Chapter 10
The Manipulability of Voting Systems

Chapter Objectives

Check off these skills when you feel that you have mastered them.

- ☐ Explain what is meant by voting manipulation.
- ☐ Determine if a voter, by a unilateral change, has manipulated the outcome of an election.
- ☐ Determine a unilateral change by a voter that causes manipulation of an election using the Borda count voting method.
- ☐ Explain the three conditions to determine if a voting system is manipulable.
- ☐ Discuss why the majority method may not be appropriate for an election in which there are more than two candidates.
- ☐ Explain four desirable properties of Condorcet's method.
- ☐ Explain why Condorcet's method is non-manipulable by a unilateral change in vote.
- ☐ Recognize when the Borda count method can be manipulated and when it can't.
- ☐ Determine a unilateral change by a voter that causes a no-winner manipulation of an election in Condorcet's method.
- ☐ Determine a unilateral change by a voter that causes manipulation of an election in the plurality runoff method.
- ☐ Determine a unilateral change by a voter that causes manipulation of an election in the Borda count voting method.
- ☐ Determine a unilateral change by a voter that causes manipulation of an election in the Hare method.
- ☐ Determine a group change by a block of voters that causes manipulation of an election in the plurality method.
- ☐ Determine an agenda change by a voter that causes manipulation of an election in the sequential pairwise voting method, with agenda.
- ☐ Explain the Gibbard-Satterthwaite theorem (GS theorem) and its weak version.
- ☐ Explain the chair's paradox and what is meant by *weakly dominates* as it relates to a voting strategy.

Guided Reading

Introduction

The expression, *Honesty is the best policy*, may not be applicable when it comes to voting. Voting in a strategic manner is called **manipulation**. This occurs when a voter casts a ballot, which does not represent his or her actual preference. These types of ballots are referred to as **insincere** or **disingenuous ballots**. In this chapter, you will be looking at the manipulability of different voting methods.

🗝 Key idea

In manipulating an outcome, a voter casts a vote that is not consistent with his or her overall preference in terms of order. His or her top choice should naturally be the one that they want to see win the election. By casting a vote in which the ordering of the non-preferred candidates are listed can change the outcome in favor of the preferred candidate. A voting system is **manipulable** if there exists *at least one* way a voter can achieve a preferred outcome by changing his or her preference ballot.

🗝 Key idea

The Borda count method is subject to manipulation under certain conditions. One of these conditions is having three voters and four candidates. Note: Other conditions will be discussed later.

Example A

Consider the following election with four candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*A*	*A*	*B*	*D*
Second	*A*	*B*	*B*	*A*	*C*
Third	*D*	*C*	*D*	*C*	*A*
Fourth	*C*	*D*	*C*	*D*	*B*

Show that if the Borda count is being used, the voter on the left can manipulate the outcome (assuming the above ballot represents his true preferences).

Solution

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	2×3	2×2	1×1	0×0	11
B	2×3	2×2	0×1	1×0	10
C	0×3	1×2	2×1	2×0	4
D	1×3	0×2	2×1	2×0	5

With the given ballots, the winner using the Borda count is *A*. However, if the leftmost voter changes his or her preference ballot, we have the following.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*A*	*A*	*B*	*D*
Second	*C*	*B*	*B*	*A*	*C*
Third	*D*	*C*	*D*	*C*	*A*
Fourth	*A*	*D*	*C*	*D*	*B*

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Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	2×3	1×2	1×1	1×0	9
B	2×3	2×2	0×1	1×0	10
C	0×3	2×2	2×1	1×0	6
D	1×3	0×2	2×1	2×0	5

With the new ballots, the winner using the Borda count is *B*.

🗝 Key idea

The term **unilateral change** is used when one voter (as opposed to a group of voters) changes his or her ballot.

🗝 Key idea

Definition of Manipulability: A voting system is said to be **manipulable** if there exist two sequences of preference list ballots and a voter (call the voter *j*) such that

- Neither election results in a tie. (Ties in an election present a problem in determining sincere preference.)
- The only ballot change is by voter *j* (This is a unilateral change)
- Voter *j* prefers the outcome (overall winner) of the second election even though the first election showed his or her true (overall order) preferences.

Section 10.1 Majority Rule and Condorcet's Method

🗝 Key idea

In this section, like in Chapter 9, it is assumed that the number of voters is odd.

🗝 Key idea

(Restated from Chapter 9) When there are only two candidates or alternatives, May's theorem states that **majority rule** is the only voting method that satisfies three desirable properties, given an odd number of voters and no ties. The three properties satisfied by majority rule are:

1. All voters are treated equally.
2. Both candidates are treated equally.
3. If a single voter who voted for the loser, *B*, changes his mind and votes for the winner, *A*, then *A* is still the winner. This is what is called **monotone.**

Because in the two-candidate case, there are only two possible rankings (*A* over *B* or *B* over *A*), the monotonic property of majority rule is equivalent to the non-manipulability of this voting system, given the voter and candidate restriction.

🗝 Key idea

Condorcet's method is non-manipulable by a unilateral change in vote. This statement **does not** consider the possibility that an election manipulation could result in no winner. It is possible to go from having a winner to having no winner by unilateral change in vote. If this is a desired outcome by the disingenuous voter, then Condorcet's method can be altered by a unilateral change in vote.

