Chapter 1
Urban Services

For All Practical Purposes: Effective Teaching

- At the beginning of the course, go out of your way to learn students' names. Naturally if you have a very large class this would be difficult. By learning at least a few students' names early on and addressing them by their name will help in making a connection between you and your students.
- Students need to be recognized as individuals. Speak to them individually or call on them during lecture. By making each student feel like a person instead of a number is very important in helping them become comfortable with their professor and will help to make a connection.

Chapter Briefing

In this chapter, you will be mainly examining graphs and determining an optimal solution. By examining features of a graph (*edges*, *vertices*, and *valances*) one can determine if an optimal solution known as an *Euler circuit* exists. If no such circuit exists, then it would be necessary to *eulerize* it by adding edges. These kinds of graph theory techniques have a variety of real-world applications as is indicated by the chapter title, *Urban Services*.

Applications in the real-world that involve constraints such as one-way streets necessitate the examination of directed graphs or *digraphs*. Because of transportation cost, many real-world applications such as snowplowing a road or delivering mail involve examinations of such graphs.

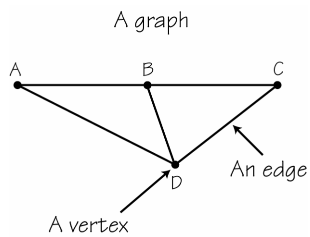
Being well prepared for class discussion with a clear understanding of the terminology, conditions, and examples is essential in order to help students focus on the topics presented in this chapter. In order to facilitate your preparation, the **Chapter Topics to the Point** has been broken down into **Graph Terminology**, **Finding Euler Circuits and Euler's Theorem**, **Altering a Graph to Obtain an Euler Circuit**, and **Digraphs**. Examples with solutions for these topics that do not appear in the text nor study guide are included in the *Teaching Guide*. You should feel free to use these examples in class, if needed.

The last section of this chapter of *The Teaching Guide for the First-Time Instructor* is **Solutions to Student Study Guide ✎ Questions**. These are the complete solutions to the three questions included in the *Student Study Guide*. Students only have the answers to these questions, not the solutions.

Chapter Topics to the Point

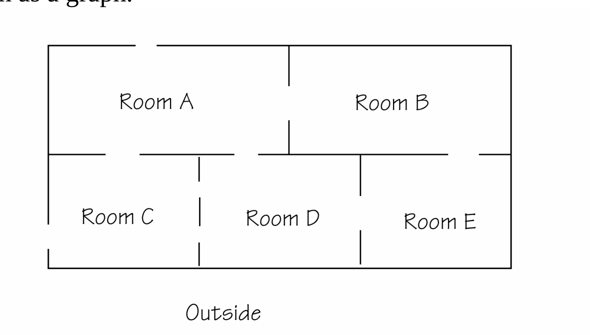
Graph Terminology

A problem such as finding an optimal route for snowplowing roads or delivering mail can be modeled abstractly as finding a best path through a **graph** that includes every edge. A graph is a finite set of dots and connecting links. A dot is called a **vertex** and the link between two vertices is called an **edge**. In trying to solve such problems, one seeks the best path through a graph that includes every edge.

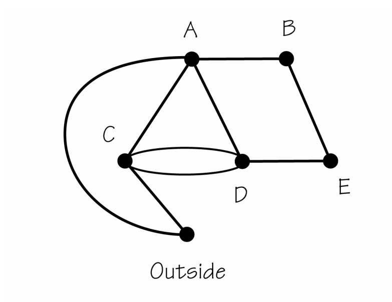


Example

Consider the following floor plan. The open space represents a door that one can pass through. Represent the floor plan as a graph.



Solution



Finding Euler Circuits and Euler's Theorem

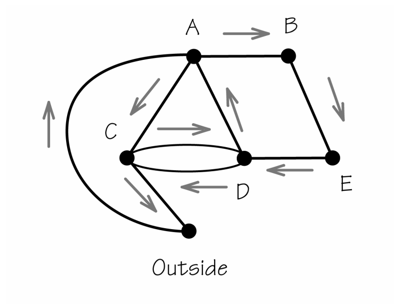
A path through a graph is a **circuit** if it starts and ends at the same vertex. A circuit is an **Euler circuit** if it covers each edge exactly once. (Euler is pronounced like "*oy'lur*")

Example

Find an Euler circuit of the graph for the floor plan.

Solution

There are many correct answers.
Starting and ending with the vertex marked "Outside", this circuit covers each edge exactly once. Any circuit drawn that meets these conditions (1. starts and ends at the same vertex, and 2. covers each edge exactly once) is an Euler circuit.



