Chapter 4
Linear Programming

For All Practical Purposes: Effective Teaching

- Occasionally during the semester remind students about your office hours. Some students can perceive that they are bothering you if they come for help. By letting students know that you are available to them and want them to seek help will communicate that you care about their understanding of the material and their overall performance.
- When a student or students do well, acknowledge their exemplary performance. Communicating to the class as a whole that the objectives are being met should help to motivate those students who may not be focused. Knowing that other students (peers) in their course are taking their work seriously should help bring them into being part of the class.

Chapter Briefing

In this chapter, you will be mainly examining applications where you find an optimal solution. Initially this will involve *linear programming* in order to find an *optimal production policy*. Although an optimal solution, in general, could be seeking a maximum or minimum value, this production policy will address only finding under what conditions the profit formula has a maximum value. In such applications closed and bounded regions, called *feasible regions*, will yield points of intersection called *corner points*. The optimal production policy will occur at one of these corner points. The geometry behind why the optimal solution occurs at a corner point is in the third section of this chapter. Since linear programming applies to cases when the problem constraints are linear inequalities and the profit function is linear in the variables, one is limited to examining a polygonal region in 2-space. If one has three products to consider, the polyhedral is in 3-space. It will have an analogous structure in higher dimensions.

Being well prepared for class discussion with examples and the knowledge that students may be lacking in the basics of graphing lines, inequalities, and finding points of intersection between lines is essential in order to help students focus on the main topics presented in this chapter (applications of optimization). In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Graphing Lines and Inequalities**, **Corner Points**, **Setting Up a Linear Program**, **Solving a Linear Program**, **Simplex Method**, **The Transportation Problem and the Northwest Corner Rule**, and **Improving on the Northwest Corner Rule**. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

Since you may demonstrate some techniques of this chapter using a graphing calculator, the *Teaching Guide* includes the feature **Teaching the Calculator**. It includes brief calculator instructions with screen shots from a TI-83.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the five questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

Graphing lines and Inequalities

An essential skill that students should master in the early part of this chapter is graphing lines, inequalities, and systems of inequalities. Some basic facts/techniques you may choose to go over are as follows.

- A line of the form $x = a$ is vertical and $y = b$ is horizontal, where *a* and *b* are real numbers.
- The graphical representation of $x = 0$ is the *y*-axis, and $y = 0$ represents the *x*-axis.
- To graph a non-vertical (or non-horizontal) line, graph by the intercept method. An *x*-intercept occurs when $y = 0$ and a *y*-intercept occurs when $x = 0$.
- To graph an inequality, first graph the line and then choose a test point to determine which half-plane to shade.
- The system of inequalities $"x \geq 0 \text{ and } y \geq 0"$ imply that the exercise is restricted to the first quadrant.

Teaching Tip

When students need to graph more than one line (inequality) in the plane, instruct them to find the *x*- and *y*- intercepts of each line that needs to be graphed (including horizontal and vertical lines) first. You may find taking a few minutes to talk about proper scaling on the coordinate axes will help greatly. Instruct students to be clear in their scaling by indicating the value of the first tick mark on the *x*- and *y*- axes (assuming it is other than 1). Also, you may let them know that the scaling does not have to be the same on each axes, but does need to be clear.

Example

Graph the region given by the following set of inequalities.

$$x \geq 0 \text{ and } y \geq 0$$
$$4x + y \leq 80$$
$$x + 2y \leq 80$$

Solution

The constraints of $x \geq 0$ and $y \geq 0$ indicate that we are restricted to the upper right quadrant created by the *x*-axis and *y*-axis.

The *y*-intercept of $4x + y = 80$ can be found by substituting $x = 0$.

$$4(0) + y = 80 \Rightarrow 0 + y = 80 \Rightarrow y = 80$$

The *y*-intercept is $(0,80)$.

The *x*-intercept of $4x + y = 80$ can be found by substituting $y = 0$.

$$4x + 0 = 80 \Rightarrow 4x = 80 \Rightarrow x = \tfrac{80}{4} = 20$$

The *x*-intercept is $(20,0)$.

The *y*-intercept of $x + 2y = 80$ can be found by substituting $x = 0$.

$$0 + 2y = 80 \Rightarrow 2y = 80 \Rightarrow y = \tfrac{80}{2} = 40$$

The *y*-intercept is $(0,40)$.

The *x*-intercept of $x + 2y = 80$ can be found by substituting $y = 0$.

$$x + 2(0) = 80 \Rightarrow x = 80$$

The *x*-intercept is $(80,0)$.

