Chapter 20
Tilings

Solutions

Exercises:

1. Exterior: $45°$. Interior: $135°$.

3. $180° - \frac{360°}{n}$.

5. The usual notation for a vertex figure is to denote a regular n-gon by n, separate the sizes of polygons by periods, and list the polygons in clockwise order starting from the smallest number of sides, so that, e.g., 3.3.3.3.3.3 denotes six equilateral triangles meeting at a vertex. The possible vertex figures are 3.3.3.3.3.3, 3.3.3.3.6, 3.3.3.4.4, 3.3.4.3.4, 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.6.3.6, 3.4.4.6, 3.4.6.4, 3.12.12, 4.4.4.4, 4.6.12, 4.8.8, 5.5.10, and 6.6.6.

7. 3.7.42, 3.9.18, 3.8.24, 3.10.15, and 4.5.20.

9. At each of the vertices except the center one, six triangles meet, with angles (in clockwise order) of $75°$, $75°$, $30°$, $30°$, $75°$, and $75°$.

11. Yes, because the half pentagon is a quadrilateral, and any quadrilateral can tile the plane.

13. (a) No.

(b) No.

(c) No.

15. Answers will vary.

17. No.; No.

19. The only way to tile by translations is to fit the outer "elbow" of one tile into the inner "elbow" of another. Labeling the corners as follows works: the corners on the top A and B, those on the rightmost side C and D, the middle of the bottom E, and the middle of the leftmost side F.

21. Just label the four corners consecutively A, B, C, and D.

23. Place the skew-tetromino on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A = (1,2)$, $B = (3,2)$, $C = (2,0)$, and $D = (0,0)$ works.

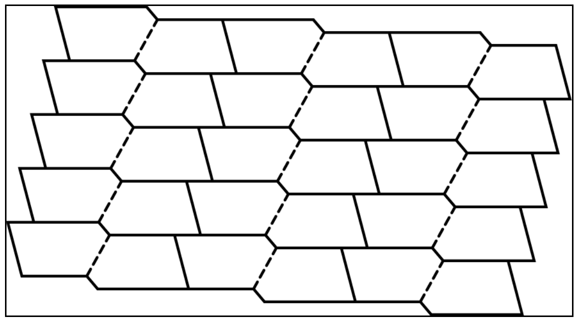
25. Place the skew-tetromino on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A = (0,1)$, $B = (1,2)$, $C = (3,2)$, $D = (3,1)$, $E = (2,0)$, and $F = (0,0)$ works.

27. (a) Yes.

(b) No.

(c) No.

29. See figure below.



31. Answers will vary.

33. N, Z, W, P, y, I, L, V, X. See www.srcf.ucam.org/~jsm28/tiling/5-omino-trans.ps.gz.

35. Place the U on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A = (2,2)$, $B = (3,2)$, $C = (3,0)$, $D = (1,0)$, $E = (0,0)$, and $F = (0,1)$ works. See www.srcf.ucam.org/~jsm28/tiling/5-omino-rot.ps.gz .

37. Answers will vary.

39. *ABAABABA*.

41. The two leftmost *A*'s would have had to come from two *B*'s in a row in the preceding month.

43. Let S_n, A_n, and B_n be the total number of symbols, the number of *A*'s, and the number of *B*'s at the *n*th stage. We note that the only *B*'s at the *n*th stage must have come from *A*'s in the previous stage, so $B_n = A_{n-1}$. Similarly, the *A*'s at the nth stage come from both *A*'s and *B*'s in the previous stage, so $A_n = A_{n\text{-}1} + B_{n\text{-}1}$. Using both of these facts together, we have $A_n = A_{n\text{-}1} + A_{n\text{-}2}$. We note that $A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 2, \ldots$. The A_n sequence obeys the same recurrence rule as the Fibonacci sequence and starts with the same values one step later; in fact, it is always just one step behind the Fibonacci sequence: $A_n = F_{n-1}$. Consequently, $B_n = A_{n-1} = F_{n-2}$, and $S_n = A_n + B_n = F_{n\text{-}1} + F_{n\text{-}2} = F_n$.

45. If a sequence ends in *AA*, its deflation ends in *BB*, which is impossible for a musical sequence. Similarly, if a sequence ends in *ABAB*, its deflation ends in *AA*, which we just showed to be impossible.

47. The first is, the second is not, part of a musical sequence: *ABAABABAAB* $\rightarrow$ *ABAABA* $\rightarrow$ *ABAB* $\rightarrow$ *AA* $\rightarrow$ (2nd special rule) *BA* $\rightarrow$ (1st special rule) *ABA* $\rightarrow$ *AB* $\rightarrow$ *A*.

ABAABABABA $\rightarrow$ *ABAAAB*, which has three *A*'s.

49. If the sequence were periodic, the limiting ratio of *A*'s to *B*'s would be the same as the ratio in the repeating part, which would be a rational number, contrary to the result of Exercise 48 (You will need to work out this exercise).