Chapter 13
Fair Division

Solutions

Exercises:

1. Donald initially receives the Palm Beach mansion (40 points) and the Trump Tower triplex (38 points) for a total of 78 points. Ivana initially receives the Connecticut estate (38 points) and the Trump Plaza apartment (30 points) for a total of 68 points. Because Ivana has fewer points than Donald, she receives the cash and jewelry (on which they both placed 2 points) bringing her total to 70 points. As Donald still has more points (78 to 70), we begin transferring items from him to her. To determine the order of transfer, we must calculate the point ratios of the items that Donald now has.

The point ratio of the Palm Beach mansion is $\frac{40}{20} = 2.0$.

The point ratio of the Trump Tower triplex is $\frac{38}{10} = 3.8$.

Because 2.0 < 3.8, the first item to be transferred is the Palm Beach mansion. However, if all of it were given to Ivana, her point total would rise to 70 + 20 = 90, and Donald's point total would fall to 78 – 40 = 38. This means that only a fraction of the Palm Beach mansion will be transferred from Donald to Ivana.

Let x be the fraction of the Palm Beach mansion that Donald retains, and let $1-x$ be the fraction of it that is given to Ivana. To equalize point totals, x must satisfy $38+40x = 70+20(1-x)$. Thus, using algebra to solve this equation yields the following.

$$\begin{aligned} 38+40x &= 70+20-20x \\ 38+40x &= 90-20x \\ 60x &= 52 \\ x &= \tfrac{52}{60} \\ x &= \tfrac{13}{15} \end{aligned}$$

Thus Donald receives the Trump Tower triplex and $\frac{13}{15}$ (about 87%) ownership of the Palm Beach mansion for a total of about 72.7 of his points, and Ivana gets the rest (for about 72.7 of her points).

3. Mike initially gets his way on the room party policy (50), the cleanliness issue (6), and lights-out time (10) for a total of 66 points. Phil initially gets his way on the stereo level issue (22), smoking rights (20), phone time (8), and the visitor policy (5) for a total of 55 points. Because Phil has fewer points that Mike, he gets his way on the alcohol use issue, on which they both placed 15 points, bringing his total to 70. To determine the order of transfer (from Phil to Mike), we must calculate the point ratios of the issues on which Phil got his way.

Point ratio of the stereo level issue is $\frac{22}{4} = 5.5.$

Point ratio of the smoking rights issue is $\frac{20}{10} = 2.0.$

Point ratio of the alcohol issue is $\frac{15}{15} = 1.0.$

Point ratio of the phone time issue is $\frac{8}{1} = 8.0.$

Point ratio of the visitor policy issue is $\frac{5}{4} = 1.25.$

The first issue to be transferred is the alcohol issue, because it has the lowest point ratio. However, if all of it were given to Mike, his point total would rise to 66 + 15 = 81, and Phil's point total would fall to 70 – 15 = 55. This means that only a fraction of the alcohol issue will be transferred from Phil to Mike.

Let x be the fraction of the alcohol issues that Phil retains, and let $1-x$ be the fraction of it that is given to Mike. To equalize point totals, x must satisfy $55+15x = 66+15(1-x)$.

Thus, using algebra to solve this equation yields the following.

$$55+15x = 66+15-15x$$
$$55+15x = 81-15x$$
$$30x = 26$$
$$x = \frac{26}{30}$$
$$x = \frac{13}{15}$$

Thus, Phil gets his way on the stereo level issue, the smoking rights issue, the phone time issue, the visitor policy issue, and $\frac{13}{15}$ (about 87%) of his way on the alcohol issue for a total of 68 points. Mike gets his way on the rest.

5. Answers will vary.

7. Allocation 1:

(a) Not proportional: Bob gets 10% in his eyes.

(b) Not envy-free: Bob, for example, envies Carol.

(c) Not equitable: Bob thinks he got 10% and Carol thinks she got 40%.

(d) Example: Give Bob X, Carol Y, and Ted Z.

Allocation 2:

(a) Not proportional: Carol gets 30% in her eyes.

(b) Not envy-free: Carol, for example, envies Bob.

(c) Not equitable: Bob thinks he got 50% and Carol thinks she got 30%.

(d) Example: Give Bob Y, Carol X, and Ted Z.

Allocation 3:

(a) Not proportional: Carol and Ted get 0% in their eyes.

(b) Not envy-free: Carol and Ted envy Bob.

(c) Not equitable: Bob thinks he got 100% and Carol thinks she got 0%.

(d) It is Pareto optimal – for Carol or Ted to get anything, Bob will have to get less.

Allocation 4:

(a) Not proportional: Carol gets 30% in her eyes.

(b) Not envy-free: Carol, for example, envies Bob.

(c) Not equitable: Bob thinks he got 50% and Carol thinks she got 30%.

Allocation 5:

(a) It is proportional.

(b) Not envy-free: Bob, for example, envies Carol.

(c) It is equitable.

9. They handle the car first, as in Exercise #8. Then Mary gets the house and places $\frac{59{,}100}{2} = 29{,}550$ in a kitty. John takes out $\frac{55{,}900}{2} = 27{,}950$ and they split the remaining $29{,}550 - 27{,}950 = 1{,}600$ equally. Thus, for the house, Mary gets it and gives John \$28,750. In total, Mary gets both the car and the house and pays John $\$15{,}081.25 + \$28{,}750 = \$43{,}831.25$.

11. First, *C* gets the house and places two-thirds of 165,000 (i.e., 110,000) in a kitty. *A* then withdraws one-third of 145,000 (i.e., 48,333) and *B* withdraws one-third of 149,999 (i.e., 50,000). They divide the remaining 11,667 equally among the three of them.

