Chapter 14
Apportionment

Chapter Objectives

Check off these skills when you feel that you have mastered them.

- ☐ State the apportionment problem.
- ☐ Explain the difference between quota and apportionment.
- ☐ State the quota condition and be able to tell which apportionment methods satisfy it and which do not.
- ☐ Do the same for the house monotone and population monotone conditions.
- ☐ Know that some methods have bias in favor of large or small states.
- ☐ Recognize the difference in computing quotas between the Hamilton method and divisor methods.
- ☐ Calculate the apportionment of seats in a representative body when the individual population sizes and number of seats are given, using the methods of Hamilton, Jefferson, Webster, and Hill-Huntington.
- ☐ Be able to give at least three reasons to support the claim that Webster's method is the "best" apportionment method.
- ☐ Calculate the critical divisor for each state.
- ☐ Determine an apportionment using the method of critical multipliers.
- ☐ Explain why the Jefferson method and the method of critical multipliers do not satisfy the quota condition.

Guided Reading

Introduction

In many situations, a fixed number of places must be divided among several groups, in a way that is proportional to the size of each of the groups. A prime example of this is the division of the 435 seats in the U.S. House of Representatives among the 50 states. A problem arises because the exact allocations will usually involve fractional seats, which are not allowed. Various methods have been proposed to round the fraction to reach the total of 435, and four different methods have been used in the apportionment of the House during the past 200 years.

Section 14.1 The Apportionment Problem

Key idea

The **apportionment problem** is to round a set of fractions so that their sum is a fixed number. An **apportionment method** is a systematic procedure that solves the apportionment problem.

Key idea

The **standard divisor** is obtained by dividing the total population by the house size. A state's **quota** is obtained by dividing its population by the average district population. It represents the exact share the state is entitled to. However, since the quota is usually not a whole number, an apportionment method must be used to change each of the fractions into an integer.

Key idea

Two notations are used in this chapter for rounding.

$\lfloor q \rfloor$ means round down to the integer value $\lceil q \rceil$ means round up to the integer value

If q is an integer value, no rounding occurs.

Example A

Suppose that a country consists of three states, A, B, and C, with populations 11,000, 17,500, and 21,500, respectively. If the congress in this country has 10 seats, find the population of the average congressional district and the quota of each of the three states.

Solution

The total population is 50,000 and there are 10 districts, so that the average size is $\frac{50,000}{10} = 5000$.

We find each state's quota by dividing its population by 5000, the size of the average congressional district. Thus, A's quota is $\frac{11,000}{5000} = 2.2$, B's quota is $\frac{17,500}{5000} = 3.5$, and C's is $\frac{21,500}{5000} = 4.3$.

Example B

States A, B, and C have populations of 11,000, 17,500, and 21,500, respectively. If their country's congress has 10 seats, which of the following assignments of seats fulfills the conditions of an apportionment method?

a) A – 3 seats, B – 3 seats, C – 4 seats.
b) A – 2 seats, B – 3 seats, C – 5 seats.
c) A – 2 seats, B – 3 seats, C – 4 seats.
d) A – 2 seats, B – 4 seats, C – 4 seats.

Solution

a), b), and d) are legitimate apportionments, while c) is not. In c), the total number of seats allocated is 9, which is short of the house size of 10. The other three apportionments allocate 10 seats.

Section 14.2 The Hamilton Method

Key idea

The **lower quota** for a state is obtained by rounding its quota *down* to the nearest integer, while to obtain the **upper quota** we round the quota *up*.

Key idea

The **Hamilton method** first assigns each state its lower quota, and then distributes any remaining seats to the states having the largest fractional parts.

Example C

Consider the country previously listed, in which state *A* has a population of 11,000, state *B*, 17,500, and state *C*, 21,500. How would Hamilton allocate the 10 seats of the house?

Solution

The quotas are: *A* – 2.2, *B* – 3.5, *C* – 4.3. Each state initially receives its lower quota, so that *A* receives two seats, *B* three, and *C* four. The one remaining seat now goes to the state with the largest fractional part, which is *B*. Hence, *B* also gets four seats.