Example B

Consider the following election with four candidates and three voters.

Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*C*	*A*	*B*
Second	*A*	*C*	*A*
Third	*D*	*D*	*D*
Fourth	*B*	*B*	*C*

Show that if Condorcet's method is being used, the voter on the left can change the outcome so that there is no winner.

Solution

There are 6 one-on-one contests as summarized below.

A* vs *B	***A:***	**2**	***B:***	**1**
A* vs *C	***A:***	**2**	***C:***	**1**
A* vs *D	***A:***	**3**	***D:***	**0**
B vs *C*	*B:*	1	*C:*	2
B vs *D*	*B:*	1	*D:*	2
C vs *D*	*C:*	2	*D:*	1

Since *A* can beat the other candidates in a one-on-one contest, *A* is declared the winner by Condorcet's method.

Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*C*	*A*	*B*
Second	*B*	*C*	*A*
Third	*A*	*D*	*D*
Fourth	*D*	*B*	*C*

A vs *B*	*A:*	1	*B:*	2
A vs *C*	*A:*	2	*C:*	1
A vs *D*	*A:*	3	*D:*	0
B vs *C*	*B:*	1	*C:*	2
B vs *D*	*B:*	2	*D:*	1
C vs *D*	*C:*	2	*D:*	1

Since no candidate can beat all other candidates in a one-on-one contest, there is no winner by Condorcet's method.

Section 10.2 Other Voting Systems for Three of More Candidates

🗝 Key idea

The Borda count method is non-manipulable for three candidates, regardless of the number of voters.

🗝 Key idea

The Borda count method is manipulable for four or more candidates (and two or more voters).

Example C

Consider the following election with four candidates and two voters.

Election 1

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*C*	*B*
Third	*B*	*A*
Fourth	*D*	*D*

Show that if the Borda count is being used, the voter on the left can manipulate the outcome (assuming the above ballot represents his true preferences).

Solution

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	0×2	1×1	0×0	4
B	0×3	1×2	1×1	0×0	3
C	1×3	1×2	0×1	0×0	5
D	0×3	0×2	0×1	2×0	0

With the given ballots, the winner using the Borda count is *C*. However, if the left-most voter changes his or her preference ballot, we have the following.

Election 2

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*D*	*B*
Third	*B*	*A*
Fourth	*C*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	0×2	1×1	0×0	4
B	0×3	1×2	1×1	0×0	3
C	1×3	0×2	0×1	1×0	3
D	0×3	1×2	0×1	1×0	2

With the new ballots, the winner using the Borda count is *A*.

Question 1

Consider Example 2 from the text. Is it possible to use the preference list ballots from Example C (last page) to create an example of manipulating the Borda count with five candidates and six voters? Justify your yes/no response.

Answer

Yes.

Key idea

The plurality runoff rule is manipulable.

Example D

Consider the following election with four candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*C*	*B*	*D*
Second	*B*	*B*	*B*	*A*	*B*
Third	*C*	*A*	*A*	*C*	*A*
Fourth	*A*	*D*	*D*	*D*	*C*

Show how the left-most voter can secure a **more preferred** outcome by a unilateral change of ballot using the plurality runoff rule.

Solution

Since *C* and *D* have the most number of first-place votes, *A* and *B* are eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*C*	*C*	*D*
Second	*C*	*D*	*D*	*D*	*C*

Since *C* has the most number of first-place votes, the winner using the plurality runoff rule is *C*. But the winner becomes *B* if the leftmost voter changes his or her ballot as the following shows.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*C*	*B*	*D*
Second	*D*	*B*	*B*	*A*	*B*
Third	*C*	*A*	*A*	*C*	*A*
Fourth	*A*	*D*	*D*	*D*	*C*

Since *B* and *C* have the most number of first-place votes, *A* and D are eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*C*	*B*	*B*
Second	*C*	*B*	*B*	*C*	*C*

Since *B* has the most number of first-place votes, the winner using the plurality runoff rule is *B*. For the first voter, having *B* win the election was **more preferred** than having *C* win the election.

🗝 Key idea

The Hare system is manipulable.

🖋 Example E

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*C*	*B*	*D*
Second	*B*	*B*	*B*	*A*	*B*
Third	*C*	*A*	*A*	*C*	*A*
Fourth	*A*	*D*	*D*	*D*	*C*

Show how the left-most voter can secure a **more preferred** outcome by a unilateral change of ballot using the Hare system.

Solution

A has the fewest first-place votes and is thus eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*C*	*B*	*D*
Second	*B*	*B*	*B*	*C*	*B*
Third	*C*	*D*	*D*	*D*	*C*

B now has the fewest first-place votes and is eliminated

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*D*	*C*	*C*	*C*	*D*
Second	*C*	*D*	*D*	*D*	*C*

D now has the fewest first-place votes and is eliminated, leaving *C* as the winner.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*C*	*B*	*D*
Second	*D*	*B*	*B*	*A*	*B*
Third	*C*	*A*	*A*	*C*	*A*
Fourth	*A*	*D*	*D*	*D*	*C*

A has the fewest first-place votes and is eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*C*	*B*	*D*
Second	*D*	*B*	*B*	*C*	*B*
Third	*C*	*D*	*D*	*D*	*C*

D now has the fewest first-place votes and is eliminated

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*C*	*B*	*B*
Second	*C*	*B*	*B*	*C*	*C*

C now has the fewest first-place votes and is eliminated, leaving *B* as the winner. For the first voter, having *B* win the election was **more preferred** than having *C* win the election.