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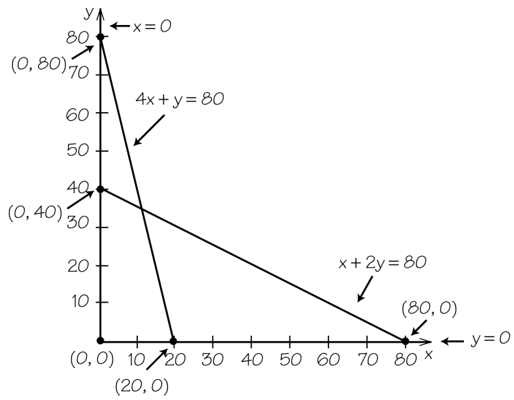
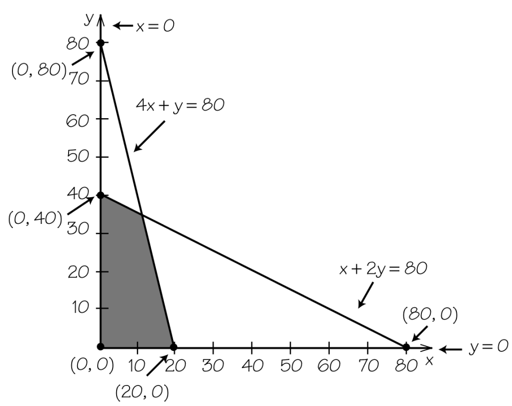
Given the four intercepts, an appropriate scaling may be 1:10 or 1:20. In this case, one may consider 1:10 because indicating 8 tick marks on each axis is reasonable and having more detail will help in the steps that follow (finding and labeling points of intersection). Plot the intercepts and connect the proper pair to determine the graph of each line. The origin is labeled along with the axes darkened because of the initial constraints.



To determine which side of a line to shade, one needs to choose a test point. Testing the point $(0,0)$ in $4x+y\le 80,$ we have the statement $4(0)+0\le 80$ or $0\le 80.$ This is a true statement. Testing the point $(0,0)$ in $x+2y\le 80,$ we have the statement $0+2(0)\le 80$ or $0\le 80.$ This is also a true statement. Thus, we shade the part of the plane in the upper right quadrant which is on the down side of both the lines $4x+y=80$ and $x+2y=80.$



☝Teaching Tip

Because the majority of the exercises in this chapter maximize profit, the region that will be graphed will be closed and bounded. Thus, the resource constraints (discussed later) will be shaded in the direction of the origin. Making the general observation and explaining why the inequality of the form $ax+by\le c$ (where a, b, and c are positive) will have shading in the direction of the origin should help students graph such inequalities in a timely manner.

Corner Points

After graphing a region defined by a system of inequalities, students will need to find points of intersection between two lines. These points will be known as corner points of the feasible region. There are three scenarios.

- Finding the point of intersection between $x = a$ and $y = b$, which is (a,b);
- Finding the point of intersection between a vertical or horizontal line and one that is neither of these (solved by the substitution method);
- Finding the point of intersection between two lines that are neither vertical or horizontal (solved by the addition method).

Example

Find the point of intersection between

a) $x = -3$ and $y = 6$.

b) $y = 1$ and $2x + 5y = 11$.

Solution

a) $(-3,6)$

b) By substituting $y = 1$ into $2x + 5y = 11$, we have the following.

$$2x + 5(1) = 11 \Rightarrow 2x + 5 = 11 \Rightarrow 2x = 6 \Rightarrow x = 3$$

Thus the point of intersection is $(3,1)$.

Example

What are the four corner points of the region in the first example?

Solution

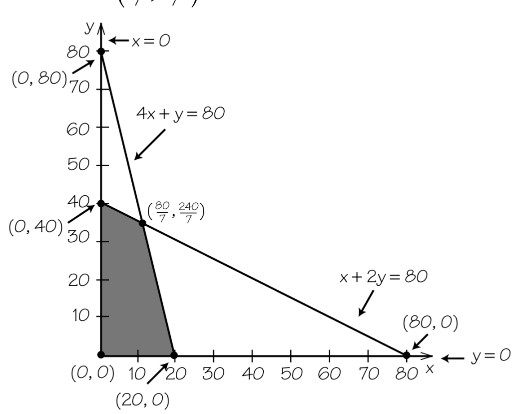
Three of the corner points, $(0,0)$, $(0,40)$, and $(20,0)$ lie on the coordinate axes. The fourth corner point is the point of intersection between the lines $4x + y = 80$ and $x + 2y = 80$. We can find this by multiplying both sides of $4x + y = 80$ by -2, and adding the result to $x + 2y = 8$.

$$\begin{array}{r} -8x - 2y = -160 \\ \underline{x + 2y = 80} \\ -7x \quad\quad = -80 \Rightarrow x = \frac{-80}{-7} = \frac{80}{7} \end{array}$$

Substitute $x = \frac{80}{7}$ into $x + 2y = 80$ and solve for y.

$$\tfrac{80}{7} + 2y = 80 \Rightarrow 2y = 80 - \tfrac{80}{7} \Rightarrow 2y = \tfrac{480}{7} \Rightarrow y = \tfrac{240}{7}$$

The point of intersection is therefore $\left(\frac{80}{7}, \frac{240}{7}\right)$.



☝Teaching Tip

In the majority of the exercises in this chapter, all the corner points will be natural numbers, unlike in the last example. You may choose to emphasize that students should take care when drawing their graphs. When a corner point needs to be solved for (not an intercept) after the feasible region is graphed, students ought to check that this point is consistent with the graph. For example, if we solved for the fourth corner point in the last example and obtained $(15,30)$ then that would imply an error occurred in either finding this corner point or perhaps in the original calculations of the intercepts.

Setting Up a Linear Program

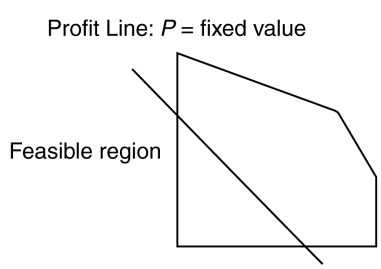
Linear programming is used to make management decisions in a business or organizational environment. We are concerned with situations where resources—time, money, raw materials, labor—are limited. These resources must be allocated with a certain goal in mind: to do a job as quickly as possible, or to keep costs to a minimum, or to make a maximum profit. In setting up a linear program one should do the following.

