Chapter 11
Weighted Voting Systems

Solutions

Exercises:

1. (a) A winning or blocking coalition would be 50 senators plus the vice president, or more than 50 senators.

(b) The vice president will not be able to break a tie. A winning or blocking coalition requires 50 or more senators.

(c) A winning coalition require at least 67 senators. A coalition of 34 or more senators can block.

3. (a) No. A dictator needs 9 votes.

(b) The weight-5 and weight-4 voters have veto power, because the coalition of all the voters has only 3 extra votes, less than they have.

(c) The weight-3 voter is a dummy, because the only winning coalition he or she he belongs to is the coalition with all the voters, and it has 3 extra votes.

5. No. If a voter X is pivotal in a permutation, then that voter is a critical voter in the winning coalition consisting of X and every voter that precedes X in the permutation. A dummy voter is not a critical voter in any winning coalition.

7. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [51: 30, 25, 24, 21]$. No voter has veto power; this means that the last voter in a permutation can never be the pivot. No voter is a dictator; thus the first voter in a permutation isn't a pivot either.

(a) The weight-30 voter (Voter *A*) is pivotal in all permutations where he or she occupies position 2 because her weight, combined with any other voter's, is enough to win. *A* is also the pivot in all permutations where he or she occupies position 3, because the two voters ahead of him or her would have a combined weight of at most 49, less than the quota. There are 12 permutations to list:

Permutations
B A C D
B A D C
B C A D
B D A C
C A B D
C A D B
C B A D
C D A B
D A B C
D A C B
D B A C
D C A B

(b) Voters other than *A* will be pivotal if and only if they are second in the permutation and *A* is first, or they are third in the permutation and *A* last. Thus, *B* is pivotal in the following four permutations:

Permutations
A B C D
A B D C
C D B A
D C B A

(c) *A* is pivotal in 12 permutations, and *B*, *C*, and *D* are each pivotal in 4. There are $4! = 24$ permutations in all. The Shapley-Shubik power index of this weighted voting system is therefore $\left(\frac{12}{24}, \frac{4}{24}, \frac{4}{24}, \frac{4}{24}\right) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$.

9. None of these voting systems have dictators, nor does anyone have veto power. Therefore the pivotal position in each permutation is in position 2 or 3. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [q: 30, 25, 24, 21]$.

(a) $[q: w_A, w_B, w_C, w_D] = [52: 30, 25, 24, 21]$

A is pivot in four permutations where he or she is in position 2, and in all six permutations where she is in position 3: that's 10 in all.

Permutations
B A C D
B A D C
C A B D
C A D B

Permutations
B C A D
B D A C
C B A D
C D A B
D B A C
D C A B

B is pivot in two positions where he or she is in position 2 and four permutations where he or she is in position 3. Thus, Voter *B* is a pivot in 6 permutations.

Permutations
A B C D
A B D C

Permutations
A D B C
D A B C
C D B A
D C B A

C has the same power as *B*, and *D* is a pivot in the remaining two permutations.

Permutations
B C D A
C B D A

The Shapley-Shubik power index of this weighted voting system is therefore the following.

$$\left(\tfrac{10}{24}, \tfrac{6}{24}, \tfrac{6}{24}, \tfrac{2}{24}\right) = \left(\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{12}\right)$$

(b) $[q: w_A, w_B, w_C, w_D] = [55: 30, 25, 24, 21]$

Now *A* is pivotal in only two permutations where he or she is in position 2. Voter *A* is still pivotal in all permutations when in position 3. Thus, Voter *A* is a pivot in 8 permutations.

Permutations
B A C D
B A D C

Permutations
B C A D
B D A C
C B A D
C D A B
D B A C
D C A B

B now has the same voting power as *A*. *C* and *D* are also equally powerful. Each is pivot in four permutations in which he or she is in third position and not preceded by *A* and *B*. Thus, the Shapley-Shubik power index of this weighted voting system is the following.

$$\left(\tfrac{8}{24}, \tfrac{8}{24}, \tfrac{4}{24}, \tfrac{4}{24}\right) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

(c) $[q: w_A, w_B, w_C, w_D] = [58: 30, 25, 24, 21]$

Any three voters have enough votes to win, and no two can win. The voters have equal power and the Shapley-Shubik power index of this weighted voting system is therefore $\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$.

11. (a) We can represent a "yes" with 1, and a "no" with 0. Then the voting combinations are the 16 four-bit binary numbers: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.