Key idea

Sequential pairwise voting, with agenda, is manipulable by having the agenda altered.

Example F

Consider the following election with four candidates and three voters.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*D*
Second	*B*	*C*	*C*
Third	*C*	*D*	*A*
Fourth	*D*	*A*	*B*

Show that sequential pairwise voting, with agenda *A*, *B*, *C*, *D*, can be manipulated by the voter on the left by a change of agenda. (assuming the above ballot represents his true preferences).

Solution

Looking at the 6 one-on-one contests we can more readily see the solution.

A vs *B*	*A:*	2	*B:*	1
A vs *C*	*A:*	1	*C:*	2
A vs *D*	*A:*	1	*D:*	2
B vs *C*	*B:*	2	*C:*	1
B vs *D*	*B:*	2	*D:*	1
C vs *D*	*C:*	2	*D:*	1

In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, we first pit *A* against *B*. Thus, *A* wins by a score of 2 to 1. *A* moves on to confront *C*. *C* wins by a score of 2 to 1. *C* moves on to confront *D*. *C* wins by a score of 2 to 1. Thus, *C* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*.

If the voter on the left changes the agenda to *B*, *C*, *D*, *A*, we have the following.

We first pit *B* against *C*. Thus, *B* wins by a score of 2 to 1. *B* moves on to confront *D*. *B* wins by a score of 2 to 1. *B* moves on to confront *A*. *A* wins by a score of 2 to 1. Thus, *A* is the winner by sequential pairwise voting with the agenda *B*, *C*, *D*, *A*.

Question 2

Consider the following election with four candidates and 3 voters.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*D*
Second	*B*	*C*	*A*
Third	*C*	*D*	*B*
Fourth	*D*	*A*	*C*

If sequential pairwise voting, with agenda is used, is it possible to make all candidates winners (i.e. four separate manipulations/agendas) by different agendas? Explain your yes/no answer.

Answer

Yes.

Key idea

Plurality voting *can be* **group-manipulable**. Group-manipulable is when a group of voters can change the outcome of an election (as a group) to something they all prefer.

Example G

Consider the following election with four candidates and 11 voters.

Election 1

	Number of voters (11)		
Rank	**2**	**4**	**5**
First	*B*	*C*	*D*
Second	*C*	*B*	*A*
Third	*A*	*D*	*C*
Fourth	*D*	*A*	*B*

Show that if plurality voting is used, the group of voters on the left can secure a **more preferred** outcome.

Solution

Since Candidate *D* has the most first-place votes, *D* is declared the winner.

Election 2

	Number of voters (11)		
Rank	**2**	**4**	**5**
First	*C*	*C*	*D*
Second	*B*	*B*	*A*
Third	*A*	*D*	*C*
Fourth	*D*	*A*	*B*

Since the group on the left changed their ballots, *C* now has 6 (the most votes) and is declared the winner. Having *C* win the election was **more preferred** by the left most group of voters, rather than having *D* win the election.

Question 3

Consider the following election with four candidates and 11 voters.

	Number of voters (11)		
Rank	**2**	**5**	**4**
First	*B*	*C*	*D*
Second	*C*	*B*	*A*
Third	*A*	*D*	*C*
Fourth	*D*	*A*	*B*

If plurality voting is used, can the group of voters on the left secure a **more preferred** outcome? Explain your yes/no answer.

Answer

No.

Section 10.3 Impossibility

🗝 Key idea

Condorcet's method has very desirable properties including the following four.

- Elections never result in ties.
- It satisfies the Pareto condition. (It states that if everyone prefers one candidate, say *A*, to another, say *B*, then *B* cannot be the winner.)
- It is non-manipulable. (In a dictatorship all ballots except that of the dictator are ignored.)

A less than desirable outcome though is that Condorcet's method could produce no winner at all.

🗝 Key idea

An important theorem in social choice is the **Gibbard-Satterthwaite Theorem** ("GS theorem" for short). It says that with three or more candidates and any number of voters, there does not exist (and never will exist) a voting system that always has all of the following features.

- a winner
- no ties
- satisfies the Pareto condition
- non-manipulable
- not a dictatorship.

🗝 Key idea

A *weak* version of the Gibbard-Satterthwaite Theorem refers to any voting system for three candidates that agrees with Condorcet's method whenever there is a Condorcet winner. This voting system must also produce a unique winner when confronted by the ballots in the Condorcet voting paradox. Given these conditions, this voting system is manipulable.

Section 10.4 The Chair's Paradox

🗝 Key idea

A (single) choice of which candidate to vote for will be called a **strategy**. If a voter is rational, he or she will not vote for their least-preferred candidate. In the text example of the chair's paradox, it is assumed that there are three candidates and three voters. If a candidate gets two or three votes, he or she wins. If each candidate gets one vote (three-way tie), then the chair has **tie-breaking** power as his or her candidate is the winner. The paradox that occurs in this voting set-up is although the chair has tie-breaking power, the eventual winner (given the different voting strategies) is his or her least-preferred candidate.