- Define the variables for the products.
- Determine the minimums of the products. If none are stated then they are assumed to be 0.
- Determine the **resource constraints**, which should be stated in the exercise.
- Set up a mixture chart which includes the profit made from each product.
- Define the **feasible region** by a set of inequalities. This will contain both inequalities from the minimums and resource constraints.
- State the profit relation
- Graph the set of inequalities to find the feasible region.
- Find all corner points.

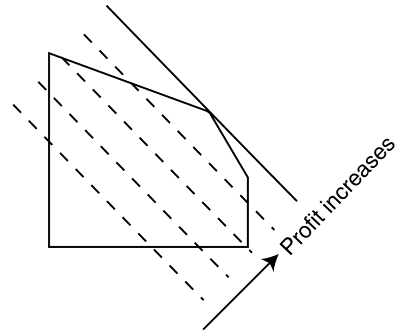
Solving a Linear Program

In a mixture problem, we seek an optimal production policy for allocating limited resources to make a maximum profit. A production policy is represented by a point in the feasible region. According to the corner point principle, the optimal production policy is represented by a corner point of the feasible region. To determine the optimal production policy, we find the corner points of our region and evaluate the profit relation. The highest value obtained will indicate the optimal production policy − that is, how many of each product should be produced for a maximum profit.

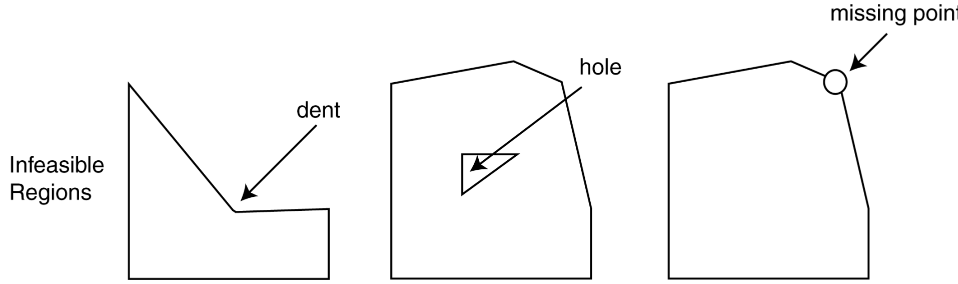
Students can envision why the corner point principle works by choosing a theoretically possible profit value, the points of the feasible region yielding that level of profit lie along a profit line cutting through the region.



Raising the profit value generally moves the profit line across the feasible region until it just touches at a corner, which will be the point with maximum profit, the optimal policy.



This explains the corner point principle; it works because the feasible region has no "holes" or "dents" or missing points along its boundary.



Note: These illustrations also appear in the *Student Study Guide*.

Example

The M & M book publishers can produce at most 2000 books in one day. The company produces Mathematics and Marketing texts. Production costs are \$10 for a mathematics text and \$30 for a marketing text. The daily operating budget is \$30,000. How many of each text should be produced if the profit is \$1.50 for each mathematics text and \$1.60 for each marketing text?
Does the policy change if they must produce at least 600 mathematics books and 200 marketing books in one day? Does the policy change (from the original set-up) if they must produce at least 600 marketing books in one day?

Solution

Let x be the number mathematics texts and y be the number of marketing texts.

	texts (2000)	costs (\$30,000)	Minimums	Profit
Mathematics, x texts	1	\$10.00	0	\$1.50
Marketing, y texts	1	\$30.00	0	\$1.60

Profit formula: $P = \$1.50x + \$1.60y$

Constraints: $x \geq 0$ and $y \geq 0$ (minimums)

$x + y \leq 2000$ (texts); $10x + 30y \leq 30{,}000$ (cost)

Feasible region:

The y-intercept of $x + y = 2000$ is $(0, 2000)$, and the x-intercept is $(2000, 0)$.

The y-intercept of $10x + 30y = 30000$ is $(0, 1000)$, and the x-intercept is $(3000, 0)$.

A scaling of 1:500 seems appropriate on both axes.
When drawing the feasible region, we are restricted to the first quadrant, and these resource constraints indicate shading towards the origin.

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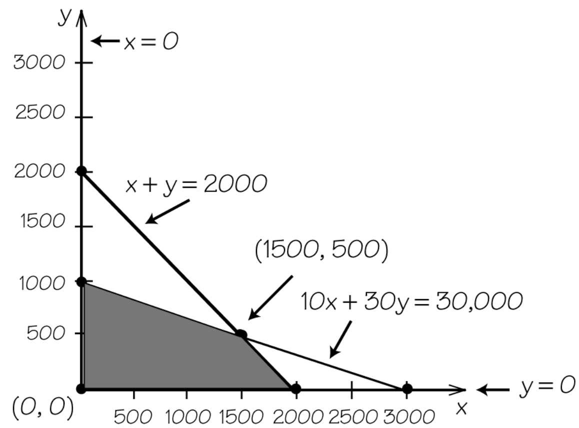

Corner points: Three corner points are $(0,0)$, $(2000,0)$, and $(0,1000)$. The fourth corner point is the point of intersection between $x+y=2000$ and $10x+30y=30,000$. We can find this by multiplying both sides of $x+y=2000$ by -10, and adding the result to $10x+30y=30,000$.

$$\begin{array}{r} -10x-10y=-20,000 \\ \underline{10x+30y=30,000} \\ 20y=10,000 \Rightarrow y=500 \end{array}$$

Substitute $y=500$ into $x+y=2000$ and solve for y. We have $x+500=2000$ or $x=1500$. The point of intersection is therefore $(1500,500)$.

