Chapter 12
Electing the President

Solutions

Exercises:

1. Assume a distribution is skewed to the left. The heavier concentration of voters on the right means that fewer voters are farther from the median. Because there are fewer voters "pulling" the mean rightward, it will be to the left of the median. Likewise, a distribution skewed to the right will have a mean to the right of the median.

3. While there is no median position such that half the voters lie to the left and half to the right, there is still a position where the middle voter (if the number of voters is odd) or the two middle voters (if the number of voters is even) are located, starting either from the left or right. In the absence of a median, less than half the voters lie to the left and less than half to the right of this middle voter's (voters') position (positions).

Hence, any departure by a candidate from a position of a middle voter to the position of a non-middle voter on the left or right will result in that candidate's getting less than half the votes – and the opponent's getting more than half. Thus, the middle position (positions) is (are) in equilibrium, making it (them) the extended median.

5. When the four voters on the left refuse to vote for a candidate at 0.6, his opponent can do better by moving to 0.7, which is worse for the dropouts.

7. The voters are spread from 0.1 to 0.9, so it is a position at 0.5 that minimizes the maximum distance (0.4) a candidate is from a voter. If the candidates are at the median of 0.6, the voter at 0.1 would be a distance of 0.5 from them. In this sense, the median is worse than the mean of 0.56, which would bring the candidates closer to the farthest-away voter and, arguably, be a better reflection of the views of the electorate.

9. The middle peak will be in equilibrium when it is the median or the extended median. Yes, it is possible that, say, the peak on the left is in equilibrium, as illustrated by the following discrete-distribution example, in which the median is 0.2:

Position *i*	1	2	3	4	5	6	7
Location (l_i) of position *i*	0.1	0.2	0.3	0.5	0.6	0.8	0.9
Number of voters (n_i) at position *i*	7	8	1	2	1	2	1

11. If the population is not uniformly distributed and, say, 80% live between $\frac{3}{8}$ and $\frac{5}{8}$ and only 10% live to the left of $\frac{3}{8}$ and 10% to right of $\frac{5}{8}$, then the bulk of the population will be well served by two stores at $\frac{1}{2}$. In fact, stores at $\frac{1}{4}$ and $\frac{3}{4}$ will be farther away for 80% of the population, so it can be argued that the two stores at $\frac{1}{2}$ provide a social optimum.

13. Presumably, the cost of travel would have to be weighed against how much lower more competitive prices are.

15. Since the districts are of equal size, the mayor's median or extended median must be between the leftmost and rightmost medians or extended medians; otherwise, at least $\frac{2}{3}$ of the voters would be on one side of the mayor's position, which would preclude it from being the median or extended median. This is not true of the mean, however, if, say, the left-district positions are much farther away from the mayor's median or extended median than the right-district positions. In such a case, the mayor's mean would be in the interval of the left-district positions.

17. If, say, A takes a position at M and B takes a position to the right of M, C should take a position just to the left of M that is closer to M than B's position, giving C essentially half the votes and enabling him or her to win the election. If neither A nor B takes a position at M, C should take a position next to the player closer to M; the position that C takes to maximize his or her vote may be either closer to M (if the candidates are far apart) or farther from M (if the players are closer together), but this position may not be winning. For example, assume the voters are uniformly distributed over [0,1]. If $\frac{3}{16}$ of the voters lie between A (to the left of M) and M, and $\frac{3}{16}$ of the voters lie between M and B (to the right of M), then C does best taking a position just to the left of A or just to the right of B, obtaining essentially $\frac{5}{16}$ of the vote. To be specific, assume C moves just to the left of A. Then A will obtain $\frac{3}{16}$ of the vote, but B will win with $\frac{1}{2}$ of the vote (that to the right of M), so C's maximizing position will not always be sufficient to win.

19. Following the hint, C will obtain $\frac{1}{3}$ of the vote by taking a position at M, as will A and B, so there will be a three-way tie among the candidates. Because a non-unimodal distribution can be bimodal, with the two modes close to M, C can win if he or she picks up most of the vote near the two modes, enabling C to win with more than $\frac{1}{3}$ of vote.