🗝 Key idea

The strategy of choosing a candidate, say *X*, **weakly dominates** another choice, say *Y*, if the choice of *X* yields outcomes that are either the same or better than the choice of *Y*.

🗝 Key idea

To examine the text example of the chair's paradox, consider the names Adam, Nadia, and Zeki as candidates. The voters are Scott (Chair), Dan, and Sami. Although each voter can cast only one vote for one candidate, they each do have overall preferences as follows.

	Number of voters (3)		
Rank	**Scott**	**Dan**	**Sami**
First	Adam	Nadia	Zeki
Second	Nadia	Zeki	Adam
Third	Zeki	Adam	Nadia

Now for Scott (Chair), voting for Adam weakly dominates voting for Nadia. The possible outcomes if Scott votes for Adam are as follows.

Scott	**Dan**	**Sami**
Adam	Nadia	Zeki

Adam wins because Scott (Chair) breaks the tie.

Scott	**Dan**	**Sami**
Adam	Nadia	Adam

Adam wins because of two-thirds vote.

Scott	**Dan**	**Sami**
Adam	Zeki	Zeki

Zeki wins because of two-thirds vote.

Scott	**Dan**	**Sami**
Adam	Zeki	Adam

Adam wins because of two-thirds vote.

The possible outcomes if Scott votes for Nadia are as follows.

Scott	**Dan**	**Sami**
Nadia	Nadia	Zeki

Nadia wins because of two-thirds vote.

Scott	**Dan**	**Sami**
Nadia	Nadia	Adam

Nadia wins because of two-thirds vote.

Scott	**Dan**	**Sami**
Nadia	Zeki	Zeki

Zeki wins because of two-thirds vote.

Scott	**Dan**	**Sami**
Nadia	Zeki	Adam

Nadia wins because Scott (Chair) breaks the tie.

Clearly for Scott (Chair) the choice of Adam yields more desirable results. Since it is assumed that Scott is rational, we know that Adam will be Scott's choice.

Now, examining Dan's options. (The text examines the case of Sami, *C*, followed by Dan, *B*.)

Scott	Dan	Sami
Adam	Nadia	Zeki

Adam wins because Scott (Chair) breaks the tie.

Scott	Dan	Sami
Adam	Nadia	Adam

Adam wins because of two-thirds vote.

Scott	Dan	Sami
Adam	Zeki	Zeki

Zeki wins because of two-thirds vote.

Scott	Dan	Sami
Adam	Zeki	Adam

Adam wins because of two-thirds vote.

Since an outcome of Zeki is more favorable to Dan, voting for Zeki weakly dominates voting for Nadia. Unfortunately, Dan's top choice of Nadia is not possible. This leaves Sami's choices to be examined.

Scott	Dan	Sami
Adam	Zeki	Zeki

Zeki wins because of two-thirds vote.

Scott	Dan	Sami
Adam	Zeki	Adam

Adam wins because of two-thirds vote.

Since Zeki is the preferred choice of Sami, voting for Zeki weakly dominates voting for Adam. So the winner would be Zeki, which is the least-preferred choice of Scott (Chair).

Homework Help

Exercises 1 – 3
Carefully read Section 10.1 before responding to these exercises. For each exercise start off by setting up Election 1 that produces a candidate, say *B*, as the winner given the voting method. Taking the ballots in the first election to be the sincere preferences of the voters, then change a ballot (one that prefers *A* to *B*) to secure a more favorable outcome by the submission of a disingenuous ballot. The following tables may be helpful in setting up the two elections.

Election 1

Rank	Number of voters (3)
First	
Second	

Election 2

Rank	Number of voters (3)
First	
Second	

Exercise 4 – 5
Carefully read Section 10.1 before responding to these exercises. Pay special attention to May's theorem. Example of voting systems should not be complicated.

Exercise 6
Review Condorcet's method and consider the three one-on-one scores of *D* versus *H*, *D* versus *J*, and *H* versus *J*.

Exercise 7
Given the preference list ballots, determine the winner by the Borda count voting method.

Election 1

	Number of voters (2)	
Rank	**1**	**1**
First	*B*	*A*
Second	*C*	*D*
Third	*A*	*C*
Fourth	*D*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Change the leftmost voter preference ballot to manipulate the election.

Election 2

	Number of voters (2)	
Rank	**1**	**1**
First		*A*
Second		*D*
Third		*C*
Fourth		*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Exercise 8

One way to get an example of manipulation of the Borda count with seven candidates and eight voters is to alter the elections in Example 2 of the text by adding *F* and *G* to the bottom of each of the six ballots in both elections, and then adding in the two rightmost columns. One could also add two ballots canceling each other out first, and then add *F* and *G* to the bottom of all eight ballots.