We wish to maximize $\$1.50x+\$1.60y$.

Corner Point	Value of the Profit Formula: $\$1.50x+\$1.60y$								
$(0,0)$	$\$1.50(0)$	+	$\$1.60(0)$	=	\$0	+	\$0	=	\$0
$(2000,0)$	$\$1.50(2000)$	+	$\$1.60(0)$	=	\$3000	+	\$0	=	\$3000
$(0,1000)$	$\$1.50(0)$	+	$\$1.60(1000)$	=	\$0	+	\$1600	=	\$1600
$(1500,500)$	$\$1.50(1500)$	+	$\$1.60(500)$	=	\$2250	+	\$800	=	\$3050*

Optimal production policy: Make 1500 mathematics texts and 500 marketing texts per day for a maximum profit of \$3050.

The policy does not change if they must produce at least 600 mathematics books and 200 marketing books in one day. The optimal solution obeys these constraints.

If the policy changes (from the original set-up) to require at least 600 marketing books in one day then we need to readdress the feasible region since our solution does not satisfy this new minimum constraint.

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Let x be the number mathematics texts and y be the number of marketing texts.

	texts (2000)	costs ($30,000)	Minimums	Profit
Mathematics, x texts	1	$10.00	0	$1.50
Marketing, y texts	1	$30.00	600	$1.60

Profit formula: $P = \$1.50x + \$1.60y$

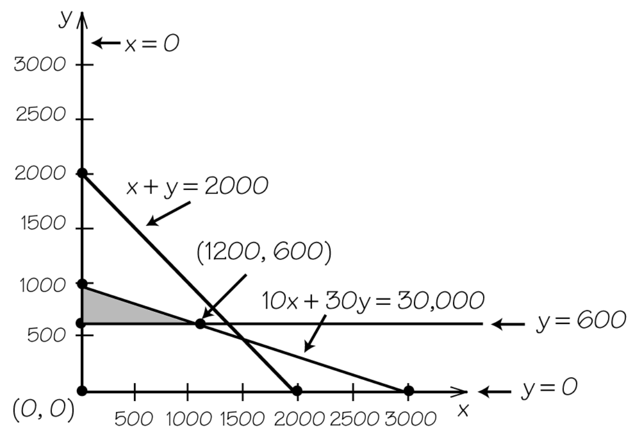
Constraints: $x \geq 0$ and $y \geq 600$ (minimums)

$x + y \leq 2000$ (texts); $10x + 30y \leq 30,000$ (costs)

Corner points: Three corner points are $(0,600)$ and $(0,1000)$. The third corner point is the point of intersection between $y = 600$ and $10x + 30y = 30,000$. We can find this by substituting $y = 600$ into $10x + 30y = 30,000$.

$$10x + 30(600) = 30,000$$
$$10x + 18,000 = 30,000$$
$$10x = 12,000 \Rightarrow x = 1200$$

The point of intersection is therefore $(1200, 600)$.



Notice that the resource constraint $x + y \leq 2000$ no longer has any effect on the corner points.

We wish to maximize $\$1.50x + \$1.60y$.

Corner Point	Value of the Profit Formula: $\$1.50x + \$1.60y$								
$(1200, 600)$	$1.50(1200)	+	$1.60(600)	=	$1800	+	$960	=	$2760*
$(0, 600)$	$1.50(0)	+	$1.60(600)	=	$0	+	$960	=	$960
$(0, 1000)$	$1.50(0)	+	$1.60(1000)	=	$0	+	$1600	=	$1600

Optimal production policy: Make 1200 mathematics texts and 600 marketing texts per day for a maximum profit of $2760.

☝Teaching Tip

If you choose to use this as the last example in class, a good question to be posed for discussion has to do with whether the additional minimum constraints could ever allow for a higher maximum profit. Why or why not?

Simplex Method

For realistic applications, the feasible region may have many variables ("products") and hundreds or thousands of corners. More sophisticated evaluation methods, such as the **simplex method**, must be used to find the optimal point.

The following Example is from the *Student Study Guide*. The solution to the problem situation posed is not provided, just the set-up as requested.

Example J (from Student Study Guide)

Dan's creamery decides to introduce a light ice cream (a third product) in his shop. Assume that a container of this light ice cream will require one-eighth pint of cream and sells at a profit of $1.00. There is no minimum on this new product, and it does not contain raspberries. Given all the constraints in Example I, determine the resource constraints and minimal constraints and the profit formula that applies to Dippy Dan's creamery shop.

Solution

Let *x* be the number of containers of ice cream, *y* be the number of containers of sherbet, and *z* be the number of containers of light ice cream.

	Cream (240 pints)	Raspberries (600 lb)	Minimums	Profit
Ice cream, *x* containers	$\frac{1}{2}$	1	100	$0.75
Sherbet, *y* containers	$\frac{1}{4}$	1	80	$0.25
Light ice cream, *z* containers	$\frac{1}{8}$	0	0	$1.00

Profit formula: $P = \$0.75x + \$0.25y + \$1.00z$

Constraints: $x \geq 100,\ y \geq 80,\ \text{and}\ z \geq 0\ (\text{minimums})$

$$\tfrac{1}{2}x + \tfrac{1}{4}y + \tfrac{1}{8}z \leq 240\ (\text{creme})$$

$$x + y + 0z \leq 600(\text{raspberries})$$

At the time of printing, the following website was used to obtain a solution to the problem situation.

