Chapter 21
Savings Models

Solutions

Exercises:

1. (a) By the pattern shown, there is an increase by a factor if 2^n every $3n$ days. By doing some calculations, we can see that $2^{232} \approx 6.9\times10^{69}$ and $2^{233} \approx 1.4\times10^{70}$. So n is between 232 and 233. Thus, $3n$ is between 696 and 699. Calculating $2^{697/3}$ and $2^{698/3}$, we see that the better choice is 698 days.

(b) We have $1000 = 10^3$. Since $\left(10^3\right)^{23} = 10^{69}$ and $\left(10^3\right)^{24} = 10^{72}$, an appropriate answer would be after 24 months.

3. (a) Since $A = P(1+rt)$, we have $A = \$1000(1+0.03\times1) = \$1000(1.03) = \$1030.00$. The annual yield is 3% since we have simple interest.

(b) Since $A = P(1+i)^n$, $n=1$, and $i = \frac{0.03}{1} = 0.03$, we have the following.

$$A = \$1000(1+0.03) = \$1000(1.03) = \$1030.00$$

The annual yield is 3% since we are dealing with simple interest because the money is compounded once during the period of one year.

(c) Since $A = P(1+i)^n$, $n=4$, and $i = \frac{0.03}{4}$, we have the following.

$$\$1000\times\left(1+\tfrac{0.03}{4}\right)^4 = \$1000(1.0075)^4 = \$1030.34$$

Since $APY = \left(1+\dfrac{r}{n}\right)^n - 1$, we have $APY = \left(1+\dfrac{0.03}{4}\right)^4 - 1 \approx 3.034\%$.

(d) $\$1000\times\left(1+\tfrac{0.03}{365}\right)^{365} = \$1000(1.000082192)^{365} = \$1030.45$, with the same result for a 360-day or 366-day year.

Since $APY = \left(1+\dfrac{r}{n}\right)^n - 1$, we have $APY = \left(1+\dfrac{0.03}{365}\right)^{365} - 1 \approx 3.045\%$, with the same result for a 360-day or 366-day year.

5. Using 365-day years: The daily interest rate $i = \frac{0.03}{365}$ is in effect for $n = 8\times365 = 2920$ days. We have in the compound interest formula $A = \$10,000 = P(1+i)^n$, so we get $P = \frac{\$10,000}{1.2712366} =$ \$7866.36. (Fine point: In fact, the 8 years must contain two Feb. 29 days. Calculating interest for $n = 6\times365 = 2190$ days at $i = \frac{0.03}{365}$ and for $n = 2\times366 = 732$ days at $i = \frac{0.03}{366}$ gives a result that differs by less than one one-hundredth of a cent.)

7. The interest is \$26.14 on a principal of \$7744.70, or $\frac{\$26.14}{\$7744.70}\times 100\% = 0.3375211435\%$ over 34 days. The daily interest rate is $\left(1.003375211435^{1/34} - 1\right)\times 100\% = 0.0099109\%$. The annual rate is then $\left(1.000099109^{365} - 1\right)\times 100\% = 3.68\%$.

9. (a) 3%: Predicted doubling time is $\dfrac{72}{100\times 0.03} = \dfrac{72}{3} = 24.$

Since $A = P(1+i)^n$, $n = 24$, and $i = \frac{0.03}{1} = 0.03$, we have the following.

$$A = \$1000(1+0.03)^{24} = \$1000(1.03)^{24} = \$2032.79$$

4%: Predicted doubling time is $\dfrac{72}{100\times 0.04} = \dfrac{72}{4} = 18.$

Since $A = P(1+i)^n$, $n = 18$, and $i = \frac{0.04}{1} = 0.04$, we have the following.

$$A = \$1000(1+0.04)^{18} = \$1000(1.04)^{18} = \$2025.82$$

6%: Predicted doubling time is $\dfrac{72}{100\times 0.06} = \dfrac{72}{6} = 12.$

Since $A = P(1+i)^n$, $n = 12$, and $i = \frac{0.06}{1} = 0.06$, we have the following.

$$A = \$1000(1+0.06)^{12} = \$1000(1.06)^{12} = \$2012.20$$

(b) 8%: Predicted doubling time is $\dfrac{72}{100\times 0.08} = \dfrac{72}{8} = 9.$

Since $A = P(1+i)^n$, $n = 9$, and $i = \frac{0.08}{1} = 0.08$, we have the following.

$$A = \$1000(1+0.08)^{9} = \$1000(1.08)^{9} = \$1999.00$$

9%: Predicted doubling time is $\dfrac{72}{100\times 0.09} = \dfrac{72}{9} = 8.$

Since $A = P(1+i)^n$, $n = 8$, and $i = \frac{0.09}{1} = 0.09$, we have the following.

$$A = \$1000(1+0.09)^{8} = \$1000(1.09)^{8} = \$1992.56$$

(c) 12%: Predicted doubling time is $\dfrac{72}{100\times 0.12} = \dfrac{72}{12} = 6.$

