Chapter 19
Symmetry and Patterns

Solutions

Exercises:

1. 5, 8 and 13

3. Answers will vary but will be Fibonacci numbers.

5. (a) The digits after the decimal point do not change.

(b) The digits after the decimal point do not change.

(c) $\phi^2 = \phi + 1$

(d) $\frac{1}{\phi} = \phi - 1$

7. (a) $\sqrt{3 \times 27} = \sqrt{81} = 9$

(b) The area of the rectangle is 4 × 64 = 256, so the square must have side $\sqrt{256}$ = 16.

9. (a) 4, 7, 11, 18, 29, 47, 76, 123.

(b) 3, 1.333, 1.75, 1.571, 1.636, 1.611, 1.621, 1.617, 1.618; The ratios approach ϕ.

11. Answers will vary

13. $m + n + (m + n) + (m + 2n) + (2m + 3n) + (3m + 5n) + (5m + 8n)$
$+ (8m + 13n) + (13m + 21n) + (21m + 34n) = 55m + 88n,$

eleven times the seventh number $5m + 8n$.

15. (a) 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1,365

(b) $B_n = B_{n-1} + 2B_{n-2}$

(c) 1, 3, 1.667, 2.2, 1.909, 2.048, 1.977, 2.012, 1.994, 2.003, 1.999

(d) x = 2,−1; we discard the −1 root.

(e) $B_n = \frac{2^n - (-1)^n}{3}$

17. Silver mean: $1 \pm \sqrt{2} \approx 2.414$; bronze mean: $\frac{1}{2}\left(3 \pm \sqrt{13}\right) \approx 3.303$;

copper mean: $2 \pm \sqrt{5} \approx 4.236$; nickel mean: $\frac{1}{2}\left(5 \pm \sqrt{29}\right) \approx 5.193$.

General expression: $\frac{1}{2}\left(m + \sqrt{m^2 + 4}\right)$.

19. All are true.

21. (a) B, C, D, E, H, I, K, O, X
(b) A, H, I, M, O, T, U, V, W, X, Y
(c) H, I, N, O, S, X, Z

23. (a) MOM, WOW; MUd and bUM reflect into each other, as do MOM and WOW.
(b) pod rotates into itself; MOM and WOW rotate into each other.
(c) Here are some possibilities: NOW NO; SWIMS; ON MON; CHECK BOOK BOX; OX HIDE.

25. For all parts, translations.
(a) Reflection in vertical lines through the centers of the A's or between them.
(b) Reflection in the horizontal midline, glide reflections.
(c) Reflection in the horizontal midline, reflections in vertical lines through the centers of the O's or between them, 180" rotation around the centers of the O's or the midpoints between them, glide reflections.
(d) None other than translations.

27. (a) *c5*
(b) *c12*
(c) *c22*

29. (a) *c6*
(b) *d2* (CBS)
(c) *d1* (Dodge Ram)

31. (a) *c4*
(b) *d2*

33. *p111*, *p1a1*, *p112*, *pm11*, *p1m1*, *pma2*, *pmm2*

35. (a) *pmm2*
(b) *p1a1*
(c) *pma2*
(d) *p112*
(e) *pmm2* (perhaps)
(f) *p1m1*
(g) *pma2*
(h) *p111*

37. (a) Half-turns are preferred on MesaVerde pottery, while reflections predominate on Begho smoking pipes.
(b) Neither culture completely excludes any strip type. Begho designs are heavily concentrated (almost 90%) in *p1m1*, *p112*, or *pmm2*, while Mesa Verde designs are more evenly distributed over the seven patterns.
(c) (a) *pm11* or *pma2*: Mesa Verde. (b) *p112*: Mesa Verde. (c) *pmm2*: Begho. (d) *pm11*: Begho. (e) *p1m1*: Difficult to say. (f) *pmm2*: Begho. (g) *pmm2*: Begho. (h) *pma2*: Mesa Verde. (i) *p1a1*: Mesa Verde.

39. (c) Smallest rotation is $90°$, there are reflections, there are reflections in lines that intersect at $45°$: *p4m*.

(d) Smallest rotation is $90°$, there are no reflections: *p4*.