21. B should enter just to the right of $\frac{3}{4}$, making it advantageous for C to enter just to the left of A, giving C essentially $\frac{1}{4}$ of the vote. With C and A almost splitting the vote to the left of M and a little beyond, B would win almost all the vote to the right of M. (If C entered at $\frac{1}{2}$, he or she would get slightly more than $\frac{1}{4}$ of the vote but lose to A, who would get $\frac{3}{8}$.)

23. If the distribution is uniform, these positions are $\frac{1}{6}$, $\frac{5}{6}$, and $\frac{1}{2}$ for A, B, and C, respectively, making D indifferent between entering just to the left of A, just to the right of B, or in between A and C at $\frac{1}{3}$ or between C and B at $\frac{2}{3}$, which would give D $\frac{1}{6}$ of the vote in any case.

25. No, because Gore would get 49%, the same as Bush, so instead of winning Gore would tie with Bush.

27. It seems far too complicated a "solution" for avoiding effects caused by the Electoral College. Why not just abolish the Electoral College?

29. By definition, more voters prefer the Condorcet winner to any other candidate. Thus, if the poll identifies the Condorcet winner as one of the top two candidates, he or she will receive more votes when voters respond to the poll by voting for one or the other of these candidates. The possibility that the Condorcet winner might not be first in the poll, but win after the poll is announced, shows that the plurality winner may not be the Condorcet winner. Some argue that the Condorcet winner is always the "proper" winner, but others counter that a non-Condorcet winner who is, say, everybody's second-most-preferred candidate is a better social choice than a 51%-Condorcet winner who is ranked last by the other 49%.

31. *D* is the Condorcet winner. It is strange in the sense that a poll that identifies either the top two or the top three candidates would not include *D*.

33. *A* would win with 4 votes to 3 votes for *B* and 3 votes for *C*. It is strange that the number of top contenders identified by a poll can result in opposite outcomes (*A* in this exercise, whereas *B* defeats *A* when only two top contenders are identified by a poll, as in Exercise 32).

35. Assume a voter votes for just a second choice. It is evident that voting for a first choice, too, can never result in a worse outcome and may sometimes result in a better outcome (if the voter's vote for a first choice causes that candidate to be elected).

37. Following the hint, the voter's vote for a first and third choice would elect either *A* or *C*. If the voter also voted for *B*, then it is possible that if *A* and *B* are tied for first place, then *B* might be elected when the tie is broken, whereas voting for just *A* and *C* in this situation would elect *A*.

39. No. Voting for a first choice can never hurt this candidate and may help elect him or her.

41. No. If class I and II voters vote for all candidates in their preferred subsets, they create a three-way tie among *A*, *B*, and *C*. To break this tie, it would be rational for the class III voter to vote for both *D* and *C* and so elect *C*, whom this voter prefers to both *A* and *B*. But now class I voters will be unhappy, because *C* is a worst choice. However, these voters cannot bring about a preferred outcome by voting for candidates different from *A* and B.

43. Without polling, *A* in case (i), *D* in case (ii), and *B* and *D* in case (iii); with polling, *B* in case (i), *D* in case (ii), and *D* in case (iii).

45. Exactly half the votes, or 9.5 votes each.

47. Substitute into the formula for r_i, in Exercise 46, $d_i = \left(n_i / N\right) D$ and $D = R$. The proportional rule is "strategy-proof" in the sense that if one player follows it, the other player can do no better than to follow it. Hence, knowing that an opponent is following the proportional rule does not help a player optimize against it by doing anything except also following it.

49. To win in states with more than half the votes, any two states will do. Thus, there is no state to which a candidate should not consider allocating resources. In the absence of information about what one's opponent is doing, all states that receive allocations should receive equal allocations since all states are equally valuable for winning.

51. The Democrat can win the election by winning in any two states or in all three. The first three expressions in the formula for PWE_D give the probabilities of winning in the three possible pairs of states, whereas the final expression gives the probability of winning in all three states.

53. Yes, but in a complicated way. Intuitively, the large states that are more pivotal, and whose citizens therefore have more voting power (as shown in Chapter 11), are more deserving of greater resources (as shown in this chapter).