(b) $\{\ \}$, $\{D\}$, $\{C\}$, $\{C,D\}$, $\{B\}$, $\{B,D\}$, $\{B,C\}$, $\{B,C,D\}$, $\{A\}$, $\{A,D\}$, $\{A,C\}$, $\{A,C,D\}$, $\{A,B\}$, $\{A,B,D\}$, $\{A,B,C\}$, and $\{A,B,C,D\}$.

(c) If the first bit of a given permutation is 1, then *A* votes "yes". If the second bit is 1, *B* votes "yes" in the corresponding coalition. The third bit tells us how *C* votes, and the fourth indicates the vote of *D*.

(d) i. 1
ii. 4
iii. 6

13. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q : w_A, w_B, w_C, w_D] = [q : 30,\ 25,\ 24,\ 21]$.

(a) $[q : w_A, w_B, w_C, w_D] = [52 : 30,\ 25,\ 24,\ 21]$

We'll copy the table of coalitions we made for Exercise 12, reducing the extra votes of each by 1. The coalition {*A*, *D*} becomes a losing coalition because its weight is only 51. It will be marked losing, and dropped when we increase the quota again.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	48	0	0	0	0
{*A*, *B*, *C*}	79	27	1	0	0	0
{*A*, *B*, *D*}	76	24	1	1	0	0
{*A*, *C*, *D*}	75	23	1	0	1	0
{*B*, *C*, *D*}	70	18	0	1	1	1
{*A*, *B*}	55	3	1	1	0	0
{*A*, *C*}	54	2	1	0	1	0
{*A*, *D*}	51	losing				
			5	3	3	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(10,6,6,2)$.

(b) $[q : w_A, w_B, w_C, w_D] = [55 : 30,\ 25,\ 24,\ 21]$

We copy the table from part (a), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	45	0	0	0	0
{*A*, *B*, *C*}	79	24	1	1	0	0
{*A*, *B*, *D*}	76	21	1	1	0	0
{*A*, *C*, *D*}	75	20	1	0	1	1
{*B*, *C*, *D*}	70	15	0	1	1	1
{*A*, *B*}	55	0	1	1	0	0
{*A*, *C*}	54	losing				
			4	4	2	2

Doubling to account for blocking coalitions, the Banzhaf power index is $(8,8,4,4)$.

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13. continued

(c) $[q : w_A, w_B, w_C, w_D] = [58 : 30,\ 25,\ 24,\ 21]$

We copy the table from part (b), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	42	0	0	0	0
{*A*, *B*, *C*}	79	21	1	1	1	0
{*A*, *B*, *D*}	76	18	1	1	0	1
{*A*, *C*, *D*}	75	17	1	0	1	1
{*B*, *C*, *D*}	70	12	0	1	1	1
{*A*, *B*}	55	losing				
			3	3	3	3

Doubling to account for blocking coalitions, the Banzhaf power index is $(6,6,6,6)$.

(d) $[q : w_A, w_B, w_C, w_D] = [73 : 30,\ 25,\ 24,\ 21]$

We copy the table from part (c), dropping the losing coalition and reducing quotas by 15. One more coalition will lose. *A* acquires veto power.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	27	1	0	0	0
{*A*, *B*, *C*}	79	6	1	1	1	0
{*A*, *B*, *D*}	76	3	1	1	0	1
{*A*, *C*, *D*}	75	2	1	0	1	1
{*B*, *C*, *D*}	70	losing				
			4	2	2	2

Doubling to account for blocking coalitions, the Banzhaf power index is $(8,4,4,4)$.

(e) $[q : w_A, w_B, w_C, w_D] = [76 : 30,\ 25,\ 24,\ 21]$

We copy the table from part (d), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	24	1	1	0	0
{*A*, *B*, *C*}	79	3	1	1	1	0
{*A*, *B*, *D*}	76	0	1	1	0	1
{*A*, *C*, *D*}	75	losing				
			3	3	1	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(6,6,2,2)$.

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13. continued

(f) $[q : w_A, w_B, w_C, w_D] = [79 : 30, 25, 24, 21]$

We copy the table from part (e), dropping the losing coalition and reducing quotas by 3. One more coalition will lose. In this system, *D* is a dummy.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	21	1	1	1	0
{*A*, *B*, *C*}	79	0	1	1	1	0
{*A*, *B*, *D*}	76	losing				
			2	2	2	0

Doubling to account for blocking coalitions, the Banzhaf power index is $(4, 4, 4, 0)$.

