Chapter 10
The Manipulability of Voting Systems

Solutions

Exercises:

1. One example of two such elections is the following:

Election 1

Rank	Number of voters (3)		
First	*A*	*A*	*B*
Second	*B*	*B*	*A*

Election 2

Rank	Number of voters (3)		
First	*B*	*A*	*B*
Second	*A*	*B*	*A*

With the voting system in which the candidate with the fewest first-place votes wins, *B* is the winner in the first election. However, if the leftmost voter changes his or her ballot as shown in the second election, then *A* becomes the winner. Taking the ballots in the first election to be the sincere preferences of the voters, the leftmost voter (who prefers *A* to *B*) has secured a more favorable outcome by the submission of a disingenuous ballot.

3. One example of two such elections is the following:

Election 1

Rank	Number of voters (3)		
First	*A*	*B*	*B*
Second	*B*	*A*	*A*

Election 2

Rank	Number of voters (3)		
First	*B*	*B*	*B*
Second	*A*	*A*	*A*

With the voting system in which the candidate receiving an even number of first-place votes wins, *B* is the winner in the first election. However, if the leftmost voter changes his or her ballot as shown in the second election, then *A* becomes the winner. Taking the ballots in the first election to be the sincere preferences of the voters, the leftmost voter (who prefers *A* to *B*) has secured a more favorable outcome by the submission of a disingenuous ballot.

5. (a) The voting system does not treat all *voters* the same.

(b) A dictatorship in which Voter #1 is the dictator.

(c) A dictatorship in which Voter #2 is the dictator and a dictatorship in which voter #3 is the dictator.

7. Election 1

	Number of voters (2)	
Rank	**1**	**1**
First	*B*	*A*
Second	*C*	*D*
Third	*A*	*C*
Fourth	*D*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	0×2	1×1	0×0	4
B	1×3	0×2	0×1	1×0	3
C	0×3	1×2	1×1	0×0	3
D	0×3	1×2	0×1	1×0	2

With the given ballots, the winner using the Borda count is *A*. However, if the leftmost voter changes his or her preference ballot, we have the following.

Election 2

	Number of voters (2)	
Rank	**1**	**1**
First	*C*	*A*
Second	*B*	*D*
Third	*D*	*C*
Fourth	*A*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	0×2	0×1	1×0	3
B	0×3	1×2	0×1	1×0	2
C	1×3	0×2	1×1	0×0	4
D	0×3	1×2	1×1	0×0	3

With the new ballots, the winner using the Borda count is *C*.

9. Election 1

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*B*
Second	*B*	*A*	*A*
Third	*C*	*C*	*C*
Fourth	*D*	*D*	*D*

With the given ballots, the winner using the Borda count is *B*.

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	2×2	0×1	0×0	7
B	2×3	1×2	0×1	0×0	8
C	0×3	0×2	3×1	0×0	3
D	0×3	0×2	0×1	3×0	0

The voter on the far left prefers *A* to *B*. By casting a disingenuous ballot (still preferring *A* to *B* though), the outcome of the election is altered.

Election 2

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*B*	*B*
Second	*C*	*A*	*A*
Third	*D*	*C*	*C*
Fourth	*B*	*D*	*D*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	2×2	0×1	0×0	7
B	2×3	0×2	0×1	1×0	6
C	0×3	1×2	2×1	0×0	4
D	0×3	0×2	1×1	2×0	1

Thus, *A* has the highest Borda score and is declared the winner.

11. The following is one such example:
Election 1

	Number of voters (9)								
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*A*	*D*	*A*	*F*	*A*	*F*
Second	*B*	*A*	*A*	*B*	*C*	*B*	*E*	*B*	*E*
Third	*C*	*C*	*C*	*C*	*B*	*C*	*D*	*C*	*D*
Fourth	*D*	*D*	*D*	*D*	*A*	*D*	*C*	*D*	*C*
Fifth	*E*	*E*	*E*	*E*	*E*	*E*	*B*	*E*	*B*
Sixth	*F*	*F*	*F*	*F*	*F*	*F*	*A*	*F*	*A*

Preference	1st place votes × 5	2nd place votes × 4	3rd place votes × 3	4th place votes × 2	5th place votes × 1	6th place votes × 0	Borda score
A	4×5	2×4	0×3	1×2	0×1	2×0	30
B	2×5	4×4	1×3	0×2	2×1	0×0	31
C	0×5	1×4	6×3	2×2	0×1	0×0	26
D	1×5	0×4	2×3	6×2	0×1	0×0	23
E	0×5	2×4	0×3	0×2	7×1	0×0	15
F	2×5	0×4	0×3	0×2	0×1	7×0	10

Thus, *B* has the highest Borda score and is declared the winner. This was the expected result.