Since $A = P(1+i)^n$, $n = 6$, and $i = \frac{0.12}{1} = 0.12$, we have the following.

$$A = \$1000(1+0.12)^{6} = \$1000(1.12)^{6} = \$1973.82$$

24%: Predicted doubling time is $\dfrac{72}{100\times 0.24} = \dfrac{72}{24} = 3.$

Since $A = P(1+i)^n$, $n = 3$, and $i = \frac{0.24}{1} = 0.24$, we have the following.

$$A = \$1000(1+0.24)^{3} = \$1000(1.24)^{3} = \$1906.62$$

36%: Predicted doubling time is $\dfrac{72}{100\times 0.36} = \dfrac{72}{36} = 2.$

Since $A = P(1+i)^n$, $n = 2$, and $i = \frac{0.36}{1} = 0.36$, we have the following.

$$A = \$1000(1+0.36)^{2} = \$1000(1.36)^{2} = \$1849.60$$

(d) For small and intermediate interest rates, the rule of 72 gives good approximations to the doubling time.

11. (a) 2, 2.59, 2.705, 2.7169, 2.718280469

(b) 3, 6.19, 7.245, 7.3743, 7.389041321

(c) $e = 2.718281828\ldots$; $e^2 = 7.389056098\ldots$. Your calculator may give slightly different answers, because of its limited precision.

13. (a) Since $A = Pe^{rt}$ we have $A = \$1000e^{(0.03)(1)} = \$1000e^{0.03} = \$1030.45$. Thus, the interest is $\$1030.45 - \$1000.00 = \$30.45$.

(b) Since $A = P\left(e^{r/360}\right)^{360}$ we have $A = \$1000\left(e^{0.03/360}\right)^{360} = \$1030.45$. Thus, the interest is $\$1030.45 - \$1000.00 = \$30.45$.

(c) Since $A = P\left(e^{r/365}\right)^{365}$ we have $A = \$1000\left(e^{0.03/365}\right)^{365} = \$1030.45$. Thus, the interest is $\$1030.45 - \$1000.00 = \$30.45$.

In all cases, \$30.45, not taking into account any rounding to the nearest cent of the daily posted interest.

15. (a) $\left(e^{0.04} - 1\right) \times 100\% = 4.08108\%$.

(b) The approximation for effective rate is $r + \frac{1}{2}r^2 = 0.05 + \frac{1}{2} \times (0.05)^2 = 0.05 + 0.00125 = 0.05125$ or 5.125%, very slightly less than the true effective rate.

17. Use the savings formula $A = d\left[\frac{(1+i)^n - 1}{i}\right]$, with A = \$2000, $i = \frac{0.05}{12}$, and n = 24.

$$\$2000 = d\left[\frac{\left(1+\frac{0.05}{12}\right)^{24} - 1}{\frac{0.05}{12}}\right] \approx 25.18592053d$$

$$d = \frac{\$2000}{25.18592053} = \$79.41$$

19. Use the savings formula $A = d\left[\frac{(1+i)^n - 1}{i}\right]$, with d = \$400, $i = \frac{0.055}{12}$, and n = 144.

$$A = d\left[\frac{(1+i)^n - 1}{i}\right] = \$400\left[\frac{\left(1+\frac{0.055}{12}\right)^{144} - 1}{\frac{0.055}{12}}\right] = \$81{,}327.45$$

21. Use the savings formula $A = d\left[\frac{(1+i)^n - 1}{i}\right]$, with A = \$1,000,000, $i = \frac{0.05}{12}$, and $n = 35 \times 12 = 420$.

$$\$1{,}000{,}000 = d\left[\frac{\left(1+\frac{0.05}{12}\right)^{420} - 1}{\frac{0.05}{12}}\right] \approx 1136.092425d$$

$$d = \frac{\$1{,}000{,}000}{1136.092425} = \$880.21$$

23. (a) $\dfrac{\$100}{1-0.32} = \$147.06$

(b) Use the savings formula $A = d\left[\dfrac{(1+i)^n - 1}{i}\right]$, with d = \$147.06, $i = \frac{0.075}{12}$, and n = 40×12 = 480 to calculate $A = \$147.06\left[\dfrac{\left(1+\frac{0.075}{12}\right)^{480} - 1}{\frac{0.075}{12}}\right] = \$444{,}683.29$.

(c) 0.68×\$444, 683.29 = \$302,384.64

25. (a) Write the series as $\frac{1}{2}\left(1+\dfrac{1}{2^1}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\dfrac{1}{2^4}\right) = \frac{1}{2}\times\dfrac{\left(\frac{1}{2}\right)^5 - 1}{\frac{1}{2}-1} = \dfrac{31}{32}$.

(b) $\dfrac{2^n - 1}{2^n}$

(c) 1

27. (a) $A = P(1.15)(1.07)(0.80) = 0.98440P$, so $r = (0.98440)^{1/3} - 1 = -0.00523 = -0.523\%$.

(b) It is the effective rate.