(g) $[q : w_A, w_B, w_C, w_D] = [82 : 30, 25, 24, 21]$

Only one winning coalition is left, with 18 extra votes. This is less than the weight of each participant. All voters are critical. In this system, a unanimous vote is required to pass a motion.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	18	1	1	1	1
			1	1	1	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(2, 2, 2, 2)$.

15. Generating powers of 2 is often helpful in such conversions.

n	0	1	2	3	4	5	6	7
2^n	1	2	4	8	16	32	64	128

n	8	9	10	11	12	13	14
2^n	256	512	1024	2048	4096	8192	16,384

(a) Since 2^9 represents the largest power of 2 that doesn't exceed 585, we start there.

$$585-512=585-2^9=73$$
$$73-64=73-2^6=9$$
$$9-8=9-2^3=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are b_9, b_6, b_3 and b_0. The binary expression is 1001001001.

(b) Since 2^{10} represents the largest power of 2 that doesn't exceed 1365, we start there.

$$1365-1024=1365-2^{10}=341$$
$$341-256=341-2^8=85$$
$$85-64=85-2^6=21$$
$$21-16=16-2^4=5$$
$$5-4=5-2^2=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are $b_{10}, b_8, b_6, b_4, b_2,$ and b_0. The binary form is 10101010101.

(c) Since 2^{10} represents the largest power of 2 that doesn't exceed 2005, we start there.

$$2005-1024=2005-2^{10}=981$$
$$981-512=981-2^9=469$$
$$469-256=469-2^8=213$$
$$213-128=213-2^7=85$$
$$85-64=85-2^6=21$$
$$21-16=16-2^4=5$$
$$5-4=5-2^2=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are $b_{10}, b_9, b_8, b_7, b_6, b_4, b_2,$ and b_0. The binary form is 11111010101.

17. (a) $C_3^6=\dfrac{6!}{3!(6-3)!}=\dfrac{6!}{3!3!}=\dfrac{6\times5\times4}{3\times2\times1}=5\times4=20.$

(b) $C_2^{100}=\dfrac{100!}{2!(100-2)!}=\dfrac{100!}{2!98!}=\dfrac{100\times99}{2\times1}=50\times99=4950.$

(c) By the duality formula, $C_{98}^{100}=C_2^{100}=4950.$ by the result of part (b).

(d) $C_5^9=\dfrac{9!}{5!(9-5)!}=\dfrac{9!}{5!4!}=\dfrac{9\times8\times7\times6}{4\times3\times2\times1}=9\times2\times7=126.$

19. (a) {A, C, D} and {A, B}

(b) A belongs to each winning coalition, so if A opposes a motion it will not pass. There are no other minimal blocking coalitions that include A, but we may notice that every winning coalition contains either B or C and D. Thus, if B can combine forces with either C or D to defeat a motion, {B, C} and {B, D} are also minimal blocking coalitions.

(c) A has veto power and thus is a critical voter in all 5 of the winning coalitions. B is critical in 3 winning coalitions: {A, B, C}, {A, B, D}, and {A, B}. Finally, C and D are only critical in one coalition: {A, C, D}. The Banzhaf power index is $(10, 6, 2, 2)$.

(d) $[q: w_A, w_B, w_C, w_D] = [5: 3, 2, 1, 1]$ is one set of weights that works, but there are many other solutions. One can reason that A, the only voter with veto power, must have the most votes, while B is more powerful than C or D (who are equally powerful).

(e) A will pivot in any permutation in which he or she comes after B or after C and D. He or she automatically pivots in the 6 permutations where he or she is in position 4, and also the 6 permutations where he or she is in position 3 because if B is not last in such a permutation, then he or she comes before A, and if Voter A is last, then C and D come before A. There are two permutations, *BACD* and *BADC* where A pivots in position 2. This adds up to $2+6+6=14$ pivots for A. D pivots in permutations where A and C appear before him or her, and B is last. There are 2 such permutations: *ACDB* and *CADB*. C has the same number of pivots as D. We have accounted for $14+2+2=18$ permutations. The remaining 6 belong to B. The Shapley-Shubik index is $\left(\frac{14}{24}, \frac{6}{24}, \frac{2}{24}, \frac{2}{24}\right) = \left(\frac{7}{12}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}\right)$.

