

# Chapter 13

## Fair Division

### Chapter Objectives

*Check off these skills when you feel that you have mastered them.*

- ☐ Describe the goal of a fair-division problem.
- ☐ Define the term “player.”
- ☐ Use the adjusted winner procedure to determine the division of a set of objects among two players.
- ☐ Define the terms equitable, envy-free, and Pareto-optimal.
- ☐ Use the Knaster inheritance procedure to determine the division of a set of objects among more than two players.
- ☐ Use cake-division procedures to provide a fair allocation to two or more players.
- ☐ Use preference lists and the “bottom-up” approach to work out optimal strategies for two players taking turns to divide a collection of assets.

## Guided Reading

### Introduction

**Fair-division problems** arise in many situations, including divorce, inheritance, or the liquidation of a business. The problem is for the individuals involved, called the **players**, to devise a scheme for dividing an object or a set of objects in such a way that each of the players obtains a share that she considers fair. Such a scheme is called a **fair-division procedure**.

## Section 13.1 The Adjusted Winner Procedure

### Key idea

In the **adjusted winner procedure** for two players, each of the players is given 100 points to distribute over the items that are to be divided. Each party is then initially given those items for which he or she placed more points than the other party. The steps are as follows.

- Each party distributes 100 points over the items in a way that reflects their relative worth to that party.
- Each item is initially given to the party that assigned it more points. Each party then assesses how many of his or her own points he or she has received. The party with the fewest points is now given each item on which both parties placed the same number of points.
- Let  $A$  denote the party with the higher point total and  $B$  be the other party. Determine the following for each item.

$$\frac{A's \text{ point value of the item}}{B's \text{ point value of the item}}$$

Transfer items from  $A$  to  $B$ , in order of increasing **point ratio**, until the point totals are equal. (The point at which equality is achieved may involve a fractional transfer of one item.)

### Example A

Suppose that Adam and Nadia place the following valuations on the three major assets, which will be divided up:

Asset	Adam	Nadia
House	40	60
Car	30	20
Boat	30	20

- Who gets each of the assets initially?
- How many points does each of the players get according to this allocation?
- Is any further exchange of property necessary in order to equalize the allocations?

### Solution

- Since the person who bids the most points on an item is initially assigned that item, Nadia gets the house, while Adam gets the car and boat.
- They have 60 points each.
- No, both Adam and Nadia have received an equal number of points.

 **Example B**

Suppose that Adam and Nadia place the following valuations on the three major assets, which will be divided up:

Asset	Adam	Nadia
House	30	60
Car	30	10
Boat	40	30

- Who gets each of the assets initially?
- How many points does each of the players get according to this allocation?
- What further exchange of property is necessary in order to equalize the allocations?
- After this final reallocation of property, how many points does each of the players receive?

**Solution**

- Nadia gets the house, while Adam gets the car and boat.
- Adam gets 70 points (30 for the car and 40 for the boat), while Nadia gets 60 for the house.
- Some of Adam's property has to be transferred to Nadia, since he has received more points initially. To determine how much, we first compute the fractions.

$$\text{Car: } \frac{30}{10} \quad \text{Boat: } \frac{40}{30}$$

We now transfer part of Adam's assets to Nadia as follows.

- Let  $x$  equal the fraction of the boat which Adam will retain.
- To equalize the number of points, we solve the following equation.

$$30 + 40x = 60 + 30(1 - x)$$

$$30 + 40x = 60 + 30 - 30x$$

$$30 + 40x = 90 - 30x$$

$$70x = 60 \Rightarrow x = \frac{6}{7}$$

- Since Adam retains  $\frac{6}{7}$  of the boat, he must transfer  $1 - \frac{6}{7} = \frac{1}{7}$  of the boat to Nadia.
- Adam gets to keep the car, which is worth 30 points to him, he also retains a  $\frac{6}{7}$  share in the boat, which is worth  $40(\frac{6}{7}) = 34\frac{2}{7}$  points, giving him a total of  $64\frac{2}{7}$  points. Nadia gets the house, worth 60 points to her, and a  $\frac{1}{7}$  share in the boat, worth  $30(\frac{1}{7}) = 4\frac{2}{7}$  points, totaling  $64\frac{2}{7}$  points.

