3834$ With pure rounding of the money, we would say between $-\$0.38$ and $\$0.38$. To not narrow the confidence interval, one could say between $-\$0.39$ and $\$0.39$.