Example D

Enrollments for four mathematics courses are as follows.

Algebra–62
Geometry–52
Trigonometry–38
Calculus–28

Ten mathematics sections will be scheduled. How many sections of each of these four courses will be allocated by the Hamilton method?

Solution

Since there are 180 students and 10 sections, the average size of a section will be $\frac{180}{10} = 18$.

$$\text{Quota for Algebra} = \frac{\text{population}}{\text{divisor}} = \frac{62}{18} = 3.44$$

$$\text{Quota for Geometry} = \frac{\text{population}}{\text{divisor}} = \frac{52}{18} = 2.89$$

$$\text{Quota for Trigonometry} = \frac{\text{population}}{\text{divisor}} = \frac{38}{18} = 2.11$$

$$\text{Quota for Calculus} = \frac{\text{population}}{\text{divisor}} = \frac{28}{18} = 1.56$$

Distributing the lower quotas, Algebra gets three sections, Geometry and Trigonometry get two each, while Calculus gets one. The Hamilton method now assigns the remaining two sections to the courses having the largest fractional parts. Thus, Geometry gets the first section, and Calculus gets the second. There will be three sections of Algebra and Geometry, and two sections of Trigonometry and Calculus.

Question 1

Four classes need to assign 12 boxes of supplies according to their size using the Hamilton method. How many boxes does each class receive?

A–28 *B*–17 *C*–24 *D*–31

Answer

A will receive 3, *B* will receive 2, *C* will receive 3, and *D* will receive 4.

Key idea

The **Alabama paradox** occurs if a state loses one or more seats when the number of seats in the house is increased. The **population paradox** occurs if a state loses at least one seat, even though its population increases, while another state gains at least one seat, even though its population decreases. The Hamilton method is susceptible to both of these paradoxes.

Section 14.3 Divisor Methods

Key idea

A **divisor method** determines each state's apportionment by dividing its population by a common divisor *d* and rounding the resulting quotient. If the total number of seats allocated, using the chosen divisor, does not equal the house size, a larger or smaller divisor must be chosen. Finding a decisive divisor for a method of apportionment depends on how the fractions are rounded.

Key idea

In the **Jefferson method** the fractions are all rounded *down.*

Example E

In the example of the three states (*A:* 11,000, *B:* 17,500, *C:* 21,500), would a divisor of 5000 be decisive in apportioning a house of 10 seats, according to the Jefferson method?

Solution

With a divisor of 5000, the quotas for *A*, *B*, and *C* are 2.2, 3.5 and 4.3, respectively. Since all fractions are rounded down in the Jefferson method, *A* would receive two seats, *B* three, and *C* four, for a total of nine, one fewer than the number to be allocated. Hence, 5000 is not a decisive divisor.

Example F

In the example of the three states (*A:* 11,000, *B:* 17,500, *C:* 21,500), would a divisor of 4350 be decisive for the Jefferson method?

Solution

A's quota is $\frac{11{,}000}{4350} = 2.53$, *B*'s quota is $\frac{17{,}500}{4350} = 4.02$, and *C*'s is $\frac{21{,}500}{4350} = 4.94$. Rounding down gives *A* two seats, and *B* and *C* four each, for a total of ten. Thus the answer is yes.

✎ Question 2
Four classes need to assign 12 boxes of supplies according to their size. Would a divisor of 9 be decisive for the Jefferson method?

A–28   *B*–17   *C*–24   *D*–31

Answer
No.

Key idea
Finding a decisive divisor by trial and error can be quite tedious. A systematic method for doing so involves calculating the critical divisor for each state.

Example G
Consider, again, the case of the four mathematics courses and their enrollment totals:

Algebra–62

Geometry–52

Trigonometry–38

Calculus–28

Ten mathematics sections will be scheduled. Use the method of critical divisors to determine how many sections of each of these courses will be allocated by the Jefferson method.