 **Key idea**

The adjusted winner procedure satisfies three important properties. The allocation must be:

- equitable:** Both players receive the same number of points.
- envy-free:** Neither player would be happier with what the other received.
- Pareto-optimal:** No other allocation, arrived at by any means, can make one party better off without making the other party worse off.

 **Question 1**

Suppose that Adam and Nadia place the following valuations on the three major assets, which will be divided up:

Asset	Adam	Nadia
House	30	40
Car	60	20
Boat	10	40

After this final reallocation of property, how many points does each of the players receive?

**Answer**

$68\frac{4}{7}$  points

## Section 13.2 The Knaster Inheritance Procedure

### Key idea

The adjusted winner procedure applies only when there are two heirs. With three or more, the **Knaster inheritance procedure** can be used.

### Example C

Ali, Bara, and Amina inherit a house. Their respective evaluations of the house are \$105,000, \$90,000, and \$120,000. Describe a fair division.

### Solution

The highest bidder gets the asset. Since Amina values the house at \$120,000, her share is  $\frac{\$120,000}{3} = \$40,000$  and pays \$80,000 into a **kitty**. Ali and Bara view their fair shares as  $\frac{\$105,000}{3} = \$35,000$  and  $\frac{\$90,000}{3} = \$30,000$ , respectively. After they take these amounts from the kitty, \$15,000 remains. This sum is then split equally among the three. Thus, Amina gets the house, and pays \$40,000 to Ali and \$35,000 to Bara.

### Example D

Ali, Bara, and Amina are heirs to an estate, which consists of a house, an antique car, and a boat (named the Baby Boo). Their evaluations of the items in the estate follow.

Asset	Ali	Bara	Amina
House	\$120,000	\$135,000	\$100,000
Car	\$40,000	\$30,000	\$22,000
Boat	\$80,000	\$60,000	\$70,000

Describe a fair division.

### Solution

It is possible to solve this problem by doing a calculation similar to the previous one for each of the three items, and then adding up the totals. However, an alternative method is to look at the entire estate. For example, Ali's bids indicate that he places a total value of \$240,000 on the estate, which means that his share is \$80,000. Similarly, Bara's estimate is \$225,000, with her share being \$75,000, while Amina's estimate is \$192,000, entitling him to \$64,000. Now Ali gets the car and boat having a total value (in his estimate) of \$120,000, which is \$40,000 over his fair share. Hence, he places \$40,000 into a kitty. In the same way, Bara gets the house and places \$60,000 in the kitty. Amina then removes her share of \$64,000, which leaves \$36,000 in the kitty. This amount is then divided equally among the three heirs.

Bara gets the house and pays \$48,000 to Amina. Ali gets the antique car and the boat (Baby Boo) and pays \$28,000 to Amina. Amina gets no items, but receives a total of \$76,000 from Bara and Ali.

## Section 13.3 Taking Turns

### Key idea

Two people often split a collection of assets between them using the simplest and most natural of fair-division schemes: **taking turns**. If they each know the other's preferences among the assets, their best strategies may not be to choose their own most highly preferred asset first.

### Example E

Rachel and Kari attend a car auction and together win the bidding for a collection of four classic cars: a '59 Chevy (C), a '48 Packard (P), a '61 Austin-Healey (A) and a '47 Hudson (H). They will take turns to split the cars between them, with Rachel choosing first. Here are their preferences:

	Rachel's Ranking	Kari's Ranking
Best	C	P
Second best	P	H
Third best	H	C
Worst	A	A

If Rachel chooses the Chevy (her favorite) first, then Kari may choose her favorite, the Packard; Rachel loses her second best choice. Is there a way for Rachel to choose strategically to end up with both of her top two?