21. Let's call the chairperson A, and the other members B_1, B_2, B_3, and B_4. A is a critical voter in all winning coalitions that include at least one other voter but not all of them. There are $2^4 - 2 = 16 - 2 = 14$ such coalitions. Each of the B_i is a critical voter in two coalitions: $\{A, B_i\}$ for $i = 1, 2, 3, 4$, and $\{B_1, B_2, B_3, B_4\}$. The Banzhaf power index is $(28, 4, 4, 4, 4)$.

23. Let's look at the minimal winning coalitions. We call the faculty members F_1, F_2, F_3, and F_4. We call the administrators A_1, A_2, and A_3. The following are two winning coalitions: $\{F_1, F_2, F_3, F_4, A_1, A_2\}$, in which the administrators are critical, but the faculty members aren't; and $\{F_1, F_2, F_3, A_1, A_2, A_3\}$ in which the faculty members are critical and the administrators are not. In any weighted voting system, the critical voters in a coalition must have more weight than those who are not critical. The first coalition that we cited indicates that the administrators should have more weight, while the second indicates that the faculty members have more weight. These contradictory requirements cannot be satisfied, so the system is not equivalent to a weighted voting system.

25. There are $7! = 5040$ permutations, so let's not make a list. Consider F_4. He or she will be critical in a permutation when he or she is fifth, followed by another faculty member and an administrator (in either order), or sixth, followed by a faculty member. If F_4 is fifth, there are 3 ways to choose the faculty member, 3 ways to choose the administrator, and 2 ways to put those two in order, The remaining 4 participants, who come before F_4 in the permutation, can be ordered 4! ways. Thus, there are $3\times3\times2\times4! = 432$ permutations where F_4 is a pivot in position 5. If F_4 is in position 6 and another faculty member in position 7, there are 5! ways to order the voters coming before the two faculty members, and 3 ways to choose the last voter in this type of permutation: $5!\times3 = 360$ permutations in all. The number of permutations in which F_4 is a pivot is thus $432 + 360 = 792$. The other three faculty members are each pivot in 792 permutations, so the faculty members are pivot in a total of $4\times792 = 3168$ permutations. That leaves $5040 - 3168 = 1872$ permutations for the administrators, 624 each. The Shapley-Shubik index of this voting system is (in lowest terms)

$$\left(\frac{11}{70}, \frac{11}{70}, \frac{11}{70}, \frac{11}{70}, \frac{13}{105}, \frac{13}{105}, \frac{13}{105}\right)$$

By this measure, each faculty member is more powerful than any administrator. A faculty member has about 15.7% of the power, and an administrator has about 12.4%.

27. All four-voter systems can be presented as weighted voting systems.

Minimal winning coalitions	Weighted voting systems
{*A*, *B*, *C*, *D*}	[4:1,1,1,1]
{*A*, *B*}, {*A*, *C*, *D*}	[5:3,2,1,1]
{*A*, *B*, *C*}, {*A*, *B*, *D*}	[5:2,2,1,1]
{*A*, *B*}, {*A*, *C*}, {*A*, *D*}	[4:3,1,1,1]
{*A*, *B*}, {*A*, *C*}, {*B*, *C*, *D*}	[5:3,2,2,1]
{*A*, *B*}, {*A*, *C*, *D*}, {*B*, *C*, *D*}	[4:2,2,1,1]
{*A*, *B*, *C*}, {*A*, *B*, *D*}, {*A*, *C*, *D*}	[4:2,1,1,1]
{*A*, *B*}, {*A*, *C*}, {*A*, *D*}, {*B*, *C*, *D*}	[4:3,2,1,1]
{*A*, *B*, *C*}, {*A*, *B*, *D*}, {*A*, *C*, *D*}, {*B*, *C*, *D*}	[3:1,1,1,1]

29. The minimal winning coalitions are {*A*, *B*} {*A*, *C*}, {*A*, *D*}, and {*B*, *C*, *D*}. Thus, *A* is more powerful than the others, and *B*, *C*, and *D* have equal power, even though their voting weights are different. Let's go through the list and see which gives the same minimal winning coalitions.

(a) Each minimal winning coalition has 3 voters. Eliminated.

(b) The minimal winning coalitions are *A* combined with another voter, or {*B*, *C*, *D*}. This matches our system.

(c) The minimal winning coalitions are *A* combined with 2 other voters. Eliminated.

(d) {*A*, *C*} and {*A*, *D*} are not winning coalitions. Eliminated.

The answer: (b).