Solution
In the Jefferson method, fractions are rounded down. Recall the following quotas.

$$\text{Quota for Algebra} = 3.44$$

$$\text{Quota for Geometry} = 2.89$$

$$\text{Quota for Trigonometry} = 2.11$$

$$\text{Quota for Calculus} = 1.56$$

The tentative allocation of sections is three to Algebra, two each to Geometry and Trigonometry, and one to Calculus. To determine which courses get the two remaining sections, we compute the critical divisors for each course. We obtain this for Algebra by dividing Algebra's enrollment, 62, by 4, which is one more than its tentative allocation.

$$\text{Critical divisor for Algebra} = \frac{62}{4} = 15.5$$

$$\text{Critical divisor for Geometry} = \frac{52}{3} = 17.33$$

$$\text{Critical divisor for Trigonometry} = \frac{38}{3} = 12.67$$

$$\text{Critical divisor for Calculus} = \frac{28}{2} = 14$$

Since Geometry's critical divisor is largest, it receives the next section. Before proceeding to allocate the next section, we recalculate Geometry's critical divisor, since it now has 3 sections. Its new divisor is $\frac{52}{4} = 13$. At this point, Algebra has the largest critical divisor, and it receives the last section. Thus, Algebra gets four sections, Geometry gets three, Trigonometry gets two and Calculus gets one.

Question 3

Four classes need to assign 12 boxes of supplies according to their size. Use the method of critical divisors to determine how many boxes each class will be allocated by the Jefferson method.

A–28 *B*–17 *C*–24 *D*–31

Answer

A will receive 3, *B* will receive 2, *C* will receive 3, and *D* will receive 4.

Key idea

The **Webster method** is also a divisor method, in which fractions greater than or equal to 0.5 $\left(\frac{1}{2}\right)$ are rounded up, while those less than 0.5 $\left(\frac{1}{2}\right)$ are rounded down.

Example H

Ten mathematics sections will be scheduled for four groups of students:

Algebra–62

Geometry–52

Trigonometry–38

Calculus–28

Use the method of critical divisors to determine how many sections of each of these courses will be allocated by the Webster method.

Solution

As we have seen, the quotas for the four subjects are 3.44, 2.89, 2.11, and 1.56, respectively. Now, however, we round the fractions in the normal way to obtain the Webster apportionment. Hence, Algebra, the fractional part of whose quota is less than 0.5, receives three seats in the tentative apportionment. Trigonometry's allocation is also rounded down to two, while Geometry and Calculus, both of whose fractional parts are greater than 0.5, have their allocations rounded up, Geometry to three and Calculus to two. Note: If the sum of the tentative allocations had been less than ten, we would have to find a smaller divisor. In either case, the method of critical divisors that we introduced in connection with the Jefferson method can be suitably modified to give the correct Webster apportionment. Trial and error can also be used to find an appropriate divisor. Algebra and Geometry get three sections each, while Trigonometry and Calculus get two.

Question 4

Four classes need to assign 12 boxes of supplies according to their size. Use the method of critical divisors to determine how many boxes each class will be allocated by the Webster method.

A–28 *B*–17 *C*–24 *D*–31

Answer

A will receive 3, *B* will receive 2, *C* will receive 3, and *D* will receive 4.

Key idea

In the Jefferson method, *all* fractions are rounded down, while in the Webster method, the cutoff point for rounding is 0.5. In the Hill-Huntington method, the cutoff point depends upon the size of the apportionment. If a state's quota is *n* seats, then its cutoff point is the geometric mean of *n* and $n+1$, which is $\sqrt{n(n+1)}$. For example, if a state's quota is between 4 and 5, then $n = 4$, so that the cutoff point is $\sqrt{4(5)} \approx 4.472$. Thus, if a state's quota is 4.37, the state would get just 4 seats, since $4.37 < 4.472$. On the other hand, if its quota is 4.48, which is greater than the cutoff point of 4.472, it would get five seats.

Example I

Ten mathematics sections will be scheduled for four groups of students:

Algebra–62
Geometry–52
Trigonometry–38
Calculus–28

Use the method of critical divisors to determine how many sections of each of these courses will be allocated by the Hill-Huntington method.