This is one possible Euler circuit for the graph. *OABEDACDCO*, where *O* represents Outside.

Teaching Tip

In finding an Euler circuit (assume the graph is connected and has all even valances), try beginning at any vertex. Once you have traveled over it, erase it. If all the edges for any particular vertex have been erased, go ahead and erase the vertex too. If an edge appears to "bridge" portions of the graph, put off going over that "bridge" until there is no other alternative. (This procedure is referenced as Fleury's Algorithm.)

The **valence** of a vertex is the number of edges that meet at that vertex. This will be either an even or odd positive integer. If the vertex is isolated then is will have valance zero.

A graph is **connected** if for every pair of vertices there is at least one path connecting these two vertices. There are no vertices with valence zero.

According to **Euler's theorem**, a connected graph has an Euler circuit if the valence at each vertex is an even number. If any vertex has an odd valence, there cannot be an Euler circuit.

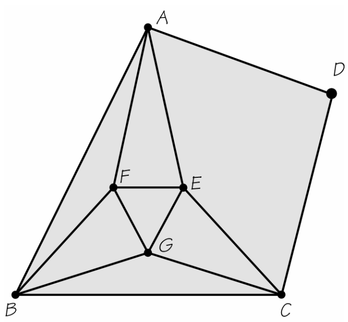
In finding an Euler circuit, never "disconnect" the graph by using an edge that is the only link between two parts of the graph not yet covered.

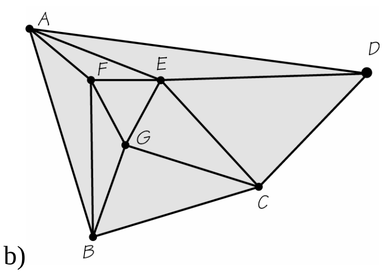
Altering a Graph to Obtain an Euler Circuit

If a graph has odd vertices, then any circuit must reuse at least one edge. Reusing an edge that joins two vertices is like adding a new edge between those vertices. Adding new edges for a circuit to produce an Euler circuit of a graph is called **eulerizing** the original graph.

Example

Determine an Euler circuit, if it exists, in the following. If it does not exist, explain why and Eulerize it.

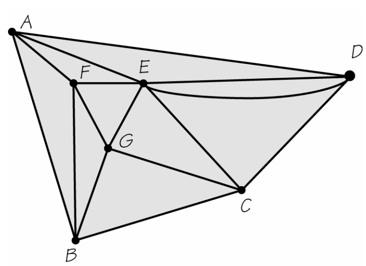
a) 

b) 

Solution

a) All the valences are even, thus an Euler circuit exists. Starting at vertex *A*, one such circuit would be *ABCDAFBGFEGCEA*.

b) Vertex *D* and *E* both have an odd valence. Adding an edge joining vertices *D* and *E* we have the following.

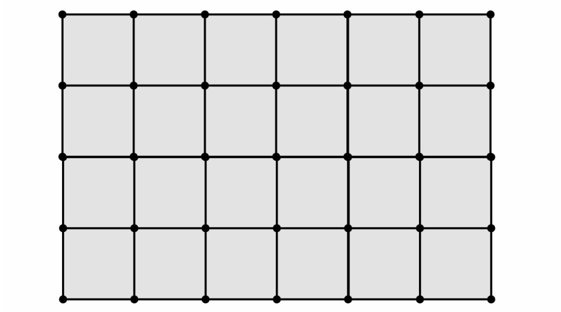


Starting at vertex A, one circuit would be *ABCDECGBFGEFAEDA*.

A systematic way to produce a good eulerization of a certain specialized "rectangular" graph is called the "**edge walker**" technique. There are basically three types of *m*-by-*n* rectangular arrays (both *m* and *n* are even, or both odd, or one even and one odd).

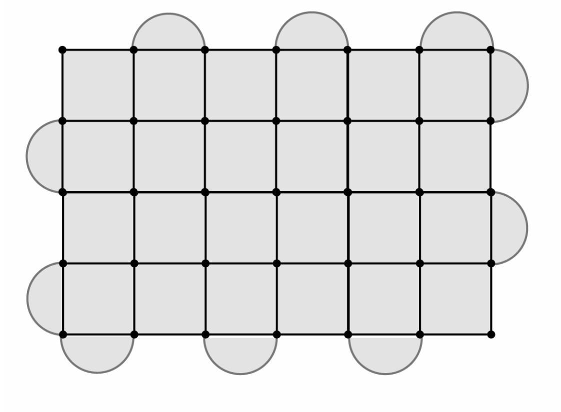
Example

Using the "edge walker" technique, eulerize this 4-by-6 rectangular graph.