31. (a) The ordinary members are equally powerful, so each gets 1 vote. The quota is 8, to make the coalition of all ordinary members winning, but 7 members losing. The chair gets 6 votes, enough to combine with 2 ordinary members and win. In our notation, the weighted voting system is

$$\left[q : w_C, w_{O_1}, w_{O_2}, w_{O_3}, w_{O_4}, w_{O_5}, w_{O_6}, w_{O_7}, w_{O_8}\right] = \left[8 : 6, 1, 1, 1, 1, 1, 1, 1, 1\right].$$

(b) The chairperson is critical in all winning coalition she belongs to, except the one in which the committee is unanimous. The number of these coalitions is $2^8 - C_0^8 - C_1^8 - C_8^8 = 256 - 1 - 8 - 1 = 246$, because there are 2^8 coalitions of ordinary members in all, of which we must eliminate $C_0^8 + C_1^8$ because they consist of 0 or 1 members, who cannot form a winning coalition with the chairperson, and C_8^8, because when all 8 ordinary members join the chairperson, the chairperson isn't critical. An ordinary member is critical in 8 winning coalitions: when joined by the rest of the ordinary members, and when joined by the chairperson and one of the other 7 ordinary members. Counting an equal number of blocking coalitions, the Banzhaf power index of this system is $(492,16,16,16,16,16,16,16,16)$.

(c) Divide the permutations into 9 groups, according to the location of the chairperson. She is pivot in groups 3, 4, 5, 6, 7, and 8. Therefore his or her Shapley-Shubik power index is $\frac{6}{9} = \frac{2}{3}$. Each ordinary member has $\frac{1}{8}$ of the remaining $1 - \frac{2}{3} = \frac{1}{3}$ of the power; hence the Shapley-Shubik power index of this system is as follows.

$$\left(\frac{2}{3}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}\right)$$

(d) In this system, the chairperson is 30.75 times as powerful as an ordinary member according to the Banzhaf index, but only 16 times as powerful by the Shapley-Shubik power index.

33. Let's determine the minimal winning coalitions. They would be of the following types:

(a) 3 city officials

(b) 2 city officials and 1 borough president

(c) 1 city official and all of the borough presidents.

Thus, the city officials all have the same power, and the borough presidents, although weaker than the city officials, also have equal power. We will assign a voting weight of 1 to each borough president. Let C denote the voting weight of a city official and let q be the quota. To make the coalition of type (i) win, and 2 city officials lose, we have

$$2C < q \le 3C.$$

To make coalitions of type (ii) win, we require $2C + 1 \ge q$. Combining these inequalities, we see that (if C is an integer), $q = 2C + 1$. The 5 borough presidents plus one city official can win, but 4 borough presidents plus a city official is a losing coalition: therefore

$$C + 4 < q \le C + 5$$

and hence $q = C + 5$. We now have two expressions for q, $2C + 1$ and $C + 5$. Equating them, $2C + 1 = C + 5$, which we can solve for C to obtain $C = 4$, and hence $q = 9$. Finally, the 5 borough presidents form a losing coalition, but win if joined by a city official: this will hold provided

$$5 < q \le C + 5$$

This is also valid for $q = 9$ and $C = 4$.

The weighted voting system is $\left[q : w_M, w_C, w_{CCP}, w_{P_1}, w_{P_2}, w_{P_3}, w_{P_4}, w_{P_5}\right] = \left[9 : 4, 4, 4, 1, 1, 1, 1, 1\right]$.

35. The three weight-3 voters, or 2 weight-3 voters and one weight-1 voter form minimal winning coalitions.

A weight-3 voter, *A*, is critical in any winning coalition with 7, 8, or 9 votes. There are 6 weight-7 coalitions that include *A*, because they are formed by assembling one of the other 2 weight-3 voters, and one of the 3 weight-1 voters. There are also 6 weight-8 coalitions with *A*: they also need one of the other 2 weight-3 voters and 2 of the 3 weight-1 voters (the number of ways to choose 2 weight-1 voters is $C_2^3 = 4$). Finally, there are 3 coalitions of weight 9 to which *A* belongs: all 3 weight-3 voters is one of them; the other 2 consist of *A* and one of the other 2 weight-3 voters, and all of the weight-1 voters. Thus *A* is critical in a total of 15 winning coalitions, and *A*'s Banzhaf power index is 30.

A weight-1 voter, *D*, is critical in 3 winning coalitions, formed by assembling *D* with 2 of the 3 weight-3 voters. Doubling, we see that the Banzhaf power index of *D* is 6.

The Banzhaf power index of this system is $(30, 30, 30, 6, 6, 6)$.