### Solution

Rachel should choose the Packard first. Kari must now resort to choosing the Hudson (her second favorite). That leaves the Chevy available to Rachel in the second round.

### Key idea

There is a general procedure for rational players, called the “bottom-up-strategy,” for optimizing individual asset allocations when taking turns. It is based on two principles:

- You should never choose your least-preferred available option.
- You should never waste a choice on an option that will come to you automatically in a later round.

### Example F

What would the allocation of cars be in the last example if Rachel and Kari use the bottom-up strategy? Assume that Rachel goes first.

	Rachel's Ranking	Kari's Ranking
Best	C	P
Second best	P	H
Third best	H	C
Worst	A	A

### Solution

Rachel: P    C  
Kari:        H    A



### Question 2

Who would get the worst choice if Kari went first?

### Answer

Rachel

## Section 13.4 Divide-and-Choose

### ✚ Key idea

A fair-division procedure known as **divide-and-choose** can be used if two people want to divide an object such as a cake or a piece of property. One of the people divides the object into two pieces, and the second person chooses either of the two pieces.

## Section 13.5 Cake-Division Procedures: Proportionality

### ✚ Key idea

The divide-and-choose method cannot be used if there are more than two players. A **cake-division procedure** is a scheme that  $n$  players can use to divide a cake among themselves in a way which satisfies each player.

### ✚ Key idea

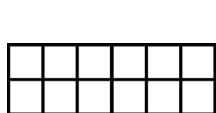
A cake-division scheme is said to be **proportional** if each player's strategy guarantees him a piece of size at least  $\frac{1}{n}$  in his own estimation. It is **envy-free** if each player feels that no other player's piece is bigger than the one he has received.

### ✚ Key idea

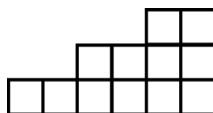
When there are three players, the **lone-divider method** guarantees proportional shares.

### 🌀 Example G

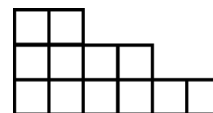
Suppose that players 1, 2, and 3 view a cake as follows.



Player 1



Player 2



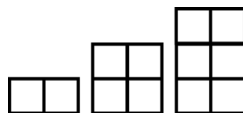
Player 3

If player 1 cuts the cake into what she perceives as three equal pieces, draw three diagrams to show how each player will view the division.

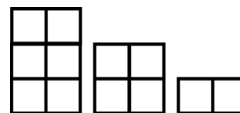
### Solution



Player 1



Player 2



Player 3

### 🌀 Example H

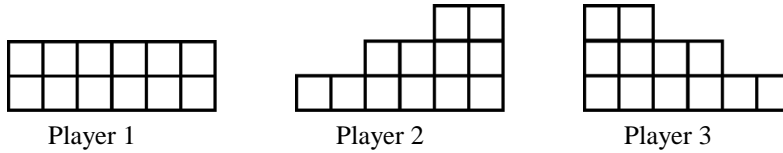
If player 2 is the first to choose, which piece will he pick, and how much of the cake does he believe he is getting?

### Solution

Player 2 believes that the pieces are cut unequally, with the right-most piece being the largest one. Thus Player 2 will select the right-most piece, which he believes is  $\frac{1}{2}$  of the cake.

### Example I

Suppose that Players 1, 2, and 3 view a cake as follows.



- a) Which of the two remaining pieces will Player 3 choose, and how much of the cake does she believe she is getting?
- b) Which piece will be left for Player 1, and how much of the cake does she believe she is getting?

### Solution

- a) Player 3 believes that the pieces are cut unequally, with the left-most piece being the largest. Player 3 will choose the left-most piece, which she believes is  $\frac{1}{2}$  of the cake.
- b) Player 1 believes that all three pieces are of equal size. The middle piece will remain for player 1, and she believes that it is  $\frac{1}{3}$  of the cake.

