Chapter 14
Apportionment

Solutions

Exercises:

1. Jane's total expenses are $71. The calculation of the percentages is shown in the table.

	Percentage	rounded
Rent	$\frac{31}{71}\times 100\% = 43.66\%$	44%
Food	$\frac{16}{71}\times 100\% = 22.54\%$	23%
Transportation	$\frac{7}{71}\times 100\% = 9.86\%$	10%
Gym	$\frac{12}{71}\times 100\% = 16.90\%$	17%
Miscellaneous	$\frac{5}{71}\times 100\% = 7.04\%$	7%

The percentages add up to 101%.

3. The new enrollments are obtained by subtracting from the enrollment of each level the number of students who are moving to a lower level, and add to each the number of students who are moving from a higher level. Here are the calculations.

Calculus I	500 + 45 =	545
Calculus II	100 − 45 + 41 =	96
Calculus III	350 − 41 + 12 =	321
Calculus IV	175 − 12 =	163

The total number of students enrolled remains 1125, and the average number of students per teaching assistant is still 56.25. Here are the new quotas.

Calculus I	545 ÷ 56.25 =	9.69
Calculus II	96 ÷ 56.25 =	1.71
Calculus III	321 ÷ 56.25 =	5.71
Calculus IV	163 ÷ 56.25 =	2.90

The new rounded quotas are as follows.

Calculus I	10
Calculus II	2
Calculus III	6
Calculus IV	3
Total	21

This calls for too many teaching assistants, so the numbers must be adjusted. The apportionment methods introduced in this chapter present a variety of approaches to solving this problem.

5. Rounding each of the summands down, the sum of the lower quotas is 0 + 1 + 0 + 2 + 2 + 2 = 7. The three numbers with the greatest fractional parts, 0.99, 1.59, and 2.38, receive their upper quotas. The apportioned sum is 0 + 2 + 1 + 2 + 3 + 2 = 10.

7. The total population is 510,000, so the standard divisor is 510,000 ÷ 102 = 5,000. The quotas are obtained by dividing each state's population by this divisor, obtaining 50.8, 30.6, and 20.6, respectively. The lower quotas add up to 100, so we must increase the apportionment of two states to their upper quotas. The first state, whose quota of 50.8 has the largest fractional part, gets an increase. The fractional parts of the quotas of the remaining two states are both equal to 0.6: they are tied for priority in receiving the last seat. A coin toss is probably the fairest way to settle this dispute.

9. The total enrollment is 115, and the standard divisor is 23. The quotas are as follows.

Geometry	77 ÷ 23 =	3.35 sections
Algebra	18 ÷ 23 =	0.78 sections
Calculus	20 ÷ 23 =	0.87 sections

The lower quota for geometry is 3, and the other two subjects have 0 lower quotas. Because they have larger fractional parts than geometry, they both receive their upper quotas, 1 each. The apportionment is as follows.

Geometry	3 sections
Algebra	1 section
Calculus	1 section

11. The states in this apportionment problem are the investors, the seats are the 100 coins, and the populations are the individual investments. Thus, the standard divisor is \$10,000 ÷ 100 coins = \$100 per coin. The quotas, which represent the number of coins each investor should receive if fractional coins were possible, are obtained by dividing each investment by this divisor.

	Quota	Lower quota
Abe	36.190	36
Beth	18.620	18
Charles	22.580	22
David	20.100	20
Esther	2.510	2
Total	100.00	98

Two investors will receive their upper quotas: Beth and Charles, who have the largest fractions. Here are the apportionments, *before the excise tax was paid.*

Abe	36
Beth	19
Charles	23
David	20
Esther	2
Total	100

When the excise tax is added, populations change, and the standard divisor changes as follows.

\$10,050 ÷ 100 = \$100.50 per coin

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11. continued

We have to recalculate the quotas. The revised investments are divided by the new standard divisor as follows.

	Investment	Quota	Lower quota
Abe	$3,635	36.169	36
Beth	$1,864	18.547	18
Charles	$2,259	22.478	22
David	$2,042	20.318	20
Esther	$250	2.488	2
Total	$10,050	100.000	98

Again, two investors will receive their upper quotas: Beth and Aunt Esther. The final apportionments are as follows.