Second, *A* gets the farm and places two-thirds of 135,000 (i.e., 90,000) in a kitty. *B* then withdraws one-third of 130,001 (i.e., 43,334) and *C* withdraws one-third of 128,000 (i.e., 42,667). They divide the remaining 3,999 equally among the three of them.

Third, *C* gets the sculpture and deposits two-thirds of 127,000 (i.e., 84,667) in a kitty. *A* then withdraws one-third of 110,000 (i.e., 36,667) and *B* withdraws one-third of 80,000 (i.e., 26,667). They divide the remaining 21,333 equally among them.

Thus, *A* gets the farm and receives \$52,222 + \$43,778 and pays \$44,667 + \$44,000, so *A*, in total, receives the farm plus \$7,333. Similarly, *B* receives \$132,334 and *C* receives both the house and the sculpture, while paying \$139,667.

13. The bottom-up strategy fills in the blanks as follows:

Bob: investments car CD player

Carol: boat television washer-dryer

Thus, Bob first chooses the investments, and the final allocation has him also receiving the car and the CD player.

15. The bottom-up strategy fills in the blanks as follows:

Mark: tractor truck tools

Fred: boat car motorcycle

Thus, Mark first chooses the tractor, and the final allocation has him also receiving the truck and the tools.

17. The bottom-up strategy fills in the blanks as follows (CT stands for Connecticut):

Donald: mansion triplex cash and jewelry

Ivana: CT estate apartment

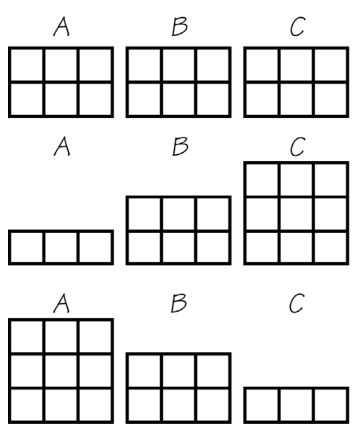
Thus, Donald first chooses the Palm Beach mansion, and the final allocation has him also receiving the Trump Tower triplex and the cash and jewelry.

19. The chooser. As divider, I'd get exactly 50% (or risk getting less). As chooser, I have a guarantee of getting at least 50% and the possibility (depending on the division) of getting more than 50%.

21. (a) Bob gets a piece whose value to him is 9 units (assuming that Bob is the divider), and Carol gets a piece whose value to her is 12 units.

(b) Carol gets a piece whose value to her is 9 units (assuming that Carol is the divider), and Ted gets a piece whose value to him is 15 units.

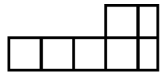
23. (a) See figures below.



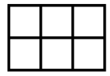
(b) Player 2 finds *B* acceptable (6 square units) and *C* acceptable (9 square units). Player 3 finds *A* acceptable (9 square units) and *B* acceptable (6 square units).

(c) Player 3 chooses *A* (9 square units). Player 2 chooses *C* (9 square units). Player 1 chooses *B* (6 square units). Yes, there is another order. Player 2 chooses *C* (9 square units). Player 3 chooses *A* (9 square units). Player 1 chooses *B* (6 square units).

25. (a) See figure below.

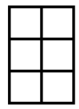


(b) Player 2 will further trim the piece:



in Player 2's eyes.

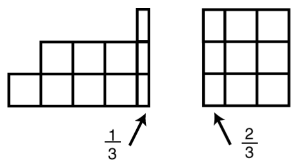
(c) Player 3 will further trim the piece:



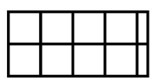
in Player 3's eyes.

(d) Player 3 receives it, and thinks it is 6 units of value. The one leaving with the first piece always thinks it is one-nth of the value with *n* players.

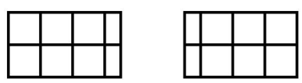
(e) Assume Player 1 is the divider. He sees it as 16 units of value, and he divides it as follows:



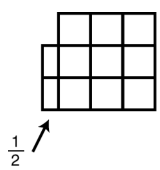
Player 2 chooses the piece on the left, which he sees as follows:



(f) Assume Player 2 is the divider. He sees it as 14 units of value, and he divides it as follows:

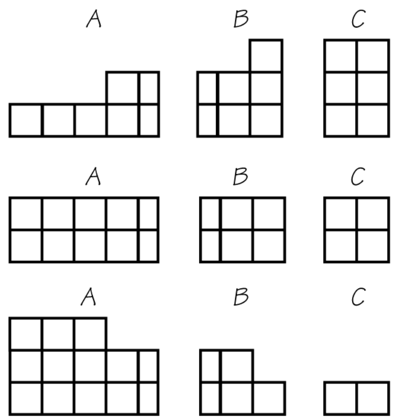


Player 1 chooses the piece on the right, which he sees as follows:

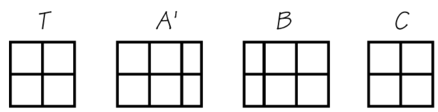


27. Bob, Carol, and Ted each divide the piece he or she has in four parts (equal in his or her own estimation). Alice then chooses one of Bob's four pieces, one of Carol's four pieces, and one of Ted's four pieces.

29. (a) See figures below.



(b) See figures below.



(c) Player 3 will choose A (which he thinks is of size 6 square units). Player 2 will choose B (which he thinks is of size 5 square units). Player 1 will receive C (which he thinks is of size 6 square units). The proviso does not come into Play (since Player 3 took the trimmed piece).