41. Certainly none of the patterns with rotations at $60°$ or $120°$ can arise from this method, which eliminates the five patterns in the two bottom branches of the identification flowchart. For the vertical motion, four orientations are possible for the second triangle in the first column, corresponding to rotations clockwise by 0, 1, 2, or 3 times a rotation of $90°$ of the triangle in the upper left square around the center of its square. Similarly, for the horizontal motion, there are four choices, which we number in the same fashion. Then there are 16 combinations, which we can denote (with vertical motion first) as 00, 01, 02, 03, 04, 10, . . . , 44. However, some horizontal motions are not compatible with some vertical motions, and some of the patterns are reflections of others along the diagonal from upper left to lower right. In fact, the patterns *p4g* and *p4* cannot be realized because the original triangle has symmetry within itself, patterns *cm* and *cmm* cannot be realized because every square contains a triangle. The remaining eight can all be formed by the technique.

43. There are no knights facing up or down (so no rotations of $90°$), no knights who are upside down (so no rotations of $180°$, and no knights at $60°$ or $120°$ angles. So the smallest rotation is $360°$, there are no reflections, but there is a glide reflection that takes yellow knights into brown knights. So the pattern is *pg*, provided color is disregarded; if not, then *p1*.

45. The block of three pentagons shaded in gray is repeated at rotations of $60°$ around the center of the “snowflake” outlined in red. There are no reflections, so the pattern is *p6*.

47. The intersection of the dashed lines are $120°$ rotation centers, but there are no reflection lines, so the pattern is *p3*

49. Answers will vary.

51. (a) *d2*.

(b) Any two of: *R* ($180°$ rotation around the center), *V* (reflection in vertical line through its center), *H* (reflection in horizontal line through its center).

(c) $\{I, R, V, H\}$

53. Let *R* denote rotation counterclockwise by $90°$, *V* reflection in vertical line through its center and *H* reflection in horizontal line through its center. Then the group can be written as $\left\{I, R, R^2, R^3, H,\ V, RH = VR, RV = HR\right\}$, where the last two elements are reflections across the diagonals.

55. Answers will vary. Here is one: $0 = (3 - 2) - 1 \neq 3 - (2 - 1) = 2$.

57. As in Example 5, number fixed positions, label with letters copies of the pattern elements in the positions, and pick a fixed position about which to make a half-turn *R*.

(a) $\left\langle T, R \mid R^2 = I,\ T \circ R = R \circ T^{-1} \right\rangle = \left\{\ldots,\ T^{-1},\ I,\ T^1,\ \ldots;\ \ldots,\ R \circ T^{-1},\ R,\ R \circ T^1,\ \ldots\right\}$.

(b) $\left\langle T, R, H \mid R^2 = H^2 = I,\ T \circ H = H \circ T,\ R \circ H = H \circ R,\ (R \circ T)^2 = I \right\rangle =$

$\left\{\ldots,\ T^{-1},\ I,\ T,\ \ldots;\ \ldots,\ R \circ T^{-1},\ R,\ R \circ T,\ \ldots;\ \ldots,\ H \circ T^{-1},\ H,\ H \circ T,\right.$

$\left.\ldots;\ \ldots, R \circ H \circ T^{-1},\ R \circ H,\ R \circ H \circ T,\ \ldots\right\}$

59. $\langle R \mid R^8 = I \rangle = \{I,\ R,\ R^2,\ R^3,\ R^4,\ R^5,\ R^6,\ R^7\}$, where R is a rotation by $45°$.

61. Get an empty cardboard box as a visual and tactile aid. The situation is analogous to the rectangle of Example 3. Place the solid at the origin of a three-dimensional coordinate system with the axes aligned through its center. The solid can be rotated by multiples of $180°$ about any of the three axes; denote these rotations as R_x, R_y, and R_z, with $R_x^2 = R_y^2 = R_z^2 = I$ and $R_z = R_x R_y = R_y R_x$, $R_y = R_x R_z = R_z R_x$, and $R_x = R_y R_z = R_z R_y$. Similarly, there are reflection symmetries across planes in each axis direction; call them *H*, *V* , and *Z*, with $H^2 = V^2 = Z^2 = I$. Because the dimensions of the solid are all unequal, there are no symmetries across diagonals through it. However, the composition *HVZ* produces one final symmetry, an "inversion" of the solid through its center: Each point goes to a point the same distance and the opposite direction from the center.

63. The carved head is reproduced in the same shape at different scales.

65. & 67. Answers will vary.