Solution

The quotas for the four courses are 3.44, 2.89, 2.11 and 1.56, respectively. We now compute the cutoff points for rounding up or down. For Algebra, the cutoff point is $\sqrt{3(4)} \approx 3.46.$ Since Algebra's quota is 3.44, which is less than the cutoff point, Algebra's tentative allocation is rounded *down.* The cutoff points for Geometry and Trigonometry are both $\sqrt{2(3)} \approx 2.45,$ so Geometry gets a third section, but Trigonometry does not. Finally, Calculus' cutoff point is $\sqrt{1(2)} \approx 1.41;$ so Calculus also gets an extra section, for a total of two. Algebra and Geometry each get three seats, while Trigonometry and Calculus get two each. Since the sum of the tentative allocations is 10, the apportionment process is completed.

Question 5

Four classes need to assign 12 boxes of supplies according to their size. Use the method of critical divisors to determine how many boxes each class will be allocated by the Hill-Huntington method.

A–28
B–17
C–24
D–31

Answer

A will receive 3, *B* will receive 2, *C* will receive 3, and *D* will receive 4.

Section 14.4 Which Divisor Method is Best?

Key idea

All of the apportionment methods attempt to minimize inequities between the states, although they all use different criteria to measure the inequities.

Key idea

The Webster method minimizes the inequity in the **absolute difference** in **representative shares**.

Key idea

The Hill-Huntington method minimizes the relative difference in **representative shares** or **district populations**.

Key idea

Given positive integers A and B with $A > B,$ the **absolute difference** is $A - B.$ The **relative difference** is $\frac{A-B}{B} \times 100\%.$

Example J
What is the relative difference between the numbers 8 and 13?

Solution
Because $A > B$, $A = 13$ and $B = 8$. The relative difference is $\frac{13-8}{8}\times 100\% = 0.625\times 100\% = 62.5\%.$

Example K
In the 1990 census, Alabama's population was 4,040,587, and Arizona's was 3,665,228. Alabama was apportioned 7 congressional seats, and Arizona received 6. What is the size of the average congressional district in each of these two states? Which of these two states is more favored by this apportionment?

Solution
Alabama, whose average district population is 577,227, is favored over Arizona, whose average district population is 610,871. Each of these averages is found by dividing the state's population by the number of seats.

Example L
What is the relative difference in average district population between Alabama (577,227) and Arizona (610,871)?

Solution
The relative difference is $\frac{610{,}871-577{,}227}{577{,}227}\times 100\% \approx 0.058\times 100\% = 5.8\%.$

Homework Help

Exercises 1 – 4
Carefully read Section 14.1 before responding to these exercises.

Exercises 5 – 13
Carefully read Section 14.2 before responding to these exercises.

Exercises 14 – 31
Carefully read Section 14.3 before responding to these exercises.

Exercises 32 – 44
Carefully read Section 14.4 before responding to these exercises.

Do You Know the Terms?

Cut out the following 27 flashcards to test yourself on Review Vocabulary. You can also find these flashcards at http://www.whfreeman.com/fapp7e.

Chapter 14 **Apportionment** $\lfloor q \rfloor$	**Chapter 14** **Apportionment** $\lceil q \rceil$
Chapter 14 **Apportionment** **Absolute difference**	**Chapter 14** **Apportionment** **Adjusted quota**
Chapter 14 **Apportionment** **Alabama paradox**	**Chapter 14** **Apportionment** **Apportionment method**
Chapter 14 **Apportionment** **Apportionment problem**	**Chapter 14** **Apportionment** **Critical divisor**

The result of rounding a number *q* up to the next integer.	**The result of rounding a number *q* down.**
The result of dividing a state's quota by a divisor other than the standard divisor. The purpose of adjusting the quotas is to correct a failure of the rounded quotas to sum to the house size.	**The result of subtracting the smaller number from the larger.**
A systematic way of computing solutions of apportionment problems.	**A state loses a representative solely because the size of the House is increased. This paradox is possible with the Hamilton method but not with divisor methods.**
The number closest to the standard divisor that can be used as a divisor of a state's population to obtain a new tentative apportionment for the state.	**To round a list of fractions to whole numbers in a way that preserves the sum of the original fractions.**

