Chapter 15
Game Theory: The Mathematics of Competition

Solutions

Exercises:

1.

$$\begin{bmatrix} 6 & 5 \\ 4 & 2 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ \boxed{5} \\ 2 \end{matrix}$$

$$\text{Column Maxima} \quad 6 \quad \boxed{5}$$

(a) - (b) Saddlepoint at row 1 (maximin strategy), column 2 (minimax strategy), giving value 5.

(c) Row 2 and column 1.

3.

$$\begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ \boxed{-2} \\ \boxed{-2} \end{matrix}$$

$$\text{Column Maxima} \quad \boxed{1} \quad 3$$

(a) No saddlepoint.

(b) Rows 1 and 2 are both maximin strategies; column 1 is the minimax strategy.

(c) None.

5.

$$\begin{bmatrix} -10 & -17 & -30 \\ -15 & -15 & -25 \\ -20 & -20 & -20 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ -30 \\ -25 \\ \boxed{-20} \end{matrix}$$

$$\text{Column Maxima} \quad -10 \quad -15 \quad \boxed{-20}$$

(a) - (b) Saddlepoint at row 3 (maximin strategy), column 3 (minimax strategy), giving value −20.

(c) Column 3 dominates columns 1 and 2, so column player should avoid strategies from columns 1 and 2.

7.

		Pitcher Fastball	Knuckleball	Row Minima
Batter	Fastball	0.500	0.200	0.200
	Knuckleball	0.200	0.300	0.200
	Column Maxima	0.500	0.300	

There is no saddlepoint.

		Pitcher Fastball	Knuckleball	
Batter	Fastball	0.500	0.200	q
	Knuckleball	0.200	0.300	$1-q$
		p	$1-p$	

Batter:
$$E_F = 0.5q + 0.2(1-q) = 0.5q + 0.2 - 0.2q = 0.2 + 0.3q$$
$$E_K = 0.2q + 0.3(1-q) = 0.2q + 0.3 - 0.3q = 0.3 - 0.1q$$

$$E_F = E_K$$
$$0.2 + 0.3q = 0.3 - 0.1q$$
$$0.4q = 0.1$$
$$q = \tfrac{0.1}{0.4} = \tfrac{1}{4}$$
$$1 - q = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

The batter's optimal mixed strategy is $(q, 1-q) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right)$.

Pitcher:
$$E_F = 0.5p + 0.2(1-p) = 0.5p + 0.2 - 0.2p = 0.2 + 0.3p$$
$$E_K = 0.2p + 0.3(1-p) = 0.2p + 0.3 - 0.3p = 0.3 - 0.1p$$

$$E_F = E_K$$
$$0.2 + 0.3p = 0.3 - 0.1p$$
$$0.4p = 0.1$$
$$p = \tfrac{0.1}{0.4} = \tfrac{1}{4}$$
$$1 - p = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

The pitcher's optimal mixed strategy is $(p, 1-p) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right)$, giving value as follows.

$$E_F = E_K = E = 0.2 + 0.3\left(\tfrac{1}{4}\right) = 0.2 + 0.075 = 0.275$$

9. The following table represents the gain or loss for the businessman.

		Tax Agency Not Audit	Audit	Row Minima
Businessman	Not Cheating	\$100	–\$100	[–\$100]
	Cheating	\$1000	–\$3000	–\$3000
	Column Maxima	\$1000	[–\$100]	

Saddlepoint is "not cheat" and "audit," giving value $-\$100$.

11. (a)

	Officer does not patrol	Officer patrols
You park in street	0	–\$40
You park in lot	–\$32	–\$16

(b)

	Officer does not patrol (*NP*)	Officer patrols (*P*)	
You park in street (*S*)	0	–\$40	q
You park in lot (*L*)	–\$32	–\$16	$1-q$
	p	$1-p$	

You:

$$E_P = (0)q + (-32)(1-q) = 0 - 32 + 32q = -32 + 32q$$
$$E_{NP} = -40q + (-16)(1-q) = -40q - 16 + 16q = -16 - 24q$$
$$E_P = E_{NP}$$
$$-32 + 32q = -16 - 24q$$
$$56q = 16$$
$$q = \tfrac{16}{56} = \tfrac{2}{7}$$
$$1 - q = 1 - \tfrac{2}{7} = \tfrac{5}{7}$$

Your optimal mixed strategy is $(q, 1-q) = \left(\tfrac{2}{7}, \tfrac{5}{7}\right)$.

Officer:

$$E_S = (0)p + (-40)(1-p) = 0 - 40 + 40p = -40 + 40p$$
$$E_L = -32p + (-16)(1-p) = -32p - 16 + 16p = -16 - 16p$$
$$E_S = E_L$$
$$-40 + 40p = -16 - 16p$$
$$56p = 24$$
$$p = \tfrac{24}{56} = \tfrac{3}{7}$$
$$1 - p = 1 - \tfrac{3}{7} = \tfrac{4}{7}$$

The officer's optimal mixed strategy is $(p, 1-p) = \left(\tfrac{3}{7}, \tfrac{4}{7}\right)$, giving the following.

$$E_S = E_L = E = -16 - 16\left(\tfrac{3}{7}\right) \approx -16 - 6.86 = -22.86$$

The value is $-\$22.86$.

(c) It is unlikely that the officer's payoffs are the opposite of yours—that she always benefits when you do not.

(d) Use some random device, such as a die with seven sides.

13. (a) Move first to the center box; if your opponent moves next to a corner box or to a side box, move to a corner box in the same row or column. There are now six more boxes to fill, and you have up to three more moves (if you or your opponent does not win before this point), but the rest of your strategy becomes quite complicated, involving choices like "move to block the completion of a row/column/diagonal by your opponent."