	Before tax	After tax
Abe	36	36
Beth	19	19
Charles	23	22
David	20	20
Esther	2	3
Total	100	100

So, Aunt Esther not only got a dollar back, but Charles had to give her one of his rare coins! At least it's still in the family. The cause of this confusion is, of course, the population paradox.

13. In the following table, the critical divisors and quotas are displayed.

House size	82	83	84	89	90	91
Divisor	220,997	218,334	215,735	203,615	201,353	199,140
	Quotas					
A	25.233	25.540	25.848	27.387	27.694	28.002
B	6.278	6.354	6.431	6.814	6.890	6.967
C	15.087	15.271	15.455	16.375	16.559	16.743
D	33.995	34.410	34.824	36.897	37.312	37.7276
E	1.407	1.424	1.442	1.527	1.544	1.562

The next table displays the lower quotas and their sum for each of the house sizes under consideration.

State	Lower Quotas					
A	25	25	25	27	27	28
B	6	6	6	6	6	6
C	15	15	15	16	16	16
D	33	34	34	36	37	37
E	1	1	1	1	1	1
Total	80	81	81	86	87	88
Shortage	2	2	3	3	3	3

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13. continued

The last row of the above table records the number of seats that still must be apportioned. These seats go to the states whose quotas have the largest fractional parts. The final apportionments are as follows.

State	State Population	Apportionments					
A	5,576,330	25	26	26	27	28	28
B	1,387,342	6	6	6	7	7	7
C	3,334,241	15	15	16	16	17	17
D	7,512,860	34	34	35	37	37	38
E	310,968	2	2	1	2	1	1
Total	18,121,741	82	83	84	89	90	91

The Alabama paradox occurs when the apportionment for the smallest state decreases from 2 to 1 as the house size increases from 83 to 84, and it occurs again as the house size increases from 89 to 90.

15. As with the Hamilton method, we have the following quotas.

Geometry	3.35 sections
Algebra	0.78 sections
Calculus	0.87 sections

The tentative apportionments are geometry, 3; algebra and calculus, 0. The critical divisors are determined by adding 1 to the tentative apportionments and dividing the result into the population of the subject, and are as follows.

Geometry	77 ÷ 4 =	19.25 students
Algebra	18 ÷ 1 =	18 students
Calculus	20 ÷ 1 =	20 students

Calculus has the greatest critical divisor, and its tentative apportionment is now 1. It receives a new critical divisor, 20 ÷ 2 = 10. Now the greatest critical divisor is that of geometry, so its apportionment is 4. The house is full, and the Jefferson apportionment is

Geometry	4 sections
Algebra	cancelled!
Calculus	1 section

17. All three divisor methods start with the quotas, which were computed in Exercise 6.

	36 pearls	37 pearls
Abe	14.25	14.65
Beth	18.36	18.87
Charles	3.38	3.48

Jefferson method: The tentative apportionments are, for 36 or 37 pearls, Abe, 14; Beth, 18; and Charles, 3. With 36 pearls, 1 is left to be apportioned; with 37 there are 2 left. Here are the critical divisors.

Abe	\$5,900 ÷ 15 =	\$393.33
Beth	\$7,600 ÷ 19 =	\$400.00
Charles	\$1,400 ÷ 4 =	\$350.00

The 36th pearl goes to Beth. When the 37th pearl is discovered, there is no need to repeat the calculations. Beth's critical divisor (only) has to be recomputed, because she has another pearl now. Now her critical divisor is \$7,600 ÷ 20 = \$380.00. The highest priority for the 37th pearl goes to Abe. Here are the final Jefferson apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	19	19
Charles	3	3

Webster method: The tentative apportionments are obtained by rounding the quotas. With 36 pearls, all the quotas are rounded down, so the tentative apportionments add up to 35. We will have to calculate critical divisors to allocate the 36th pearl.