Chapter 14 **Apportionment** **Critical divisor for the Jefferson method causing change in tentative apportionment**	**Chapter 14** **Apportionment** **Critical divisor for the Webster method causing change in tentative apportionment**
Chapter 14 **Apportionment** **Critical divisor for the Hill-Huntington method causing change in tentative apportionment**	**Chapter 14** **Apportionment** **District population**
Chapter 14 **Apportionment** **Divisor method**	**Chapter 14** **Apportionment** **Geometric mean**
Chapter 14 **Apportionment** **Hamilton method**	**Chapter 14** **Apportionment** **Hill – Huntington method**

p stands for the state's population, and n is its tentative apportionment. to increase: $\frac{p}{n+0.5}$ to decrease: $\frac{p}{n-0.5}$	p stands for the state's population, and n is its tentative apportionment. to increase: $\frac{p}{n+1}$ to decrease: not necessary
A state's population divided by its apportionment.	p stands for the state's population, and n is its tentative apportionment. to increase: $\frac{p}{\sqrt{n(n+1)}}$ to decrease: $\frac{p}{\sqrt{n(n-1)}}$
For positive numbers A and B, the geometric mean is defined to be $\sqrt{AB}$.	One of many apportionment methods in which the apportionments are determined by dividing the population of each state by a common divisor to obtain adjusted quotas. The apportionments are calculated by rounding the adjusted quotas.
A divisor method that minimizes relative differences in both representative shares and district populations.	An apportionment method that assigns to each state either its lower quota or its upper quota. The states that receive their upper quotas are those whose quotas have the largest fractional parts.

Chapter 14 **Apportionment** **Jefferson method**	**Chapter 14** **Apportionment** **Lower quota**
Chapter 14 **Apportionment** **Population paradox**	**Chapter 14** **Apportionment** **Quota**
Chapter 14 **Apportionment** **Quota condition**	**Chapter 14** **Apportionment** **Relative difference**
Chapter 14 **Apportionment** **Representative share**	**Chapter 14** **Apportionment** **Standard divisor**

The integer part $\lfloor q_i \rfloor$ of a state's quota q_i.	**A divisor method invented by Thomas Jefferson, based on rounding all fractions down. Thus, if u_i is the adjusted quota of state i, the state's apportionment is $\lfloor u_i \rfloor$.**
The quotient p/s of a state's population divided by the standard divisor. The quota is the number of seats a state would receive if fractional seats could be awarded.	**A situation is which state A gains population and loses a congressional seat, while state B loses population (or increases population proportionally less than state A) and gains a seat. This paradox is possible with all apportionment methods *except* divisor methods.**
Subtracting the smaller number from the larger of two positive numbers, and expressing the result as a percentage of the smaller number.	**A requirement that an apportionment method should assign to each state either its lower quota or its upper quota in every situation. The Hamilton method satisfies this condition, but none of the divisor methods do.**
The ratio p/h of the total population p to the house size h. In a congressional apportionment problem, the standard divisor represents the average district population.	**A state's representative share is the state's apportionment divided by its population. It is intended to represent the amount of influence a citizen of that state would have on his or her representative.**

Chapter 14 **Apportionment** **Tentative apportionment**	**Chapter 14** **Apportionment** **Upper quota**
Chapter 14 **Apportionment** **Webster method**	

The result of rounding a state's quota *up* to a whole number. A state whose quota is *q* has an upper quota equal to $\lceil q \rceil$.	**The result of rounding a state's quota or adjusted quota to obtain a whole number.**
	A divisor method based on rounding fractions the usual way.

Practice Quiz

1. A county is divided into three districts with populations: Central, 3100; Western, 3500; Eastern, 1700. There are six seats on the county council to be apportioned. What is the quota for the Eastern district?

a. less than 1

b. 1

c. more than 1

2. The Hamilton method of apportionment can display

a. the population paradox, but not the Alabama paradox.

b. the Alabama paradox, but not the population paradox.

c. both the Alabama paradox and the population paradox.