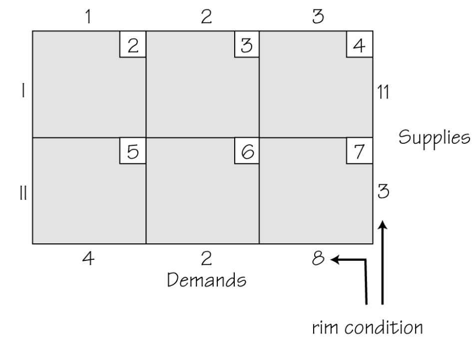
http://www-fp.mcs.anl.gov/otc/Guide/CaseStudies/simplex/applet/SimplexTool.html

Given this information, the optimal production policy is 100 containers of ice cream, 80 containers of sherbet, and 1360 containers of light ice cream, giving Dan a maximum profit of $1455.

The Transportation Problem and the Northwest Corner Rule

A **transportation problem** involves supply, demand, and transportation costs. A supplier makes enough of a product to meet the demands of other companies. The supply must be delivered to the different companies, and the supplier wishes to minimize the shipping cost, while satisfying demand.

The amount of product available and the requirements are shown on the right side and the bottom of a table. These numbers are called **rim conditions**. A table showing costs (in the upper right-hand corner of a cell) and rim conditions form a tableau.



Each cell is indicated by its row and column. For example the cell with a cost of 3 is cell (I, 2).

Teaching Tip

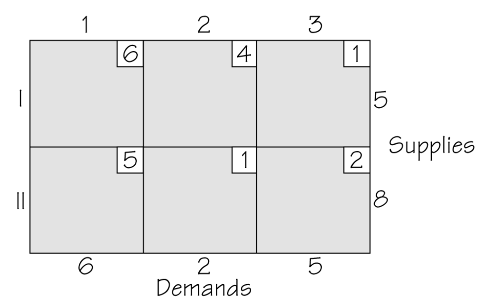
Point out early on to the students that the sum of the rim conditions on a row must equal the sum on the column.

The **northwest corner rule (NCR)** involves the following.

- Locate the cell in the far top left (initially that will be cell (I,1)).
- Cross out the row or column that has the smallest rim value for that cell.
- Place that rim value in that cell and reduce the other rim value by that smaller value.
- Continue that process until you get down to a single cell.
- Calculate the cost of this solution.

Example

Apply the Northwest Corner Rule to the following tableau and determine the cost associated with the solution.



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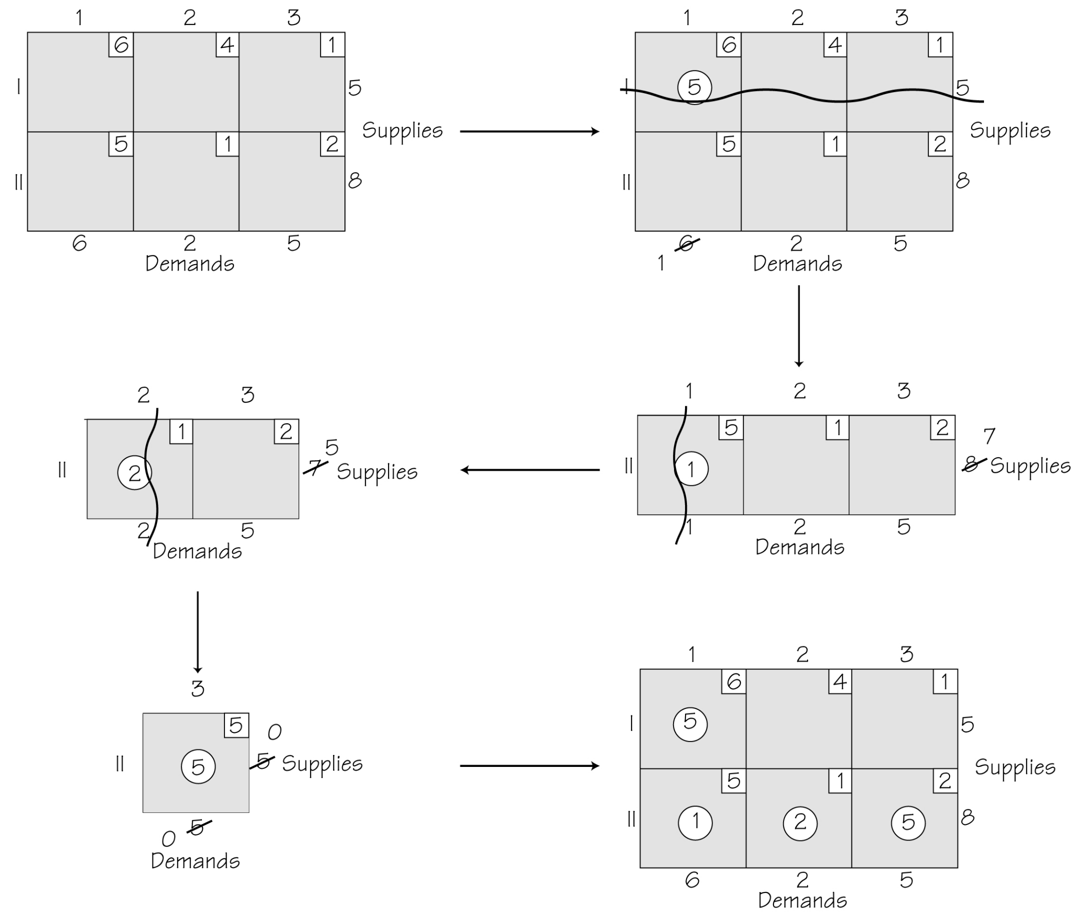
Solution