Abe	\$5,900 ÷ 14.5 =	\$406.90
Beth	\$7,600 ÷ 18.5 =	\$410.81
Charles	\$1,400 ÷ 3.5 =	\$400.00

Beth, with the greatest critical divisor, gets the 36th pearl. With 37 pearls, Abe's and Beth's quotas are both rounded up, and Charles's is rounded down. These tentative apportionments, 15, 19, and 3, add up to 37. Abe receives the 37th pearl. Here are the final Webster apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	19	19
Charles	3	3

Hill-Huntington method: The rounding point for numbers between 3 and 4 is $\sqrt{3\times 4}=3.464$; for numbers between 14 and 15 it is $\sqrt{210}=14.491$; and for numbers between 18 and 19 it is $\sqrt{342}=18.493$. Rounding *a la* Hill-Huntington, we obtain the following tentative apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	18	19
Charles	3	4
Total	35	38

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17. continued

When the calculation is done with 36 pearls, only 35 are accounted for by the tentative apportionments, and with 37, the apportionments add up to 38. Let's calculate critical divisors to determine who gets the 36th and 37th pearls.

Abe	$\$5,900 \div \sqrt{14\times15}$ =	\$407.14
Beth	$\$7,600 \div \sqrt{352}$ =	\$405.08
Charles	$\$1,400 \div \sqrt{12}$ =	\$404.15

Abe has priority for the 36th pearl, and once he receives it, his critical divisor is recomputed as $\$5,900 \div \sqrt{15\times16} = \$380.84$. The priority for the 37th pearl goes to Beth. Here are the final Webster apportionments.

	36 pearls	37 pearls
Abe	15	15
Beth	18	19
Charles	3	3

With 36 pearls, there is a difference between the Hill-Huntington apportionment and the others, but with 37, the three methods produce the same results. If there is a principle on which to choose a method, it would probably be to choose the method by which the cost per pearl is as close as possible to the same for each of the friends. The cost per pearl is the district size. The method that minimizes relative differences in the cost per pearl is Hill-Huntington method. If the friends would prefer to minimize absolute differences, they would have to use the Dean method, which was not covered in this chapter. Charles might want to study up on it, though, because it allocates the 36th pearl to Beth, and the 37th to him!

19. The percentages are the quotas.

Hamilton method: Start with the lower quotas, $87 + 10\times1$, whose sum is 97. The three percentages with the greatest fractional parts, 87.85, 1.26, and 1.25, are rounded up to get the upper quotas; the remaining percentages are rounded down. The final apportionment is

$$88 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 100\%.$$

The first three percentages are rounded to upper quotas, and the remaining percentages are rounded to lower quotas. The quota condition is satisfied.

Jefferson method: Tentatively apportion to each percentage its lower quota. The critical divisors are then the unrounded percentage divided by (1 + the tentative apportionment). Thus, the critical divisor belonging to 87.85% is 87.85 ÷ 88 = 0.9983, while the critical divisors belonging to the smaller percentages range from 1.26 ÷ 2 = 0.63 down to 1.17 ÷ 2 = 0.585. The largest critical divisor belongs to 87.85%, so its tentative apportionment is increased to 88 and its new critical divisor is 87.85 ÷ 89 = 0.9871. This is still the largest critical divisor, so the apportionment of 87.85% is increased to 89. The new critical divisor, 87.85 ÷ 90 = 0.9761, is still the largest, so its apportionment is increased to 90. Now the house is full, and the Jefferson apportionment is

$$90 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 100\%.$$

This apportionment rounds 87.85% to 90%, more than the upper quota. The quota condition is violated.

Webster method: The rounded percentages add up to 98, so we need to calculate critical divisors. The critical divisor belonging to 87.85% is 87.85 ÷ 88.5 = 0.9927. Among the smaller percentages, the largest critical divisor is that of 1.26%, which is 1.26 ÷ 1.5 = 0.84. The point goes to 87.85%, whose apportionment increases to 89. This calls for a new critical divisor, 87.85 ÷ 89.5 = 0.9816, which exceeds the critical divisors of the smaller percentages. The apportionment of 87.85% is therefore increased again to 90. The final apportionment is the same as the Jefferson apportionment, so it too violates the quota condition.