### Key idea

The **last-diminisher method** is a more complicated procedure, which guarantees proportional shares with any number of players. Like the lone-divider method, it is proportional but not envy-free.

## Section 13.6 Cake-Division Procedures: The Problem of Envy

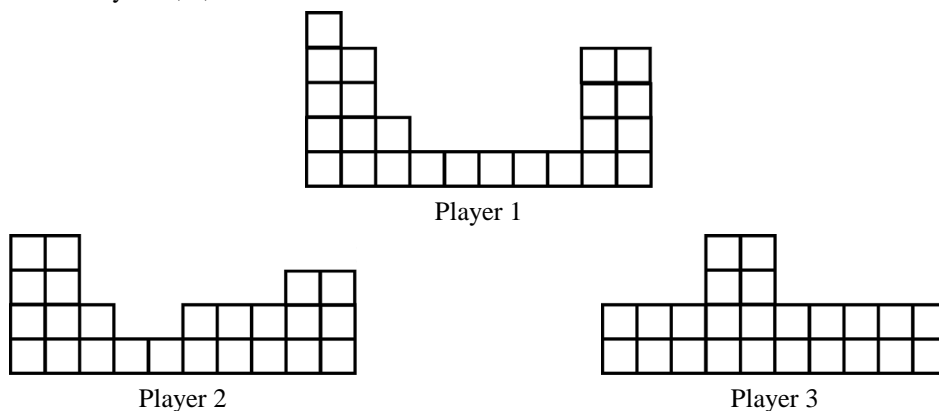
### Key idea

Envy-free cake-division schemes are still more complicated. The **Selfridge-Conway procedure** solves this problem for the case of three players. For more than three players, a scheme which involves a **trimming procedure** has been developed. In it, proportions of the cake are successively allocated in an envy-free fashion, with the remaining portions diminishing in size. Eventually, the remainder of the cake is so small that it will not affect the perception of each player that he or she has obtained the largest piece. If players cannot find a way to make an item fairly divisible, there may be no alternative but to sell it and share the proceeds equally. The procedure is as follows.

- Step 1 Player 1 cuts the cake into three pieces he considers to be the same size. He hands the three pieces to Player 2.
- Step 2 Player 2 trims at most one of the three pieces so as to create at least a two-way tie for largest. Setting the trimmings aside, Player 2 hands the three pieces (one of which may have been trimmed) to Player 3.
- Step 3 Player 3 now chooses, from among the three pieces, one that he considers to be at least tied for largest.
- Step 4 Player 2 next chooses, from the two remaining pieces, one that she considers to be at least tied for largest, with the proviso that if she trimmed a piece in Step 2, and Player 3 did not choose this piece, then she must now choose it.
- Step 5 Player 1 receives the remaining piece.