Solution

Starting with the upper left-hand corner of the graph and traveling clockwise around the boundary of the rectangle, connect each odd vertex you encounter to the next vertex using a new edge.



☝Teaching Tip

Given the three types of rectangular arrays, ask students if they see any patterns in terms of the number of added edges or any types of symmetries that might exist.

The **Chinese postman problem** involves finding a circuit that reuses as few edges as possible.

☝Teaching Tip

Ask students in class to give examples as to where in real life one would want to reuse as few edges as possible. What would be the consequences if an optimal route is found? (decreasing cost, increasing profits, saving time etc.)

Digraphs

Applying the techniques described in this chapter to specific real tasks such as collecting garbage and reading electric meters results in complications that require modification of theories. Types of complications include one-way streets, multi-lane roads, obstructions, and a wide variety of human factors. When edges of a graph indicate a direction that must be traveled, we have a directed graph or **digraph**.

Example

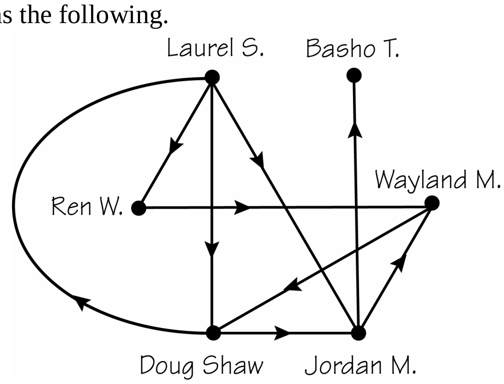
Dr. Doug Shaw is usually blamed for spreading mathematical rumors. After investigating rumor sources we have the following.

Basho T. heard the rumor from Jordan M. Wayland M. heard it from Ren W. and Jordan M. Doug Shaw heard it from Laurel S. and Wayland M. Ren W. heard it from Laurel S. Laurel S. heard it from Doug Shaw.

a) Can one conclude that Doug Shaw is the true culprit?
b) If Laurel S. incorrectly stated that Doug Shaw was the source of the rumor she heard, then what can one conclude?

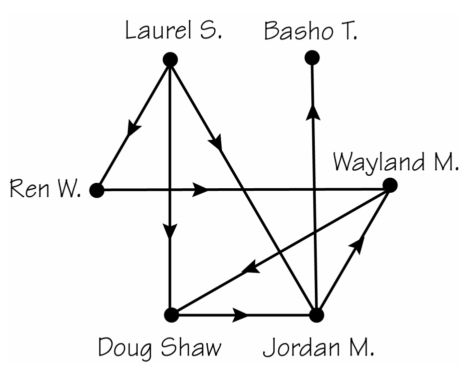
Solution

a) Make a digraph such as the following.



It is possible that Doug Shaw could have started the rumor. However, the only one that can be excluded from starting the rumor is Basho T.

b) Altering the digraph we have the following.

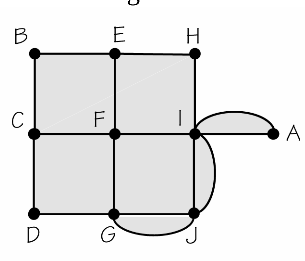


We can conclude that Doug Shaw can be exonerated and Laurel S. was the person that started the rumor.

Solutions to Student Study Guide ✎ Questions

Question 1

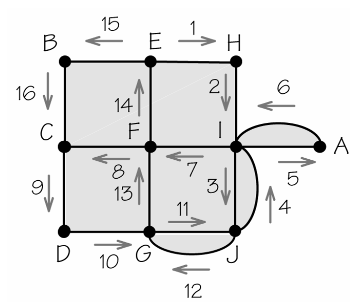
Given the following graph, which of the following is true?



a) An Euler circuit can be found
b) A circuit can be found if you are allowed to repeat using edges.

Solution

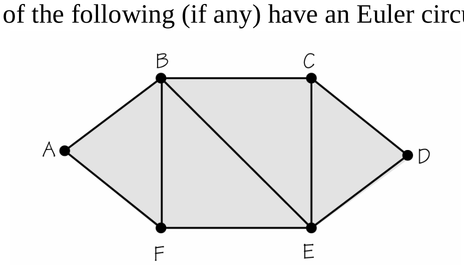
Consider the following path that begins at *E* as indicated by the numbers. An Euler circuit cannot be found. However, if one can repeat edges, then a circuit can be formed by continuing from *C* and going *CDGJIFE*.