21. (a) $\sqrt{0\times 1} = 0$

(b) $\sqrt{1\times 2} = 1.4142$

(c) $\sqrt{2\times 3} = 2.4495$

(d) $\sqrt{3\times 4} = 3.4641$

23. The standard divisor is $(36+61+3)\div 5 = 20$ students. The quotas are as follows.

Algebra	36 ÷ 20 =	1.8 sections
Geometry	61 ÷ 20 =	3.05 sections
Calculus	3 ÷ 20 =	0.15 sections

Webster would round the quotas to 2, 3, and 0, respectively. These tentative apportionments add up to 5, the house size, and are the final Webster apportionments. Because Hill-Huntington rounds all numbers between 0 and 1 to 1, its tentative apportionment would be 2, 3, and 1. This would exceed the house size by 1, so we have to reduce one of the tentative apportionments. This requires critical divisors. They are as follows.

Algebra	$36 \div \sqrt{2\times 1}$	=	25.456 students
Geometry	$62 \div \sqrt{3\times 2}$	=	24.903 students
Calculus	$7 \div \sqrt{1\times 0}$	=	∞ students

The least critical divisor belongs to Geometry, so its apportionment is decreased to 2. In summary, here are the apportionments.

	Webster	Hill-Huntington
Algebra	2	2
Geometry	3	2
Calculus	cancelled!	1

It's likely that the principal would prefer the Webster method, because classes as small as the calculus class, with 3 students, should be cancelled. Notice that the Hill-Huntington apportionment gives Geometry less than its lower quota in order to accommodate Calculus.

25. Let's start by taking a seat from California, putting it in play. This leaves 52 seats for California, and California's priority for getting the extra seat is measured by its critical divisor,

$$\frac{\text{Population of California}}{\sqrt{52\times 53}} = 646{,}330.227.$$

To secure the seat in play, Utah's population has to increase enough so that its critical divisor,

$$\frac{\text{Revised population of Utah}}{\sqrt{3\times 4}},$$

surpasses California's. Thus, Utah needs a population of more than the following.

$$646{,}330.227\times\sqrt{12} = 2{,}238{,}954$$

The 2000 census recorded Utah's population as 2,236,714, so an additional 2241 residents would be needed.

27. Before the excise tax was included, the quotas, calculated as in Exercise 11, are rounded to obtain a tentative apportionment.

	Quota	Rounded quota
Abe	36.19	36
Beth	18.62	19
Charles	22.58	23
David	20.10	20
Esther	2.51	3
Total	100.00	101

One quota must be reduced, so we calculate critical divisors as follows.

Abe	$\$3619 \div (36 - 0.5) =$	\$101.94
Beth	$\$1862 \div (19 - 0.5) =$	\$100.65
Charles	$\$2258 \div (23 - 0.5) =$	\$100.36
David	$\$2010 \div (20 - 0.5) =$	\$103.08
Esther	$\$251 \div (3 - 0.5) =$	\$100.40

The least critical divisor is Charles's, so his apportionment is 22. After the tax is added, new rounded quotas are calculated.

	Quota	Rounded quota
Abe	36.17	36
Beth	18.55	19
Charles	22.48	22
David	20.32	20
Esther	2.49	2
Total	100.01	99

Now one of the tentative apportionments must increase, so we must again compute critical divisors.

Abe	$\$3635 \div (36 + 0.5) =$	\$99.589
Beth	$\$1864 \div (19 + 0.5) =$	\$95.590
Charles	$\$2259 \div (22 + 0.5) =$	\$100.400
David	$\$2042 \div (20 + 0.5) =$	\$99.610
Esther	$\$250 \div (2 + 0.5) =$	\$100.000

Charles has the largest critical divisor, so his apportionment is increased to 23. The final apportionments are as follows.

	Before tax	After tax
Abe	36	36
Beth	19	19
Charles	22	23
David	20	20
Esther	3	2
Total	100	100

Esther must give one of her three rare coins to her nephew.