The cost is $5(6)+1(5)+2(1)+5(2)=30+5+2+10=47.$

The **indicator value of a cell *C*** (not currently a circled cell) is the cost change associated with increasing or decreasing the amount shipped in a circuit of cells starting at *C*. It is computed with alternating signs and the costs of the cells in the circuit.

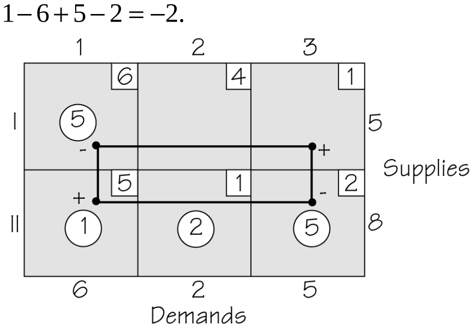
Example

Determine the indicator value of the non-circled cells in the last example.

Solution

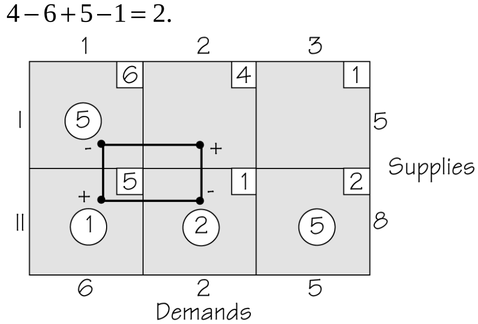
The indicator value for cell $(\mathrm{I},3)$ is

$1-6+5-2=-2.$



The indicator value for cell $(\mathrm{I},2)$ is

$4-6+5-1=2.$



Improving on the Northwest Corner Rule.

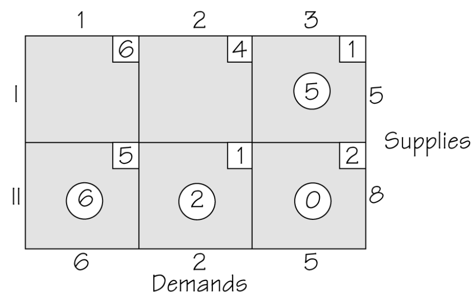
If some indicator cells are positive and some are zero, there are multiple solutions for an optimal value. The **stepping stone method** improves on some non-optimal feasible solution to a transportation problem. This is done by shipping additional amounts using a cell with a negative indicator value.

Example

Apply the stepping stone algorithm to determine an optimal solution. Consider both cells with negative indicator values, determine the new cost for each consideration and compare to the solution found using the Northwest Corner Rule.

Solution

Increasing the amount shipped through cell $(\text{I},3)$, we have the following.



The cost is $5(1)+0(2)+2(1)+6(5)=5+0+2+30=37.$

The cost can be reduced to 37 (compared to 47) when increasing the amount shipped through cell $(\text{I},3)$.

Teaching the Calculator

The graphing calculator can be used to graph a set of constraints and determine corner points.

Example

Use the graphic calculator to find the corner points for the following set of inequalities.

$$x \geq 0,\ \ y \geq 0,\ 2x+3y \leq 160,\ x+y \leq 60$$

Solution

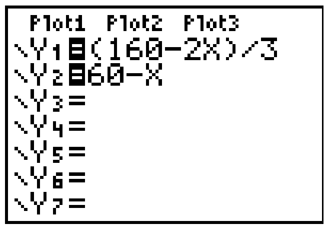
Although the graphing calculator can locate points on the coordinate axes, because a proper window needs to be determined, it is easier to go ahead and calculate *x*- and *y*- intercepts of the lines.

In order to enter $2x+3y \leq 160$ and $x+y \leq 60$ into the calculator, we must solve each one for *y*.

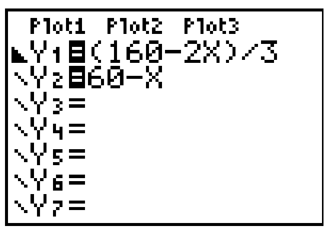
$$2x+3y \leq 160 \Rightarrow 3y \leq 160-2x \Rightarrow y \leq (160-2x)/3$$

$$x+y \leq 60 \Rightarrow y \leq 60-x$$

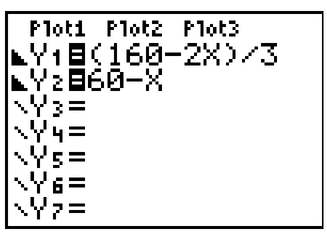
After pressing Y=, you can enter the equations.



To graph the first equation as an inequality, toggle over to the left of the line and press ENTER three times.



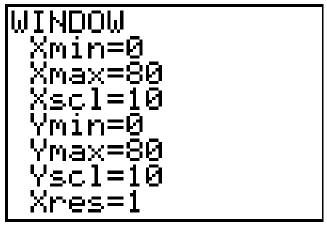
Repeat for the other inequality.