Election 1

	Number of voters (8)							
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*A*	*E*	*A*	*E*		
Second	*B*	*C*	*B*	*D*	*B*	*D*		
Third	*C*	*A*	*C*	*C*	*C*	*C*		
Fourth	*D*	*D*	*D*	*B*	*D*	*B*		
Fifth	*E*	*E*	*E*	*A*	*E*	*A*		
Sixth								
Seventh								

Preference	1st place votes × 6	2nd place votes × 5	3rd place votes × 4	4th place votes × 3	5th place votes × 2	6th place votes × 1	7th place votes × 0	Borda score
A	×6	×5	×4	×3	×2	×1	×0	
B	×6	×5	×4	×3	×2	×1	×0	
C	×6	×5	×4	×3	×2	×1	×0	
D	×6	×5	×4	×3	×2	×1	×0	
E	×6	×5	×4	×3	×2	×1	×0	
F	×6	×5	×4	×3	×2	×1	×0	
G	×6	×5	×4	×3	×2	×1	×0	

Election 2

	Number of voters (8)							
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First		*B*	*A*	*E*	*A*	*E*		
Second		*C*	*B*	*D*	*B*	*D*		
Third		*A*	*C*	*C*	*C*	*C*		
Fourth		*D*	*D*	*B*	*D*	*B*		
Fifth		*E*	*E*	*A*	*E*	*A*		
Sixth								
Seventh								

Preference	1st place votes × 6	2nd place votes × 5	3rd place votes × 4	4th place votes × 3	5th place votes × 2	6th place votes × 1	7th place votes × 0	Borda score
A	×6	×5	×4	×3	×2	×1	×0	
B	×6	×5	×4	×3	×2	×1	×0	
C	×6	×5	×4	×3	×2	×1	×0	
D	×6	×5	×4	×3	×2	×1	×0	
E	×6	×5	×4	×3	×2	×1	×0	
F	×6	×5	×4	×3	×2	×1	×0	
G	×6	×5	×4	×3	×2	×1	×0	

Exercise 9
Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*B*
Second	*B*	*A*	*A*
Third	*C*	*C*	*C*
Fourth	*D*	*D*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First		*B*	*B*
Second		*A*	*A*
Third		*C*	*C*
Fourth		*D*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Exercise 10
Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*		
Second	*B*	*A*	*A*		
Third	*C*	*C*	*C*		
Fourth	*D*	*D*	*D*		

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

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Exercise 10 continued
Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First		*B*	*B*		
Second		*A*	*A*		
Third		*C*	*C*		
Fourth		*D*	*D*		

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Exercise 11
Election 1

	Number of voters (9)								
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*						
Second	*B*	*A*	*A*						
Third	*C*	*C*	*C*						
Fourth	*D*	*D*	*D*						
Fifth									
Sixth									

Preference	1st place votes × 5	2nd place votes × 4	3rd place votes × 3	4th place votes × 2	5th place votes × 1	6th place votes × 0	Borda score
A	×5	×4	×3	×2	×1	×0	
B	×5	×4	×3	×2	×1	×0	
C	×5	×4	×3	×2	×1	×0	
D	×5	×4	×3	×2	×1	×0	
E	×5	×4	×3	×2	×1	×0	
F	×5	×4	×3	×2	×1	×0	

Election 2

	Number of voters (9)								
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First		*B*	*B*						
Second		*A*	*A*						
Third		*C*	*C*						
Fourth		*D*	*D*						
Fifth									
Sixth									

Preference	1st place votes × 5	2nd place votes × 4	3rd place votes × 3	4th place votes × 2	5th place votes × 1	6th place votes × 0	Borda score
A	×5	×4	×3	×2	×1	×0	
B	×5	×4	×3	×2	×1	×0	
C	×5	×4	×3	×2	×1	×0	
D	×5	×4	×3	×2	×1	×0	
E	×5	×4	×3	×2	×1	×0	
F	×5	×4	×3	×2	×1	×0	

Exercise 12
Election 1

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*B*	*D*	*C*	*B*
Second	*C*	*C*	*A*	*A*
Third	*D*	*A*	*B*	*C*
Fourth	*A*	*B*	*D*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Election 2

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First		*D*	*C*	*B*
Second		*C*	*A*	*A*
Third		*A*	*B*	*C*
Fourth		*B*	*D*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Exercise 13
Election 1

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*C*	*B*	*D*
Second	*B*	*A*	*D*	*C*
Third	*C*	*B*	*C*	*A*
Fourth	*D*	*D*	*A*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Continued on next page

Exercise 13 continued

Election 2

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First		*C*	*B*	*D*
Second		*A*	*D*	*C*
Third		*B*	*C*	*A*
Fourth		*D*	*A*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	×3	×2	×1	×0	
B	×3	×2	×1	×0	
C	×3	×2	×1	×0	
D	×3	×2	×1	×0	

Exercise 14

In this exercise, award 1 point to the winner of the one-on-one competition and 0 to the loser. If it is a tie, award ½ point to each.

Election 1

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*C*	*A*	*D*
Second	*B*	*E*	*E*	*B*
Third	*C*	*D*	*D*	*E*
Fourth	*D*	*B*	*C*	*C*
Fifth	*E*	*A*	*B*	*A*

There are 10 one-to-one contests. Ties are possible since we have an even number of voters.

A versus *B*: _____ *B* versus *D*: _____

A versus *C*: _____ *B* versus *E*: _____

A versus *D*: _____ *C* versus *D*: _____

A versus *E*: _____ *C* versus *E*: _____

B versus *C*: _____ *D* versus *E*: _____

You may find it helpful to summarize your results in the following table.