29. (a) $\$(1.04)^3 = \$1.12$

(b) \$1/1.12 = \$0.89

31. $\$10{,}000 \times (1-0.12)^6 \times \left(\dfrac{1}{1+0.03}\right)^6 \approx \$3900$

33. (a) Since $\dfrac{\text{cost in 2006}}{\$10.75} = \dfrac{\text{CPI for 2006}}{\text{CPI for 1962}} = \dfrac{200.5}{30.9} \approx 6.4898673139$, we have the following.

$$\text{cost in 2006} = \$10.75(6.4898673139) = \$69.75 \approx \$70$$

$$\$10.75 \times (6.4898673) = \$69.75 \approx \$70$$

Additional answers will vary.

(b) Since $\dfrac{\text{cost in 2006}}{\$0.25} = \dfrac{\text{CPI for 2006}}{\text{CPI for 1970}} = \dfrac{200.5}{38.8} \approx 5.167525773$, we have the following.

$$\text{cost in 2006} = \$0.25(5.167525773) = \$1.29$$

Since $\dfrac{\text{cost in 2006}}{\$0.25} = \dfrac{\text{CPI for 2006}}{\text{CPI for 1974}} = \dfrac{200.5}{49.3} \approx 4.06693712$, we have the following.

$$\text{cost in 2006} = \$0.70(4.06693712) = \$2.85$$

35. Let the purchasing power of the original salary be P. Then the purchasing power of the new salary is $P \times 1.10 \times \dfrac{1}{1+0.20} \approx 0.917P$, an 8.3% loss.

37. Nowhere close

(a) As she ends her 35th year of service, her salary will be $166,973.02, which we multiply by $\frac{1}{1.03^{35}}$ to get the equivalent in today's dollars: $59,339.44. (We do not take into account that annual salaries are normally rounded to the nearest dollar or hundred dollars.) The result is easily obtained by use of a spreadsheet, proceeding through her salary year by year and then adjusting at the end for inflation. Here is the corresponding formula, using Fisher's effect with $r = 0.04$ and $a = 0.03$.

$$\left\{\left[\$42{,}000\left(1+\frac{0.01}{1.03}\right)^{7}+\$1500\left(\frac{1}{1.03}\right)^{7}\right]\left(1+\frac{0.01}{1.03}\right)^{7}+\$1500\left(\frac{1}{1.03}\right)^{14}\right\}$$
$$\times\left\{\left(1+\frac{0.01}{1.03}\right)^{20}\left(\frac{1}{1.03}\right)\right\}$$

The last factor is for inflation during her 35th year of service.

(b) $57,394.20.

39. $2,000,000 × $\left(1+x+\ldots+x^{19}\right)$, with $x=\frac{1}{1+a}=\frac{1}{1.03}$, giving $29.8 million. If you can expect to earn interest rate r on funds once you receive them, through the last payment, then the present value of your stream of income of annual lottery payments P plus interest (with inflation rate a) is as follows.

$$P\left[\left(\frac{1+r}{1+a}\right)^{19}\cdot 1+\left(\frac{1+r}{1+a}\right)^{18}\left(\frac{1}{1+a}\right)^{1}+\left(\frac{1+r}{1+a}\right)^{17}\left(\frac{1}{1+a}\right)^{2}+\ldots\right.$$
$$\left.\ldots+\left(\frac{1+r}{1+a}\right)^{1}\left(\frac{1}{1+a}\right)^{18}+1\cdot\left(\frac{1}{1+a}\right)^{19}\right]=P\frac{1}{(1+a)^{19}}\left[\frac{(1+r)^{20}-1}{r}\right]$$

For P = $2 million, r = 4%, and a = 3%, we get $33.4 million. If you can earn 4% forever but inflation stays at 3%, the present value is infinite!

41. (a) Use the savings formula with A = $100,000, n = 35 × 4 = 140 quarters, and $i=\frac{0.072}{4}$ per quarter. You find d = $161.39.

(b). $\dfrac{\$100{,}000}{(1.04)^{35}}=\$25{,}341.55$

(c) $\dfrac{\$100{,}000}{(1.04)^{65}}=\$7813.27$

43. (a) $\frac{1.0453}{1.031}-1$ = 0.01387 = 1.39%.

(b) $(1 - 0.30)\times$ 1.39% = 0.97% (however, some states and cities do not tax interest earned on U.S. government securities).

45. The price before should have been about $\dfrac{D(1.03)}{0.15-0.03}=8.583D$, the price after should have been $\dfrac{D(1.03)}{0.1475-0.03}=8.766D$, so the percentage change expected was $\dfrac{8.766D-8.583D}{8.583D}=2.13\%$, which applied to the Dow Jones should have produced a rise of 188 pts. The answer does not depend on the value of D.

47. Programming the savings formula into the spreadsheet and varying the value of i until you find $A \geq \$5000$, using the Solver command in Excel, or otherwise: i = 1.60% per month, or an annual rate of 12 × 1.60% = 19.2%.

49. 4.97%. It is the effective rate.