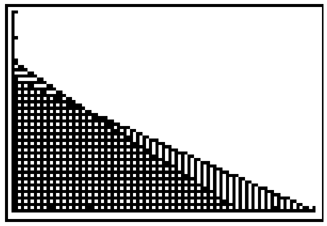
By pressing WINDOW you will enter an appropriate window for Xmin, Xmax, Ymin and Ymax. Xmin and Ymin would both be 0 in most cases. Determine the *x*- and *y*-intercepts of the lines to assist you in determining a window as well as corner points.

$2x+3y=160$ has an *x*-intercept of $(80,0)$ and a *y*-intercept of $\left(0,53\frac{1}{3}\right)$. $x+y=60$ has an *x*-intercept of $(60,0)$ and a *y*-intercept of $(0,60)$.

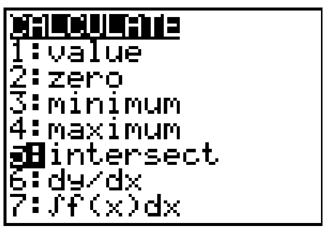
The following window is appropriate.

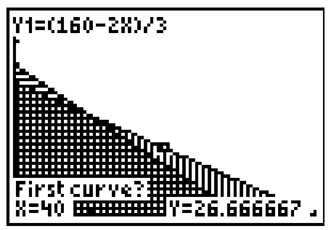


Next, we display the feasible region by pressing the GRAPH button.

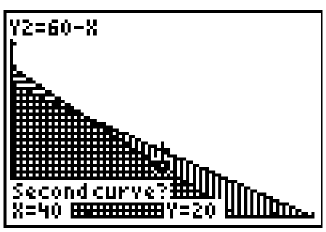


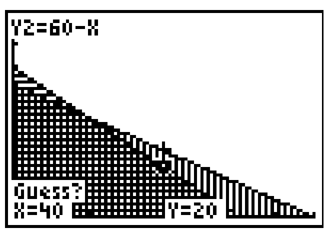
We know three of the four corner points. They are $(0,0)$, $(60,0)$, and $\left(0,53\frac{1}{3}\right)$. To find the fourth corner point, you will need to find the point of intersection between the lines $2x+3y=160$ and $x+y=60$. To do this, you will need to press 2nd followed by TRACE. You then need to toggle down or press 5 followed by ENTER.

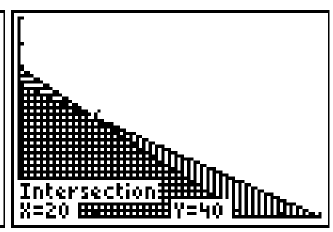




Press ENTER three times and the following three screens will be displayed.







The fourth corner point is therefore $(20,40)$.

Solutions to Student Study Guide Questions

Question 1

a) Find the point of intersection between $x = 20$ and $y = 30$.

b) Find the point of intersection between $y = 5$ and $13x + 21y = 1678$.

Solution

a) The point of intersection will be the given *x*- and *y*-components, as given, respectively. It will therefore be $(20, 30)$.

b) By substituting $y = 5$ into $13x + 21y = 1678$ we have the following.

$$13x + 21(5) = 1678 \Rightarrow 13x + 105 = 1678 \Rightarrow 13x = 1573 \Rightarrow x = 121$$

Thus, the point of intersection is $(121, 5)$.

Question 2

Find the point of intersection between $7x + 3y = 43$ and $8x + 7y = 67$.

Solution

We can find this by multiplying both sides of $8x + 7y = 67$ by -3, and multiplying both sides of $7x + 3y = 43$ by 7. These results can be added as follows.

$$\begin{array}{rl} -24x - 21y &= -201 \\ 49x + 21y &= 301 \\ \hline 25x \qquad &= 100 \Rightarrow x = \tfrac{100}{25} = 4 \end{array}$$

Substitute $x = 4$ into $7x + 3y = 43$ and solve for *y*. We have the following.

$$7(4) + 3y = 43 \Rightarrow 28 + 3y = 43 \Rightarrow 3y = 15 \Rightarrow y = \tfrac{15}{3} = 5$$

Thus, the point of intersection is $(4, 5)$.

Question 3

Apply the Northwest Corner Rule to the following tableau to determine the cost associated with the solution.