	A	*B*	*C*	*D*	*E*
Total					

Continued on next page

Exercise 14 continued
Election 2

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First		*C*	*A*	*D*
Second		*E*	*E*	*B*
Third		*D*	*D*	*E*
Fourth		*B*	*C*	*C*
Fifth		*A*	*B*	*A*

A versus *B*: _____ *B* versus *D*: _____

A versus *C*: _____ *B* versus *E*: _____

A versus *D*: _____ *C* versus *D*: _____

A versus *E*: _____ *C* versus *E*: _____

B versus *C*: _____ *D* versus *E*: _____

You may find it helpful to summarize your results in the following table.

	A	*B*	*C*	*D*	*E*
Total					

Exercise 15
Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*A*	*A*
Second	*B*	*C*	*C*	*C*	*C*
Third	*C*	*A*	*A*	*B*	*B*

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First		*B*	*B*	*A*	*A*
Second		*C*	*C*	*C*	*C*
Third		*A*	*A*	*B*	*B*

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First					
Second					

Exercise 16
Review the Hare voting system before starting this exercise.
Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*C*	*D*
Second	*B*	*A*	*B*	*B*	*B*
Third	*C*	*C*	*A*	*A*	*C*
Fourth	*D*	*D*	*D*	*D*	*A*

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First		*B*	*C*	*C*	*D*
Second		*A*	*B*	*B*	*B*
Third		*C*	*A*	*A*	*C*
Fourth		*D*	*D*	*D*	*A*

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First					
Second					
Third					

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First					
Second					

Exercise 17
Review the plurality runoff rule before starting this exercise.
Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*C*	*C*	*B*
Second	*B*	*B*	*A*	*A*	*C*
Third	*C*	*C*	*B*	*B*	*A*

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First					
Second					

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First		*A*	*C*	*C*	*B*
Second		*B*	*A*	*A*	*C*
Third		*C*	*B*	*B*	*A*

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First					
Second					

Exercise 18
Review sequential pairwise, with agenda, voting method before starting this exercise.
Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First		*B*	*C*
Second		*C*	*A*
Third		*A*	*B*

Exercise 19

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*C*	*B*
Second	*B*	*A*	*D*
Third	*D*	*B*	*C*
Fourth	*C*	*D*	*A*

There are 12 different possible agendas to consider with four candidates.

A, *B*, *C*, *D* (equivalent to *B*, *A*, *C*, *D*)
A, *B*, *D*, *C* (equivalent to *B*, *A*, *D*, *C*)
A, *C*, *B*, *D* (equivalent to *C*, *A*, *B*, *D*)
A, *C*, *D*, *B* (equivalent to *C*, *A*, *B*, *D*)
A, *D*, *B*, *C* (equivalent to *D*, *A*, *B*, *C*)
A, *D*, *C*, *B* (equivalent to *D*, *A*, *C*, *B*)
B, *C*, *A*, *D* (equivalent to *C*, *B*, *A*, *D*)
B, *C*, *D*, *A* (equivalent to *C*, *B*, *D*, *A*)
B, *D*, *A*, *C* (equivalent to *D*, *B*, *A*, *C*)
B, *D*, *C*, *A* (equivalent to *D*, *B*, *C*, *A*)
C, *D*, *A*, *B* (equivalent to *D*, *C*, *A*, *B*)
C, *D*, *B*, *A* (equivalent to *D*, *C*, *B*, *A*)

Exercise 20
Review the Pareto condition before starting this exercise. Look under the first 🗝 Key idea from Section 10.3 in this Study Guide.

Exercise 21
Review the plurality rule before starting this exercise.
Election 1

22%	**23%**	**15%**	**29%**	**7%**	**4%**
D	*D*	*H*	*H*	*J*	*J*
H	*J*	*D*	*J*	*H*	*D*
J	*H*	*J*	*D*	*D*	*H*

Consider what would happen if the voters in the 7% group all change their ballots.
Election 2

22%	**23%**	**15%**	**29%**	**7%**	**4%**
D	*D*	*H*	*H*		*J*
H	*J*	*D*	*J*		*D*
J	*H*	*J*	*D*		*H*

Exercise 22

(a) Assume that the winner with the voting paradox ballots is *A*. Consider the following two elections:

Election 1

Rank	Number of voters (3)		
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Election 2

Rank	Number of voters (3)	
First	*A*	*C*
Second	*B*	*A*
Third	*C*	*B*

In Election 1, the winner is *A* (our assumption in this case) and in Election 2, the winner is *C* (because we are assuming that our voting system agrees with Condorcet's method when there is a Condorcet winner, as *C* is here).

(b) Assume that the winner with the voting paradox ballots is *B*. Consider the following two elections:

Election 1

Rank	Number of voters (3)		
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Election 2

Rank	Number of voters (3)	
First	*A*	*B*
Second	*B*	*C*
Third	*C*	*A*

In Election 1, the winner is *B* (our assumption in this case) and in Election 2, the winner is *A* (because we are assuming that our voting system agrees with Condorcet's method when there is a Condorcet winner, as *A* is here).

Exercise 23

Review what a dictator is before starting this exercise. Look under the first 🗝 Key idea from Section 10.3 in this Study Guide.