The voter on the far left prefers *A* to *B*. By casting a disingenuous ballot (still preferring *A* to *B* though), the outcome of the election is altered.

Election 2

	Number of voters (9)								
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*A*	*D*	*A*	*F*	*A*	*F*
Second	*D*	*A*	*A*	*B*	*C*	*B*	*E*	*B*	*E*
Third	*C*	*C*	*C*	*C*	*B*	*C*	*D*	*C*	*D*
Fourth	*B*	*D*	*D*	*D*	*A*	*D*	*C*	*D*	*C*
Fifth	*E*	*E*	*E*	*E*	*E*	*E*	*B*	*E*	*B*
Sixth	*F*	*F*	*F*	*F*	*F*	*F*	*A*	*F*	*A*

Preference	1st place votes × 5	2nd place votes × 4	3rd place votes × 3	4th place votes × 2	5th place votes × 1	6th place votes × 0	Borda score
A	4×5	2×4	0×3	1×2	0×1	2×0	30
B	2×5	3×4	1×3	1×2	2×1	0×0	29
C	0×5	1×4	6×3	2×2	0×1	0×0	26
D	1×5	1×4	2×3	5×2	0×1	0×0	25
E	0×5	2×4	0×3	0×2	7×1	0×0	15
F	2×5	0×4	0×3	0×2	0×1	7×0	10

Thus, *A* has the highest Borda score and is declared the winner.

13. Election 1

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*C*	*B*	*D*
Second	*B*	*A*	*D*	*C*
Third	*C*	*B*	*C*	*A*
Fourth	*D*	*D*	*A*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	1×2	1×1	1×0	6
B	1×3	1×2	1×1	1×0	6
C	1×3	1×2	2×1	0×0	7
D	1×3	1×2	0×1	2×0	5

Thus, *C* has the highest Borda score and is declared the winner. But the winner becomes *B* if the leftmost voter changes his or her ballot as follows.

Election 2

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*B*	*C*	*B*	*D*
Second	*A*	*A*	*D*	*C*
Third	*D*	*B*	*C*	*A*
Fourth	*C*	*D*	*A*	*B*

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	0×3	2×2	1×1	1×0	5
B	2×3	0×2	1×1	1×0	7
C	1×3	1×2	1×1	1×0	6
D	1×3	1×2	1×1	1×0	6

Thus, *B* has the highest Borda score and is declared the winner.

15. Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*A*	*A*
Second	*B*	*C*	*C*	*C*	*C*
Third	*C*	*A*	*A*	*B*	*B*

Since Candidates *A* and *B* both have the same (high) number of last-place votes, they are both eliminated, leaving Candidate *C* as the winner using Coombs rule. But the winner becomes *A* if the leftmost voter changes his or her ballot as the following shows.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*B*	*A*	*A*
Second	*C*	*C*	*C*	*C*	*C*
Third	*B*	*A*	*A*	*B*	*B*

B has the most last-place votes, thus Candidate *B* is eliminated, and we have the following.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*C*	*C*	*A*	*A*
Second	*C*	*A*	*A*	*C*	*C*

C now has the most last-place votes, thus Candidate *C* is eliminated, and *A* becomes the winner by the Coombs method.

17. Election 1

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*C*	*C*	*B*
Second	*B*	*B*	*A*	*A*	*C*
Third	*C*	*C*	*B*	*B*	*A*

Since *A* and *C* have the most number of first-place votes, *B* is eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*C*	*C*	*C*
Second	*C*	*C*	*A*	*A*	*A*

Since *C* has the most number of first-place votes, the winner using the plurality runoff rule is *C*. But the winner becomes *B* if the leftmost voter changes his or her ballot as the following shows.

Election 2

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*A*	*C*	*C*	*B*
Second	*A*	*B*	*A*	*A*	*C*
Third	*C*	*C*	*B*	*B*	*A*

Since *B* and *C* have the most number of first-place votes, *A* is eliminated.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*B*	*B*	*C*	*C*	*B*
Second	*C*	*C*	*B*	*B*	*C*

Since *B* has the most number of first-place votes, the winner using the plurality runoff rule is *B*.