(b) Showing that your strategy is optimal involves showing that it guarantees at least a tie, no matter what choices your opponent makes.

15. Player II will choose H $\frac{1}{2}$ of the time and T $\frac{1}{2}$ of the time.

For player I, $E_H = 8\left(\frac{1}{2}\right) - 3 = 4 - 3 = 1$ and $E_T = -4\left(\frac{1}{2}\right) + 1 = -2 + 1 = -1.$

Thus, player I should always play *H*, winning \$1 on average.

17. Rewriting the matrix using abbreviations we have the following.

		Player II		
		F	*C*	*R*
	F	−.25	0	.25
Player I	*BF*	0	0	−.25
	BC	−.25	−.25	0

(a) Player I should avoid "Bet, then call" because it is dominated by "fold" (all entries in *F* row are bigger than corresponding entries in *BC*). Player II should avoid "call" because "fold" dominates it (all entries in *F* column are smaller than corresponding entries in *C*).

(b) Player I will never use "Bet, then call", and Player II will never use "Calls". Removing these, we are left with the following.

		Player II		
		F	R	
Player I	*F*	−.25	.25	q
	BF	0	−.25	$1-q$
		p	$1-p$	

$E_F = -.25q + 0(1-q) = -.25q$

$E_R = .25q + (-.25)(1-q) = .50q - .25$

$$E_F = E_R$$
$$-.25q = .50q - .25$$
$$-.75q = -.25$$
$$q = \tfrac{-.25}{-.75} = \tfrac{1}{3} \Rightarrow 1-q = 1-\tfrac{1}{3} = \tfrac{2}{3}$$

Player I's strategy for (F, BF, BC) is $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

$E_F = -.25p + .25(1-p) = -.25p + .25 - .25p = .25 - .50p$

$E_{BF} = (0)p + (-.25)(1-p) = -.25 + .25p$

$$E_F = E_{BF}$$
$$.25 - .50p = -.25 + .25p$$
$$-.75p = -.50$$
$$p = \tfrac{-.50}{-.75} = \tfrac{2}{3} \Rightarrow 1-p = 1-\tfrac{2}{3} = \tfrac{1}{3}$$

Player II's strategy for (F, C, R) is $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$.

$E_F = E_{BF} = E = -.25 + .25\left(\frac{2}{3}\right) = -\frac{1}{4} + \frac{1}{4}\left(\frac{2}{3}\right) = -\frac{3}{12} + \frac{2}{12} = -\frac{1}{12}$

The value is value $-\frac{1}{12}$.

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17. continued

(c) Player II. Since the value is negative, player II's average earnings are positive and player I's are negative.

(d) Yes. Player I bets first while holding *L* with probability $\frac{2}{3}$. Player II raises while holding *L* with probability $\frac{1}{3}$, so sometimes player II raises while holding *L*.

19. (a) Leave umbrella at home if there is a 50% chance of rain; carry umbrella if there is a 75% chance of rain.

(b) Carry umbrella in case it rains.

(c) Saddlepoint at "carry umbrella" and "rain," giving value –2.

(d) Leave umbrella at home.

21. The Nash equilibrium outcomes are $(4,3)$ and $(3,4)$. [It would be better if the players could flip a coin to decide between $(4,3)$ and $(3,4)$.]

23. The Nash equilibrium outcome is $(2,4)$, which is the product of dominant strategies by both players.

25. The players would have no incentive to lie about the value of their own weapons unless they were sure about the preferences of their opponents and could manipulate them to their advantage. But if they do not have such information, lying could cause them to lose more than 10% of their weapons, as they value them, in any year.

27. The sophisticated outcome, *x*, is found as follows: *Y*'s strategy of *y* is dominated; with this strategy of *Y* eliminated, *X*'s strategy of *x* is dominated; with this strategy of *X* eliminated, *Z*'s strategy of *z* is dominated, which is eliminated. This leaves *X* voting for *xy* (both *x* and *y*), *Y* voting for *yz*, and *Z* voting for *zx*, creating a three-way tie for *x*, *y*, and *z*, which *X* will break in favor of *x*.

29. The payoff matrix is as follows:

		Even		
		2	**4**	**6**
	1	$(2,1)$	$(2,1)$	$(2,1)$
Odd	**2**	$(2,4)$	$(6,3)$	$(6,3)$
	3	$(2,4)$	$(4,8)$	$(10,5)$

Odd will eliminate strategy 1, and Even will eliminate strategy 6, because they are dominated. In the reduced 2×2 game, Odd will eliminate strategy 5. In the reduced 1×2 game, Even will eliminate strategy 4. The resulting outcome will be $(2,4)$, in which Odd chooses strategy 3 and Even chooses strategy 2. The outcome $(2,1)$, in which Odd chooses strategy 1 and Even chooses strategy 2, is also in equilibrium.

31. If the first player shoots in the air, he will be no threat to the two other players, who will then be in a duel and shoot each other. If a second player fires in the air, then the third player will shoot one of these two, so the two who fire in the air will each have a 50–50 chance of survival. Clearly, the third player, who will definitely survive and eliminate one of her opponents, is in the best position.

33. In a duel, each player has incentive to fire – preferably first – because he or she does better whether the other player fires (leaving no survivors, which is better than being the sole victim) or does not fire (you are the sole survivor, which is better than surviving with the other player). In a truel, if you fire first, then the player not shot will kill you in turn, so nobody wants to fire first. In a four-person shoot-out, if you fire first, then you leave two survivors, who will not worry about you because you have no more bullets, leading them to duel. Thus, the incentive in a four-person shoot-out—to fire first—is the same as that in a duel.

35. Nobody will shoot.

37. The possibility of retaliation deters earlier shooting.