Exercises 24 – 25

Carefully read the conditions being used as a voting rules.

Exercises 26 – 27

Carefully read in Section 10.4 what it means to *weakly dominate*. Reading through the example in the text along with the similar version in this Study Guide should help you in coming up with the scenario to show that voting for a certain candidate does not weakly dominate your strategy of voting for another.

Do You Know the Terms?

Cut out the following 11 flashcards to test yourself on Review Vocabulary. You can also find these flashcards at http://www.whfreeman.com/fapp7e.

Chapter 10 **The Manipulability of Voting Systems** **Agenda manipulation**	**Chapter 10** **The Manipulability of Voting Systems** **Chair's paradox**
Chapter 10 **The Manipulability of Voting Systems** **Disingenuous ballot**	**Chapter 10** **The Manipulability of Voting Systems** **Gibbard-Satterthwaite theorem**
Chapter 10 **The Manipulability of Voting Systems** **Group manipulability**	**Chapter 10** **The Manipulability of Voting Systems** **Manipulation**
Chapter 10 **The Manipulability of Voting Systems** **May's theorem for manipulability**	**Chapter 10** **The Manipulability of Voting Systems** **Strategy**

The fact that with three voters and three candidates, the voter with tie-breaking power (the "chair") can -– if all three voters act rationally in their own self-interest -– end up with his least-preferred candidate as the election winner.	**The ability to control who wins an election with sequential pairwise voting by a choice of the agenda -– that is, a choice of the order in which the one-on-one contests will be held.**
Alan Gibbard's and Mark Satterthwaite's independent discovery that every voting system for three or more candidates and any number of voters that satisfies the Pareto condition, always produces a unique winner, and is not a dictatorship, can be manipulated.	**Any ballot that does not represent a voter's true preferences.**
A voting system is manipulable if there exists at least one election in which a voter can change his or her ballot (with the ballots of all other voters left unchanged) in such a way that he or she prefers the winner of the new election to the winner of the old election, assuming that the original ballots represent the true preferences of the voters.	**A voting system is group manipulable if there exists at least one election in which a group of voters can change their ballots (with the ballots of voters not in the group left unchanged) in such a way that they all prefer the winner of the new election to the winner of the old election, assuming that the original ballots represent the true preferences of these voters.**
In the chair's paradox, a choice of which candidate to vote for is called a strategy. This is a special case of the use of the term in general game-theoretic situations.	**Kenneth May's discovery that for two candidates and an odd number of voters, majority rule is the only voting system that treats both candidates equally, all voters equally, and is non-manipulable.**

Chapter 10
The Manipulability of Voting Systems

Tie-breaking power

Chapter 10
The Manipulability of Voting Systems

Unilateral change

Chapter 10
The Manipulability of Voting Systems

Weak-dominance

A change (in ballot) by a voter while every other voter keeps his or her ballots exactly as it was.	**That aspect of the voting rule used in the chair's paradox that says the winner will be whichever candidate the chair votes for if there is a tie (which only happens if each candidate gets exactly one vote).**
	One strategy (for example, a choice of whom to vote for) weakly dominates another if it yields an outcome that is at least as good, and sometimes better, than the other.

Practice Quiz

1. Which of the following is *not* part of the conditions that define manipulability?

a. The voter that changes his or her preference list ballot must manipulate the election so that his or her top choice is declared the winner.

b. The voter that changes his or her preference list ballot must force a tie in the second election.

c. The voter that changes his or her preference list ballot prefers the outcome of the second election.

2. "Among all two-candidate voting systems that never result in a tie, majority rule is the only one that treats all voters equally, both candidates equally, and is non-manipulable." This statement is known as

a. the chair's paradox.

b. May's theorem.

c. The Gibbard-Satterthwaite theorem.

3. Consider the following election with four candidates and three voters. If the Borda count method is used, the winner of the first election is *B*. What can be said about the left-most voter?

Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*C*
Second	*D*	*C*	*B*
Third	*B*	*A*	*A*
Fourth	*C*	*D*	*D*

Borda Scores	
A	5
B	6
C	5
D	2

a. In Election 2, the left-most voter cannot manipulate his or her ballot for a more favorable outcome.

b. In Election 2, the left-most voter can manipulate his or her ballot as follows.

Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*C*
Second	*C*	*C*	*B*
Third	*D*	*A*	*A*
Fourth	*B*	*D*	*D*

Borda Scores	
A	5
B	5
C	7
D	1

c. The left-most voter can manipulate the vote if the right-most voter will also change his or her preference ballot.