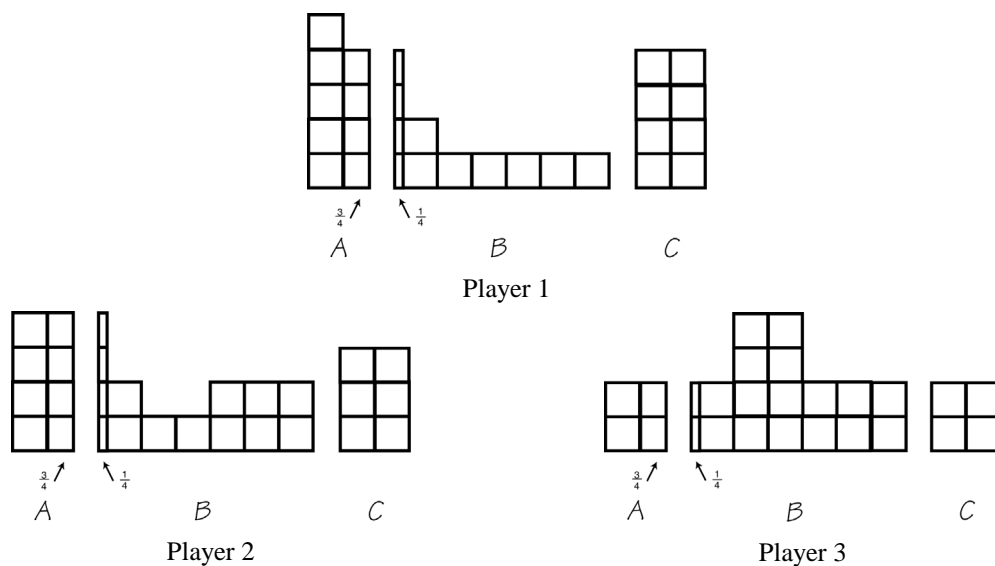
### Example J

Suppose that Players 1, 2, and 3 view a cake as follows.



Provide a total of three drawings that show how each player views the division by Player 1 into three pieces he or she considers to be the same size or value. Label the pieces *A*, *B*, and *C*. Assume the cuts correspond to vertical lines. Identify pieces that are acceptable to Players 2 and 3.

### Solution



Since an acceptable piece would be at least of size  $\frac{24}{3} = 8$ , Player 2 would find piece *B* acceptable. Player 3 also would find piece *B* acceptable.

### Question 3

If Player 2 did the cutting, how many pieces would be acceptable to Players 1 and 3?

### Answer

Two for Player 1 and 1 for Player 3.

## Homework Help

Exercises 1 – 6

Carefully read Section 13.1 before responding to these exercises. Remember when calculating point ratios you calculate them relative to the person that has more total points in order to share with the other person. In Exercises 5 and 6, answers will vary.

Exercise 7

Carefully read Section 13.1 and pay special attention to the theorem on page 480.

Exercises 8 – 12

Carefully read Section 13.2 before responding to these exercises. In Exercise 10, answers will vary.

Exercises 13 – 18

Carefully read Section 13.3 before responding to these exercises. The following may be helpful.

### Exercise 13

Bob: \_\_\_\_\_  
Carol: \_\_\_\_\_

### Exercise 14

Carol: \_\_\_\_\_  
Bob: \_\_\_\_\_

### Exercise 15

Mark: \_\_\_\_\_  
Fred: \_\_\_\_\_

### Exercise 16

Fred: \_\_\_\_\_  
Mark: \_\_\_\_\_

### Exercise 17

Donald: \_\_\_\_\_  
Ivana: \_\_\_\_\_

### Exercise 18

Ivana: \_\_\_\_\_  
Donald: \_\_\_\_\_

Exercises 19 – 22

Carefully read Section 13.4 before responding to these exercises.

Exercises 23 – 28

Carefully read Section 13.4 before responding to these exercises.

Exercises 29 – 30

Carefully read Section 13.5 before responding to these exercises.

## Do You Know the Terms?

Cut out the following 16 flashcards to test yourself on Review Vocabulary. You can also find these flashcards at <http://www.whfreeman.com/fapp7e>.

<b>Chapter 13</b> <b>Fair Division</b>  <b>Adjusted winner procedure</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Bottom-up strategy</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Cake-division procedure</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Convention of the Law of the Sea</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Divide-and-choose</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Envy-free</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Equitable</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Knaster inheritance procedure</b>

<p>A bottom-up strategy is a strategy under an alternating procedure in which sophisticated choices are determined by working backwards.</p>	<p>The allocation resulting from this procedure is equitable, envy-free, and Pareto-optimal. It requires no cash from either player, but one of the objects may have to be divided or shared by the two players.</p>
<p>An agreement based on divide-and-choose that protects the interests of developing countries in mining operations under the sea.</p>	<p>Such procedures involve finding allocations of a single object that is finely divisible, as opposed to the situation encountered with either the adjusted winner procedure or Knaster's procedure. Each player has a strategy that will guarantee that player a piece with which he or she is "satisfied," even in the face of collusion by the others.</p>
<p>A fair-division procedure is said to be envy-free if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large (or at least as desirable) as that received by any other player, no matter what the other players do.</p>	<p>A fair-division procedure for dividing an object or several objects between two players. This method produces an allocation that is both proportional and envy-free (the two being equivalent when there are only two players).</p>
<p>A fair-division procedure for any number of parties that begins by having each player (independently) assign a dollar value (a "bid") to the item or items to be divided so as to reflect the absolute worth of each object to that player. It never requires the dividing or sharing of an object, but it may require that the players have a large amount of cash on hand.</p>	<p>An allocation (resulting from a fair-division procedure like adjusted winner) is said to be equitable if each player believes he or she received the same fractional part of the total value</p>