37. Let's start with a weight-1 voter, *A*.

Case I: 3 weight-1 voters

A will be pivot in permutations where he or she is in position 3, and positions 1 and 2 are occupied by weight-3 voters. There are $C_2^3 = 3$ ways to choose the weight-3 voters who come first, 2 ways to put them in order, and 3! ways to put the voters following *A* in order. Thus the Shapley-Shubik power index of *A* is $\frac{3\times2\times3!}{6\times5\times4\times3!} = \frac{3\times2}{6\times5\times4} = \frac{1}{5\times4} = \frac{1}{20}$.

The other weight-1 voters have the same power, and the weight-3 voters share the remaining $1-3\times\frac{1}{20} = 1-\frac{3}{20} = \frac{17}{20}$ of the power.

Case II: 4 weight-1 voters

Now *A* will be pivot in permutations where he or she is in position 3, and positions 1 and 2 are occupied by weight-3 voters. There are still 3 ways to select the 2 weight-3 voters and 2 ways to put them in order, but now there are 4! ways to arrange the voters who follow *A* in the permutation. This gives $6\times4!$ permutations in which *A* is pivot.

A will also be pivot in any permutation where he or she is in position 5 and the final two positions are occupied by weight-3 voters. There are the same number of these permutations. The Shapley-Shubik power index for *A* is therefore $\frac{2\times6\times4!}{7\times6\times5\times4!} = \frac{2\times6}{7\times6\times5} = \frac{2}{7\times5} = \frac{2}{35}$. The other 3 weight-1 voters have the same power, and the remaining $1-4\times\frac{2}{35} = 1-\frac{8}{35} = \frac{27}{35}$ of the power belongs to the weight-3 voters.

Although each voter's share of power decreased proportionally in the Banzhaf model when a new voter joined the system, in this particular situation, each weight-1 voter's power, measured by the Shapley-Shubik model, increased when the new voter was included, because $\frac{2}{35} > \frac{1}{20}$.

39. Consider Maine, which has 2 congressional districts. Ignoring the rest of the country, Maine would be a 3-voter system, in which the state has 2 votes, and each congressional district has 1. There are 3! permutations of voters, but only 2 are possible: 1M2 and 2M1, where the congressional districts are identified by numerals, and the state is M. The reason is as follows. Each entity (district or statewide) is given a score, which is the number of votes recorded for the Bush-Cheney ticket, divided by the number of votes cast in that entity for the Kerry-Edwards ticket. The entity's position in the permutation is determined by that ratio.

Let $r_1, r_2,$ and r_M denote the ratios for the two districts and the state as a whole, respectively, and let $y_1, y_2,$ and y_M be the number of votes cast for the Kerry-Edwards ticket in each entity. The reason that some electoral permutations are impossible is that $y_M = y_1 + y_2$. Each vote is actually counted twice: once for the elector representing the voter's congressional district, and once for the two statewide electors. The number of votes for the Bush-Cheney ticket in the three entities were $r_1 y_1,\ r_2 y_2,$ and $r_M\left(y_1 + y_2\right)$. Because the number of statewide votes for the Bush-Cheney ticket can also be determined by adding the votes in the two districts,

$$r_M\left(y_1 + y_2\right) = r_1 y_1 + r_2 y_2.$$

Dividing by $\left(y_1 + y_2\right)$ we can obtain a formula for r_M:

$$r_M = \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2}.$$

Suppose that $r_1 > r_2$. Then $r_1 y_2 > r_2 y_2$, and thus

$$r_1 = \frac{r_1\left(y_1 + y_2\right)}{y_1 + y_2} = \frac{r_1 y_1 + r_1 y_2}{y_1 + y_2} > \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2} = r_M$$

Also, $r_1 y_1 > r_2 y_1$, so

$$r_M = \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2} > \frac{r_2 y_1 + r_2 y_2}{y_1 + y_2} = \frac{r_2\left(y_1 + y_2\right)}{y_1 + y_2} = r_2.$$

These inequalities, taken together show that $r_1 > r_M > r_2$, if $r_1 > r_2$. Of course, if $r_1 > r_2$, the same argument would show $r_2 > r_M > r_1$.

Any permutation of the states in which the statewide electors for Maine do not fall between the electors for the two congressional districts is therefore impossible. The same is true for Nebraska: at least one district elector must come before the statewide electors, and one must come after them.

The Shapley-Shubik power index should be computed by using only the possible permutations. A given entity's Shapley-Shubik power index would be the number of possible permutations in which it is pivot, divided by the total number of possible permutations.