4. Consider the following election with four candidates and three voters. Assume that these represent true preferences and sequential pairwise voting, with agenda *A*, *B*, *C*, *D*.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*C*
Second	*D*	*C*	*D*
Third	*B*	*D*	*B*
Fourth	*C*	*A*	*A*

Which of the following agendas will allow the voter on the right to manipulate the outcome to a *more* favorable one?

a. *A*, *C*, *D*, *B*

b. *D*, *B*, *C*, *A*

c. *C*, *B*, *A*, *D*

5. Consider the following election with four candidates and three voters.

Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*B*	*C*	*D*
Second	*C*	*B*	*B*
Third	*D*	*D*	*C*
Fourth	*A*	*A*	*A*

Which of the following Election 2 preference list ballots demonstrates that if Condorcet's method is being used, the voter on the right can change the outcome from having a winner to having no winner.

a. Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*B*	*C*	*D*
Second	*C*	*B*	*B*
Third	*D*	*D*	*A*
Fourth	*A*	*A*	*C*

b. Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*B*	*C*	*D*
Second	*C*	*B*	*A*
Third	*D*	*D*	*B*
Fourth	*A*	*A*	*C*

c. Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*B*	*C*	*D*
Second	*C*	*B*	*C*
Third	*D*	*D*	*B*
Fourth	*A*	*A*	*A*

6. Which of the following voting systems are not manipulable by a single voter but is by a group of voters?

a. Plurality voting

b. Borda count

c. Hare system

7. Consider the following election with three candidates and five voters.

Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*B*	*A*	*C*
Second	*A*	*A*	*C*	*C*	*A*
Third	*C*	*B*	*A*	*B*	*B*

Assume that these represent true preferences and the Hare voting method is used. Which of the following is true regarding the left-most voter.

a. He or she can obtain a more favorable outcome by the following unilateral change.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*C*	*B*	*A*	*C*
Second	*C*	*A*	*C*	*C*	*A*
Third	*A*	*B*	*A*	*B*	*B*

b. He or she can obtain a more favorable outcome by the following unilateral change.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*B*	*A*	*C*
Second	*B*	*A*	*C*	*C*	*A*
Third	*C*	*B*	*A*	*B*	*B*

c. He or she cannot obtain a more favorable outcome by a unilateral change

8. The chair's paradox refers to

a. the manipulability of an election by the chair of a committee.

b. the chair obtaining a more favorable outcome by actually handing over tie-breaking power to one of the other voters.

c. having an election with three voters and three candidates.

9. As discussed in the chair's paradox, voting for a candidate, say *X*, that *weakly dominates* another, say *Y*, refers to

a. *X* having more potential to win the election over *Y*.

b. choosing *X* over *Y* would yield a better outcome for the voter.

c. choosing *X* over *Y* would yield a better or the same outcome for the voter.

10. Consider the following election with four candidates and 13 voters.

Election 1

	Number of voters (13)		
Rank	**3**	**4**	**6**
First	*B*	*C*	*D*
Second	*A*	*B*	*A*
Third	*C*	*A*	*C*
Fourth	*D*	*D*	*B*

If plurality voting is used, which of the following demonstrates that a group of voters secured a more preferred outcome?

a. Election 2

	Number of voters (13)		
Rank	**3**	**4**	**6**
First	*B*	*D*	*D*
Second	*A*	*B*	*A*
Third	*C*	*A*	*C*
Fourth	*D*	*C*	*B*

b. Election 2

	Number of voters (13)		
Rank	**3**	**4**	**6**
First	*C*	*C*	*D*
Second	*B*	*B*	*A*
Third	*A*	*A*	*C*
Fourth	*D*	*D*	*B*

c. Election 2

	Number of voters (13)		
Rank	**3**	**4**	**6**
First	*B*	*B*	*D*
Second	*A*	*C*	*A*
Third	*C*	*A*	*C*
Fourth	*D*	*D*	*B*

Word Search

Refer to page 384 of your text to obtain the Review Vocabulary. There are 9 hidden vocabulary words/expressions in the word search below. *Gibbard-Satterthwaite theorem* and *May's theorem for manipulability* were both omitted from the word search. It should be noted that spaces are removed as well as apostrophes.

S R M E Q E O Y C D R R R G O L E N F F B U X A U
I E B S A M M A E G S E P O C T T E E N J O I Q O
I N M L P W A S F S B X O D A R A P S R I A H C O
K I K S O T G J K S H S B I E T E B A J V D E I I
T B M F A H E A S O Z G V A D O F G V O L W D P X
V E L T E N N A M I O N P G P O H G E Q T V P E F
Q X Y H K P D I S I N G E N U O U S B A L L O T O
F F L A W E A K D O M I N A N C E T S A F B P I Y
U F P B G E M N C Q S E F M Z R O C P M N F H E S
I O S P R D A F O M X H O A X E Q E A E S L P B N
A F K U A C N D I E T A P L R R T S Z E W L R R T
P I O S S A I N E S C E Z C P E E O D D M N F E V
N T O F V I P O F B C T O I P K A O I E A S A A T
N O I T A L U P I N A M Q J A A E L L O C F N K L
M O H T O L L A B E R E C N I S N I V O S R R I D
S F K J A L A O H E G N A H C L A R E T A L I N U
H C U I C Y T I L I B A L U P I N A M P U O R G F
W K L E M G I V T Z A Y Q P H C I D L K H J A P N
I H T N T E O Z E X P E X N B T I S E O P W E O R
S E F Y A T N G N H O T I A G R R I W S E C S W A
E I S U V A O I E U O Z I C S I F I M B O O U E P
N G C S T R I S M L N P I Q G Z G I U F B H A R O
E T R J A T S E Q A E P O J I S F A L H Q I S E S
V H N E N S L E R F A G I M I T Y O T P E T R C I
M W F N O U F D Q I P P S O T P N Q R E F K R U S

1. _____
2. _____
3. _____
4. _____
5. _____
6. _____
7. _____
8. _____
9. _____