<b>Chapter 13</b> <b>Fair Division</b>  <b>Last-diminisher method</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Lone-divider method</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Pareto-optimal</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Point ratio</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Preference lists</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Proportional</b>
<b>Chapter 13</b> <b>Fair Division</b>  <b>Selfridge-Conway envy-free procedure</b>	<b>Chapter 13</b> <b>Fair Division</b>  <b>Taking turns</b>

<p>A cake-division procedure introduced by Hugo Steinhaus in the 1940s. It works only for three players and produces an allocation that is proportional but not, in general, envy-free.</p>	<p>A cake-division procedure introduced by Stefan Banach and Bronislaw Knaster in the 1940s. It works for any number of players and produces an allocation that is proportional but not, in general, envy-free.</p>
<p>The fraction in which the numerator is the number of points in one party placed on an object and the denominator is the number of points the other party placed on the object.</p>	<p>When no other allocation, achieved by any means whatsoever, can make any one player better off without making some other player worse off.</p>
<p>A fair-division procedure is said to be proportional if each of <math>n</math> players has a strategy that can guarantee that player a share of whatever is being divided that he or she considers to be at least <math>1/n</math> of the whole in size or value.</p>	<p>Rankings of the items to be allocated, from best to worst, by each of the participants.</p>
<p>A fair-division procedure in which two or more parties alternate selecting objects.</p>	<p>A cake-division procedure introduced independently by John Selfridge and John Conway around 1960. It works only for three players but produces an allocation that is envy-free (as well as proportional).</p>

## Practice Quiz

1. In a fair-division procedure, the goal is for each person to receive
  - a. an identical portion.
  - b. what is perceived as an identical portion.
  - c. what is perceived as an acceptable portion.
2. Using the adjusted winner procedure to divide property between two people, each person always receives
  - a. exactly 50 points of value.
  - b. at least 50 points of value.
  - c. no more than 50 points of value.

3. Amina and Ali must make a fair division of three cars. They assign points to the cars, and use the adjusted winner procedure. Which car is divided between Amina and Ali?

	Amina	Ali
Red car	40	20
White car	25	30
Blue car	35	50

- a. Red car
  - b. White car
  - c. Blue car
4. Bara and Jenna must make a fair division of three art items. They assign points to the items, and use the adjusted winner procedure. Who ends up with the Sculpture?

	Bara	Jenna
Painting	35	50
Sculpture	40	35
Tapestry	25	15

- a. Bara
  - b. Jenna
  - c. Bara and Jenna share the Sculpture
5. Alex and Kari use the Knaster inheritance procedure to fairly divide a coin collection. Alex bids \$700 and Kari bids \$860. What is the outcome?
  - a. Kari gets the coins and pays Alex \$430.
  - b. Kari gets the coins and pays Alex \$390.
  - c. Kari gets the coins and pays Alex \$160.

6. Four children bid on two objects. Using the Knaster inheritance procedure, who ends up with the most cash money?

	Caleb	Quinn	Nadia	Adam
House	\$81,000	\$75,000	\$82,000	\$78,000
Car	\$12,000	\$11,000	\$10,000	\$13,000

- a. Caleb  
b. Quinn  
c. Nadia
7. Four children bid on two objects. Using the Knaster inheritance procedure, what does Adam do?