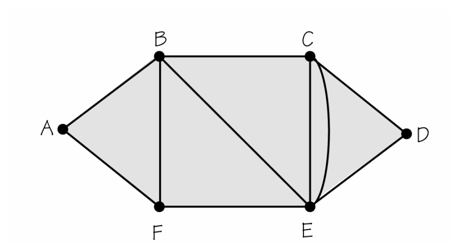
You may choose to note that an Euler circuit could be found if only we had an additional edge that connects *C* to *E*.

Question 2

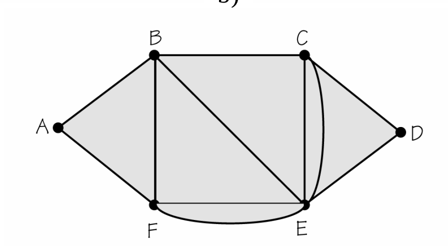
Which of the following (if any) have an Euler circuit?



a)



b)



c)

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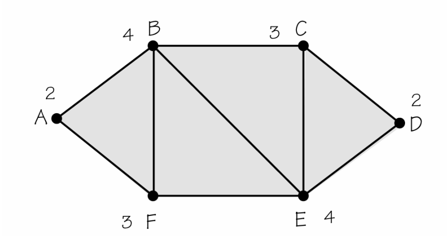
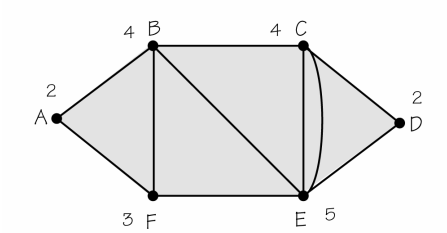
Solution

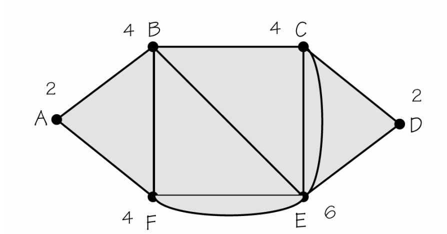
a) This graph has two odd vertices, so it cannot have an Euler circuit.



b) This graph has two odd vertices, so it cannot have an Euler circuit.



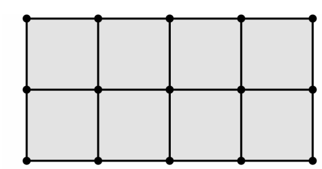
c) This graph has no odd vertices and is connected, so it must have an Euler circuit.



The only one that has an Euler circuit is therefore c).

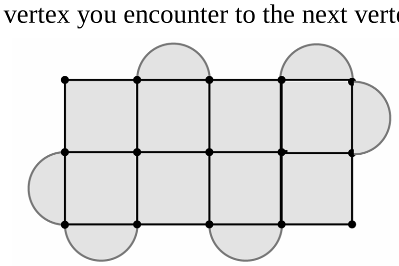
Question 3

Use the "edge walker" technique to eulerize this 2-by-4 rectangular graph. How many edges are added?



Solution

Starting with the upper left-hand corner of the graph and traveling clockwise around the boundary of the rectangle, connect each odd vertex you encounter to the next vertex using a new edge.



Thus, there are six additional edges.

Question 4

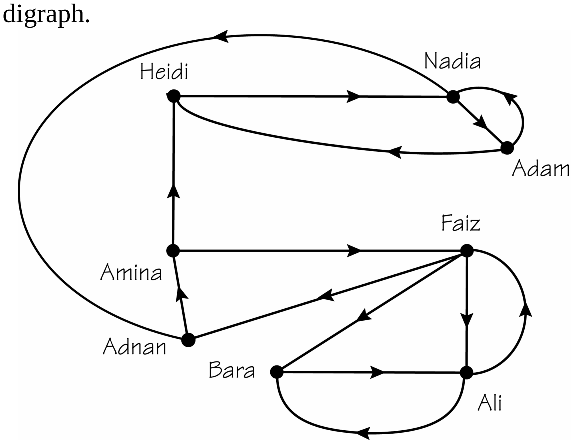
Suppose Amina needs to spread a message among friends. Because of a restriction in the flow of communications, her 7 friends can only talk as follows.

Friend	Can Talk to
Amina	Heidi, Faiz
Nadia	Adam
Ali	Faiz, Bara
Adnan	Amina
Faiz	Bara, Ali, Adnan
Adam	Heidi, Nadia
Bara	Ali
Heidi	Nadia

If Amina initiates the message and a continuous path is created, what is the minimum number of times the message is passed on in order to reach all of her friends? (The person that Amina originally delivers the message to counts as the first and the last person receiving it counts.)

Solution

Consider the following digraph.



The message could have been passed starting with Amina, then Heidi, Nadia, Adam, Nadia, Adnan, Amina, Faiz, Bara, Ali, for a total of 9 times