29. The quota for the Liberals is $99 \times 49\% = 48.51$, and the Tories' quota is 50.49. With the Hamilton method, the lower quotas add up to 98, and the additional seat goes to the party whose quota has the largest fractional part. This gives the Liberals 49 votes, and the Tories have 50. The Webster method yields the same result because it would round the Liberals' quota up, and the Tories' down.

The Jefferson starts by giving each party its lower quota, 48 for the Liberals and 50 for the Tories. The last seat is given to the party with the largest critical divisor. The formula for critical divisors is (percent of vote received) ÷ (1 + tentative apportionment). Thus the critical divisor for the Liberals is 49 ÷ (1 + 48) = 1, and the critical divisor for the Tories is 51 ÷ (1 + 50) = 1. There is a tie for the 99th seat.

31. The following table displays the quotas and tentative apportionment due to the Webster method.

State	Population	Quota	Tentative apportionment
Virginia	630,560	18.310	18
Massachusetts	475,327	13.803	14
Pennsylvania	432,879	12.570	13
North Carolina	353,523	10.266	10
New York	331,589	9.629	10
Maryland	278,514	8.088	8
Connecticut	236,841	6.877	7
South Carolina	206,236	5.989	6
New Jersey	179,570	5.214	5
New Hampshire	141,822	4.118	4
Vermont	85,533	2.484	2
Georgia	70,835	2.057	2
Kentucky	68,705	1.995	2
Rhode Island	68,446	1.988	2
Delaware	55,540	1.613	2
Totals	3,615,920	105	105

Because the tentative apportionment results in the assignment of 105 seats, there is no need for critical divisors: it is the final apportionment. In effect, a seat that had been assigned to Vermont moves to Pennsylvania.

33. Jim is 7 inches taller than Alice. The relative difference of their heights is 7 inches divided by Alice's height, 65 inches: $\frac{7}{65}$ = 10.77%.

35. (a) California, $33{,}930{,}798 \div 53 = 640{,}204$; Utah, $2{,}236{,}714 \div 3 = 745{,}571$.

(b) Absolute difference, 745, 571 − 640,204 = 105,367,

Relative difference, $105{,}367 \div 640{,}204 = 16.46\%$

(c) The district size for California would be 33,930,798 ÷ 52 = 652,515, and the district size for Utah would be 2,236,714 ÷ 4 = 559,178.5. The absolute difference is 652,515 − 559,178 = 93,336.5. The relative difference is 93,336.5 ÷ 559,178.5 = 16.69%

(d) The absolute difference in district populations would be less if California had 52 seats, and Utah had 4. With that revised apportionment, the relative differences would be greater. Thus, the Hill-Huntington method, which was used in apportioning Congress after the 2000 census, did not minimize absolute differences in district population. It minimized relative differences.

37. With 10 seats for Massachusetts, and 6 for Oklahoma, the representative shares (per million population) for these states are 10 ÷ 6.029051 = 1.6586 seats per million for Massachusetts, and 6 ÷ 3.145585 = 1.9074 for Oklahoma. The inequity in representative share is in favor of Oklahoma, by 0.2488 seats per million population. If Massachusetts had 11 seats, and Oklahoma 5, the respective representative shares would be 1.8245 and 1.5895. The inequity, in favor of Massachusetts, is 0.235 seats per million population. Therefore, the Webster apportionment would give Massachusetts the seat.

39. (a) Lowndes favors small states, because in computing the relative difference, the fractional part of the quota will be divided by the lower quota. If a large state had a quota of 20.9, the Lowndes relative difference works out to be 0.045. A state with a quota of 1.05 would have priority for the next seat.

(b) Yes, because like the Hamilton method, the Lowndes method presents a way to decide, for each state, if the lower or upper quota should be awarded.

(c) Yes. Since the method is not a divisor method, the population paradox is inevitable.

(d) Let r_i denote the relative difference between the quota and lower quota for state *i*. The following table displays the numbers r_i for each state. Because the lower quotas add up to 97, the 8 states with the largest values in the r_i column will receive their upper quotas.