19.

	Number of voters (3)		
Rank	**1**	**1**	**1**
First	*A*	*C*	*B*
Second	*B*	*A*	*D*
Third	*D*	*B*	*C*
Fourth	*C*	*D*	*A*

(a) For *B* to win, consider the agenda *D*, *A*, *C*, *B*.

In sequential pairwise voting with the agenda *D*, *A*, *C*, *B*, we first pit *D* against *A*. There is 1 voter that prefers *D* to *A* and 2 prefer *A* to *D*. Thus, *A* wins by a score of 2 to 1. *D* is therefore eliminated, and *A* moves on to confront *C*.

There is 1 voter who prefers *A* to *C* and 2 prefer *C* to *A*. Thus, *C* wins by a score of 2 to 1. *A* is therefore eliminated, and *C* moves on to confront *B*.

There is 1 voter who prefers *C* to *B* and 2 prefer *B* to *C*. Thus, *B* wins by a score of 2 to 1.

Thus, *B* is the winner by sequential pairwise voting with the agenda *D*, *A*, *C*, *B*.

(b) For *C* to win, consider the agenda *B*, *D*, *A*, *C*.

In sequential pairwise voting with the agenda *B*, *D*, *A*, *C*, we first pit *B* against *D*. There are 3 voters that prefer *B* to *D* and 0 prefer *D* to *B*. Thus, *B* wins by a score of 3 to 0. *D* is therefore eliminated, and *B* moves on to confront *A*.

There is 1 voter who prefers *B* to *A* and 2 prefer *A* to *B*. Thus, *A* wins by a score of 2 to 1. *B* is therefore eliminated, and *A* moves on to confront *C*.

There is 1 voter who prefers *A* to *C* and 2 prefer *C* to *A*. Thus, *C* wins by a score of 2 to 1.

Thus, *C* is the winner by sequential pairwise voting with the agenda *B*, *D*, *A*, *C*.

(c) For *D* to win, consider the agenda *B*, *A*, *C*, *D*.

In sequential pairwise voting with the agenda *B*, *A*, *C*, *D*, we first pit *B* against *A*. There is 1 voter that prefers *B* to *A* and 2 prefer *A* to *B*. Thus, *A* wins by a score of 2 to 1. *B* is therefore eliminated, and *A* moves on to confront *C*.

There is 1 voter who prefers *A* to *C* and 2 prefer *C* to *A*. Thus, *C* wins by a score of 2 to 1. *A* is therefore eliminated, and *C* moves on to confront *D*.

There is 1 voter who prefers *C* to *D* and 2 prefer *D* to *C*. Thus, *D* wins by a score of 2 to 1.

Thus, *D* is the winner by sequential pairwise voting with the agenda *B*, *A*, *C*, *D*.

Note: In any of the three parts, the first two candidates can be switched and the outcome will be the same.

21. Election 1

22%	23%	15%	29%	7%	4%
D	*D*	*H*	*H*	*J*	*J*
H	*J*	*D*	*J*	*H*	*D*
J	*H*	*J*	*D*	*D*	*H*

D has 22% + 23% = 45% of the first-place votes. *H* has 15% + 29% = 44% of the first-place votes. *J* has 7% + 4% = 11% of the first-place votes. Since *D* has the most first-place votes, Alfonse D'Amato (*D*) is the winner by plurality voting. The plurality rule is group manipulable as the following shows if the voters in the 7% group all change their ballots.

Election 2

22%	23%	15%	29%	7%	4%
D	*D*	*H*	*H*	*H*	*J*
H	*J*	*D*	*J*	*J*	*D*
J	*H*	*J*	*D*	*D*	*H*

D has 22% + 23% = 45% of the first-place votes. *H* has 15% + 29% + 7%= 51% of the first-place votes. *J* has 4% of the first-place votes. Since *H* has the most first-place votes, Elizabeth Holtzman (*H*) is the winner by plurality voting.

23. Properties 1, 2, and 3.

25. Properties 1, 2, and 4.

27. Consider the following scenario: The chair votes for *A* and I vote for *B*. If you vote for *C*, the winner is *A* (your least preferred outcome) while the winner is *B* if you vote for *B*. This shows that voting for *C* does not weakly dominate your strategy of voting for *B*.