	Caleb	Quinn	Nadia	Adam
House	\$81,000	\$75,000	\$82,000	\$78,000
Car	\$12,000	\$11,000	\$10,000	\$13,000

- a. takes car and takes cash  
b. takes car and pays cash  
c. takes car only
8. Suppose John and Dean use the bottom-up strategy, taking turns to divide several items, ranked in order as shown below. If John goes first, what is his first choice?

	John's ranking	Dean's Ranking
1 <sup>st</sup> choice	Clock	Radio
2 <sup>nd</sup> choice	Radio	Phone
3 <sup>rd</sup> choice	Toaster	Toaster
4 <sup>th</sup> choice	Phone	Clock

- a. Clock  
b. Radio  
c. Toaster
9. Suppose seven people will share a cake using the last-diminisher method. To begin, Matthew cuts a piece and passes it to Dani. Dani trims the piece and passes it on to each of the remaining five people, but no one else trims the piece. Then
- a. Matthew gets this piece.  
b. Dani gets this piece.  
c. the last person who is handed the piece keeps it.
10. Which of the following procedures is envy-free?
- a. Lone-divider  
b. Selfridge-Conway  
c. Last-diminisher

## Word Search

Refer to pages 493 – 494 of your text to obtain the Review Vocabulary. There are 15 hidden vocabulary words/expressions in the word search below. *Selfridge-Conway envy-free procedure* does not appear in the word search. It should be noted that spaces are removed as well as hyphens.

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V T N A Z A L R T C A O S E P O E J J A O O R J B E Y Q I L
H T E K I E Z W M V I K M C Z A N P P S O G Z N D Z K Q G D
E O T L P S N O I I S A M I Q E E A U O N H M X P E O L W D
V I T O R Y P R S A E Z S S H E Q I P T D R L Q T M S H T C
R C C P I Z S T G J I C F T O D G S I Y T C U W C C C M E T
E C H O R M A Y P N K X R G F O O R L K M A F T S W A D P T
E O R E N Z N E M Y N M T E O V F X C D S H E E G Q N I O W
A L E C D V P D P I D H I M N E L E Y Y E R Z R E N D C D P
O P E R U D E C O R P R E N N I W D E T S U J D A O I A R V
M R I H L C C N L C I X G T O B S O O X Y E H C H L E K D C
E E J T Q T L N T C P E O L T C V R N A L N H T R I E E A S
Z F L A S T D I M I N I S H E R M E T H O D E S O E C D M T
E E M S E F S C E Q O Y G E T A R T S P U M O T T O B I U C
E R U D E C O R P E C N A T I R E H N I R E T S A N K V E T
V E H H Z N N N G S E S O O H C D N A E D I V I D J D I L G
N N L M A S I N H L G R G F S Z C E D X O I X E N Z R S B E
F C R P Z O R Z A R F V I Q T E S I T R O R P X B W L I A S
S E N B K P S K C A Q P M S A H V S O Y D T A I T N M O T S
Q L I E F E O A F M E R O X H I E V Y P E N R D L U E N I A
F I D E S N S T W I V E O I D W S L D Q N R E T A I C P U H
T S G L Y S E E R F Y V N E F G G R A V F E T Y O H Z R Q E
D T O P S R N F S A R M N L P D S S C W R B O X S E Y O E R
O S N I N Q I E M W U O R L A N O I T R O P O R P S P C T W
M I E C T R I O A F L A L M M L O S G H B F P D O E S E W I
F I V J A K E Z H F E G I N H D H F Y O S T T I W I G D E B
L P R H S R D S S J E E B D D K Q F Q K I E I H O E L U S U
A P D L T T T V F Z N E E H U X Z T I H Z E M K E L R R T N
O X V F H C K K U P O I P O I N T R A T I O A E P S Q E Q H
Q P H C T A H C R E G L H E E B W M W T N O L S J A E C R I
H E R E A N E D L M Y P C M E E S E E H E E S E C E A A K N

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1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
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