State	p_i	q_i	$\lfloor q_i \rfloor$	r_i	rank	a_i
Virginia	630,560	18.310	18	1.7%	14	18
Massachusetts	475,327	13.803	13	6.2%	8	14
Pennsylvania	432,879	12.570	12	4.8%	9	12
North Carolina	353,523	10.266	10	2.7%	13	10
New York	331,589	9.629	9	7.0%	7	10
Maryland	278,514	8.088	8	1.1%	15	8
Connecticut	236,841	6.877	6	14.6%	6	7
South Carolina	206,236	5.989	5	19.8%	5	6
New Jersey	179,570	5.214	5	4.3%	10	5
New Hampshire	141,822	4.118	4	3.0%	11	4
Vermont	85,533	2.484	2	24.2%	4	3
Georgia	70,835	2.057	2	2.9%	12	2
Kentucky	68,705	1.995	1	99.5%	1	2
Rhode Island	68,446	1.988	1	98.8%	2	2
Delaware	55,540	1.613	1	61.3%	3	2
Totals	3,615,920	105	97	–	–	105

41. (a) Let $n = \lfloor q \rfloor$. If q is between n and $n+0.4$, then the Condorcet rounding of q is equal to n. Since $q+0.6 < n+1$ in this case, it is also true that $\lfloor q+0.6 \rfloor = n$. On the other hand, if $n+0.4 \le q < n+1$, then the Condorcet rounding of q is $n+1$, and also $n+1 \le q+0.6 < n+1.6$, so $\lfloor q+0.6 \rfloor = n+1$.

(b) The method favors small states, since numbers will be rounded up more often than down; and this makes it more likely that the quotas will be adjusted downward.

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41. continued

(c) If the sum of the tentative apportionments is less than the house size, the critical divisor for state *i*, with population p_i, is the greatest divisor d_i that would apportion another seat to the state. Thus, if the tentative apportionment is n_i, then $n_i + 0.4 = \frac{p_i}{d_i}$, and hence $d_i = \frac{p_i}{n_i + 0.4}$. The state with the largest critical divisor gets the next seat, and then its critical divisor is recomputed. The process stops when the house is full.

If the total apportionment is more than the house size, then the critical divisor for state *i* is the least divisor that would cause the state's tentative apportionment to decrease. Thus $n_i - 1 + 0.4 = \frac{p_i}{d_i}$, so $d_i = \frac{p_i}{n_i - 0.6}$. The state with the least critical divisor of all has its tentative apportionment decreased by 1. Its critical divisor is then recomputed. The process stops when enough seats have been removed so that the number of seats apportioned is equal to the house size.

43. Let $f_i = q_i - \lfloor q_i \rfloor$ denote the fractional part of the quota for state *i*. Since the Hamilton method assigns to each state either its lower or its upper quota, each absolute deviation is equal to either f_i (if state *i* received its lower quota) or 1 − f_i (if it received its upper quota). For convenience, let's assume that the states are ordered so that the fractions are decreasing, with f_1 the largest and f_n the smallest. If the lower quotas add up to $h - k$, where *h* is the house size, then states 1 through *k* will receive their upper quotas. The maximum absolute deviation will be the larger of $1 - f_k$ and f_{k+1}.

The maximum absolute deviation for the Hamilton method is less than 1, because each fractional part f_i and its complement, $1 - f_i$, is less than 1. If a particular apportionment fails to satisfy the quota condition, then for al least one state, the absolute deviation exceeds 1, and hence the maximum absolute deviation is greater than that of the Hamilton apportionment.

If an apportionment satisfies the quota condition then — as with the Hamilton method — *k* states receive their upper quotas and $n - k$ states receive their lower quotas.

If a state *j*, where $j \le k$, receives its lower quota, then — to compensate — a state *l*, where *l* > *k*, must get its upper quota. The absolute deviations for these states would be f_j and $1 - f_l$, respectively. Because of the way the fractions have been ordered, we have $1 - f_l \ge 1 - f_k$. Therefore, the absolute deviation for one of states *j* and *l* will be equal to or exceed the maximum absolute deviation of the Hamilton apportionment. We conclude that no apportionment is better than Hamilton's, if what we mean by "better" is "smaller maximum absolute deviation."