3. The Jefferson method of apportionment

a. is a divisor method.

b. satisfies the quota condition.

c. is biased in favor of smaller states.

4. The Webster method of apportionment

a. is susceptible to the Alabama paradox.

b. favors smaller states.

c. can have ties.

5. A county is divided into three districts with populations Central, 3100; Western, 3500; Eastern, 1700. There are nine seats on the school board to be apportioned. What is the apportionment for the Eastern district using the Hamilton method?

a. 1

b. 2

c. 3

6. A county is divided into three districts with populations: Central, 3100; Western, 3500; Eastern, 1700. There are nine seats on the school board to be apportioned. What is the apportionment for the Eastern district using the Jefferson method?

a. 0

b. 1

c. 2

7. A county is divided into three districts with populations: Central, 3100; Western, 3500; Eastern, 1700. There are nine seats on the school board to be apportioned. What is the apportionment for the Eastern district using the Webster method?

a. 1

b. 2

c. 3

8. A county is divided into three districts with populations: Central, 3100; Northern, 1900; Southern, 2800. There are five seats on the zoning board to be apportioned. What is the apportionment for the Southern district using the Hill-Huntington method?

a. 1

b. 2

c. 3

9. The geometric mean of 7 and 8 is
a. 7.5.
b. more than 7.5.
c. less than 7.5.

10. The relative difference between 7 and 8 is

a. 1.

b. 12.5%.

c. 14.3%.

Word Search

Refer to pages 531 – 532 of your text to obtain the Review Vocabulary. There are 22 hidden vocabulary words/expressions in the word search below. $\lfloor q \rfloor$ and $\lceil q \rceil$ do not appear in the word search. It should be noted that spaces are removed as well as hyphens. *Critical Divisor* appears in the word search.

```
D N O D O H T E M R O S I V I D D R A D N A T S A
E L E U E C N E R E F F I D E V I T A L E R D N M
S E L L D R V T L R N H H L T A S N D O G O R G L
Q A Z X M N N O I T I D N O C A T O U Q H R O G T
N O A G W J E L M N T N E R G Z R P W T V E V L M
P A R S D A T O U Q R E W O L N I E E L R P G A C
P B I G R P O E M N H H E Y U C C M E R R R O L W
K S S M O P R E P R E S E N T A T I V E S H A R E
P O G T J O P O P U L A T I O N P A R A D O X E B
B L T D E R H A M M V C R A E C O E A R N P U T S
R U G O U T D T D T C T O M S R P H D A J X E A T
V T E H P I B K M Q F O N E O I U N J E E J S E E
S E O T P O N D O R E O M F K T L N U H F J G R R
R D M E E N R G M E I E W V A I A A S E F H A E M
R I E M R M E H M T T N J A E C T N T F E C B S E
H F T N Q E I T R K Y P L E J A I S E T R Q E P T
F F R O U N E O N H H J E F O L O H D E S E K O H
U E I T O T P E J C M F E T M D N K Q U O F S C O
Q R C L T P O P S X X S E M A I S L U H N H A G D
R E M I A R E A T O U Q N W G V E M O C M T A H B
T N E M N O I T R O P P A E V I T A T N E T M F L
E C A A P B E N X U S E H P T S Z R A G T E T G W
S E N H I L L H U N T I N G T O N M E T H O D C F
O I E X A E N L Q I C M T O I R G O O R O F R E H
A L A B A M A P A R A D O X L I S C G R D C A N E
```

1. ____________________
2. ____________________
3. ____________________
4. ____________________
5. ____________________
6. ____________________
7. ____________________
8. ____________________
9. ____________________
10. ____________________
11. ____________________
12. ____________________
13. ____________________
14. ____________________
15. ____________________
16. ____________________
17. ____________________
18. ____________________
19. ____________________
20. ____________________
21. ____________________
22. ____________________