	1	2	Supplies
I	10	9	5
II	8	2	4
III	3	5	2
Demands	3	8	

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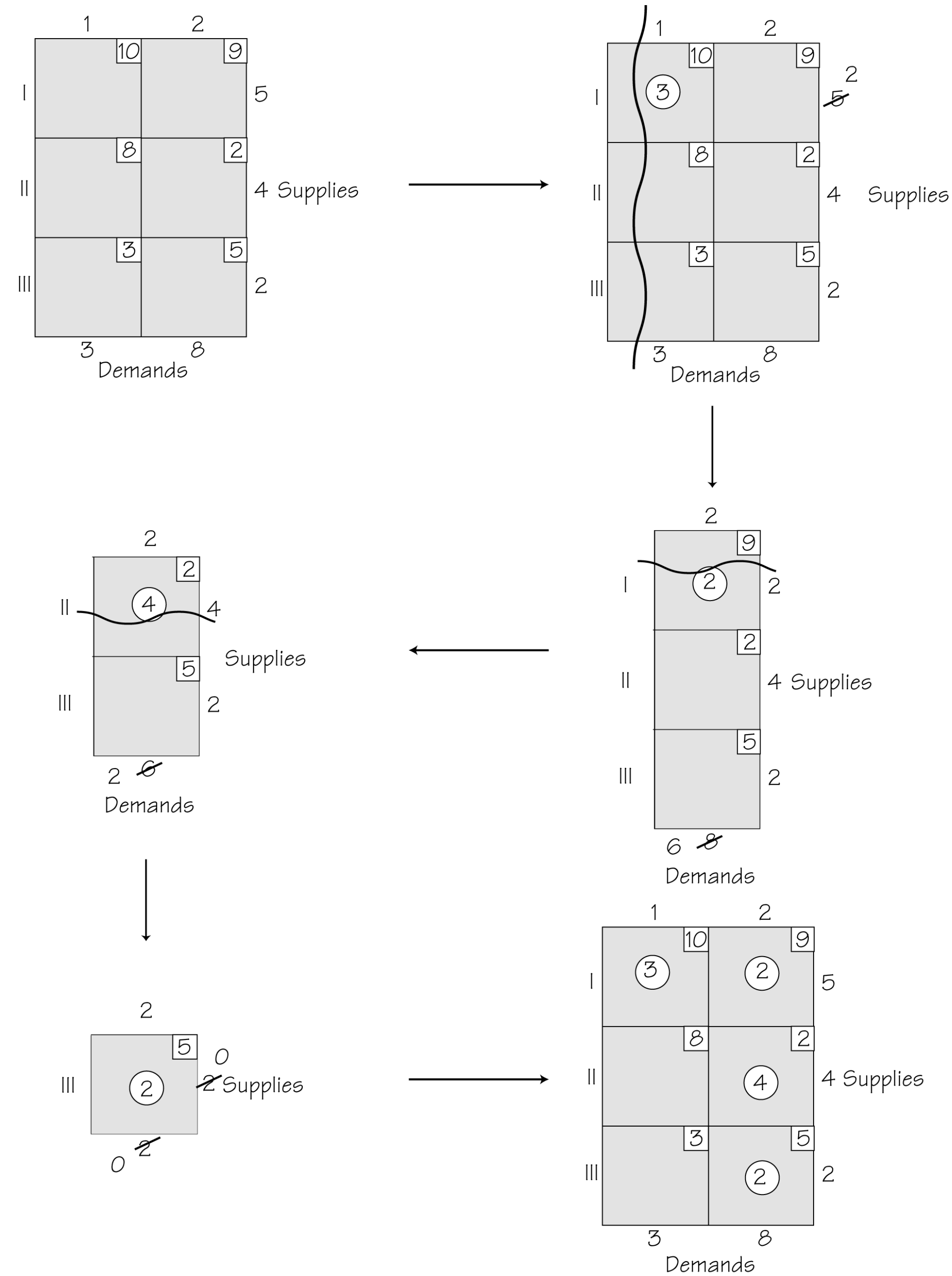
Solution



$$3(10)+2(9)+4(2)+2(5)=30+18+8+10=66$$

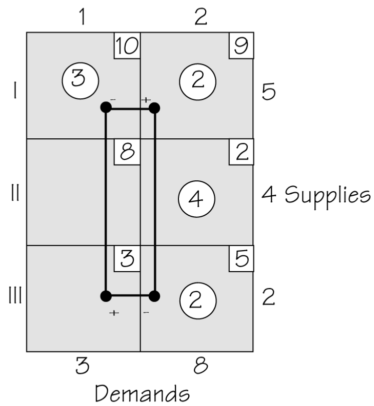
Question 4

What is the indicator value of each of the non-circled cells in Question 3?

Solution

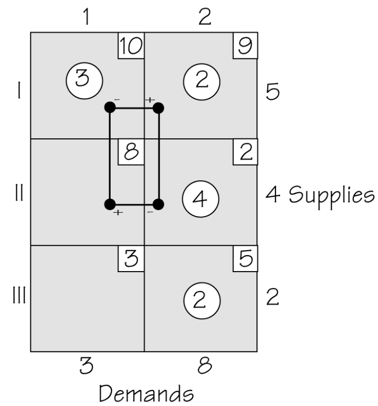
The indicator value for cell $(\text{III},1)$ is

$3-5+9-10=-3.$



The indicator value for cell $(\text{II},1)$ is

$8-2+9-10=5.$

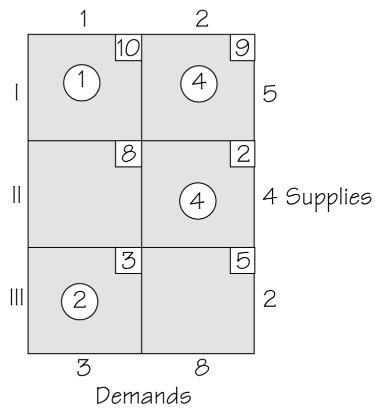


Question 5

Determine the cost involved when applying the stepping stone method to determine an optimal solution for Question 4 when compared to the cost found by the Northwest Corner Rule.

Solution

Cell $(\text{III},1)$ has a negative indicator value. Increasing the amount shipped through cell $(\text{III},1)$, we have the following.



The cost is $1(10)+4(9)+4(2)+2(3)=10+36+8+6=60.$ The cost can be reduced to 60, compared to 66.