Chapter 20
Tilings

Chapter Outline

Chapter Summary

A mosaic uses repeated shapes to cover a flat surface without overlap and, ideally, without gaps. If we call the shapes tiles, we have an example of a *tiling* (or *tessellation*). Tilings can be of practical interest: If we need to cut pieces of a certain shape and this shape can be used in a tiling, then we have a way to cut our pieces with little or no waste.

Monohedral tilings are the simplest type. They use tiles having the same size and shape. A tiling is called *regular* if it is monohedral and its tile is a regular polygon. For simplicity, only edge-to-edge tilings are considered. In such tilings, the edge of one tile coincides with the edge of a neighboring tile. The *vertex figure* for such a tiling is the tile configuration at points where the edges of different tiles meet. Examining this configuration, we can show that only three regular polygons can be used in a regular tiling of the plane: the equilateral triangle, the square, and the regular hexagon.

It is possible to mix shapes and obtain edge-to-edge tilings. A semiregular tiling uses different regular polygons arranged in such a way that all vertex figures are alike. There are only eight edge-to-edge tilings of this type. If the restriction on identical vertex figures is dropped, then there are infinitely many.

Natural extensions to consider are tilings with irregular polygons. Again, considering only monohedral edge-to-edge tilings, we can reach several conclusions: any triangle tiles; any quadrilateral, *convex* or not, tiles; exactly three classes of convex hexagons tile; and a convex polygon with at least seven sides cannot tile. What happens if convex pentagons are used is not completely understood.

Tiling by translation is accomplished by laying tiles edge-to-edge in rows. Of course, the tile must be able to fit exactly into each of its neighbors, including those above and below. A shape is suitable for such a tiling if it satisfies one of two conditions on its shape. If we are also allowed to turn our shape upside-down, we can consider tiling by translation and half-turns. Conway’s criterion singles out those shapes that can be used in this way.

The Dutch artist *M. C. Escher* used tilings, some with two or more interlocking pieces, in his work. He also used tilings of the Poincare disk. Many of his tilings are *periodic*, the pattern repeating at regular intervals in one or more directions. Tilings whose patterns do not repeat by translation, or whose repetition is not completely regular, are called nonperiodic. In all known cases, if a shape can be used to make a *nonperiodic* tiling, then it can be used to make a periodic one. It had been conjectured that the same holds for any set of shapes. *Roger Penrose* discovered a counterexample in 1975 requiring a set of only two shapes.

Penrose tilings have arbitrarily large regions with 5-fold and 10-fold rotational symmetry. These tilings have been generalized to three dimensions and these space tilings are nonperiodic with 5-fold rotational symmetry. This feature has provided an unexpected connection with crystallography. Barlow's law shows that periodic tiling of the plane or of three-space cannot have 5-fold symmetry. However, Daniel Shectman discovered just such symmetry in examining crystals in a certain manganese-aluminum alloy. The apparent "forbidden symmetry" is due to the nonperiodic nature of the crystal array in the alloy. Since crystals have always been thought of as corresponding to periodic tilings, the new structures have been dubbed *quasicrystals*. Here is another instance of mathematical research "anticipating" scientific discovery.

Skill Objectives

1. Calculate the number of degrees in each angle of a given regular polygon.

2. Given the number of degrees in each angle of a regular polygon, determine its number of sides.

3. Define the term tiling (tessellation).

4. List the three regular polygons for which a monohedral tiling exists.

5. When given a mix of regular polygons, determine whether a tiling of these polygons could exist.

6. Explain the difference between a periodic and a nonperiodic tiling.

7. Discuss the importance of the Penrose tiles.

8. Explain why 5-fold symmetry in a crystal structure was thought to be impossible.

Teaching Tips

1. The tessellations of artist M. C. Escher are of great interest to many students. They often enjoy trying to create some of their own. The following instructions will produce the boundary of a rotational tessellation. After several attempts, students will find a boundary in which they visualize an object. They may add any markings inside the boundary to make their design more attractive or convincing, so long as they don't alter the boundary in any way. The important concepts used in this approach are rotational symmetry and preservation of area (when you add area onto the triangle by drawing a curve outside the triangle, and compensate for it by taking away the same amount on the inside of the triangle).

- **a.** Draw equilateral triangle *ABC* with a pencil.
- **b.** Beginning at vertex *A*, draw a curve that goes both inside and outside the triangle and ends at vertex *B*.
- **c.** Repeat this same curve identically along side *CB*, making sure that the point of the curve at vertex *A* is now at vertex *C* and *B* remains in the same place. (You are actually rotating the curve around point *B*.)
- **d.** Starting at vertex *C*, draw a curve inside the triangle that stops at the midpoint of side *AC*.
- **e.** Repeat this curve on the outside of the triangle by rotating it around the midpoint of side *AC*.
- **f.** Erase the lines of the triangle and focus only on the boundary you have created. You may turn it in any direction. Does this boundary suggest an object or a design to you? If so, add any marks you wish inside the boundary to enhance the image. Just be careful not to alter the boundary itself. If not, try the process again by starting with another equilateral triangle.
- **g.** When you have a design you like, make six copies of it on a copy machine. It's important to have all six identical.
- **h.** Arrange the six designs in a rotational pattern. (Hint: point *C* is critical; it is the point about which the rotation takes place. All six designs will have this point in the center of the overall pattern.)

2. Sets of tiles are now available commercially. They are typically used by elementary school teachers to explain the idea of tiling.

Research Paper

Have students research the life and contributions of Maurits Cornelis Escher (1898 – 1972). As the son of an engineer he had an early interest in carpentry and music. Escher's works are loved by millions, and many examples are available on the Internet.

Collaborative Learning

Creating Tilings

1. A good ice-breaker for this chapter is to cut out many copies of a quadrilateral (not a parallelogram) from sheets of paper. Distribute them to your class and ask your students to determine whether it is possible to tile the plane with copies of this quadrilateral.

2. Even though the author gives a detailed description of how to produce Escher-type tilings, it is unlikely that the students will attempt to make their own outside of class. It might be instructive to devote a few minutes of class time to the construction of such tilings. Try to keep things relatively simple, perhaps by starting with rectangles and having the students make minor modifications in just two sides, following the recipe in the book. After getting them started in class, you can ask the students to complete their "projects" at home.

Solutions

Skills Check:

1. b	2. b/c	3. c	4. b	5. a	6. c	7. b	8. c	9. a	10. a
11. c	12. b	13. c	14. b	15. b	16. c	17. b	18. b	19. c	20. a

Cooperative Learning:

The plane may be tiled with any shape quadrilateral.

Exercises:

1. Exterior: $45°$. Interior: $135°$.

2. Exterior: $36°$. Interior: $144°$.

3. $180° - \dfrac{360°}{n}$.

4. 3, $60°$; 4, $90°$; 5, $108°$; 6, $120°$; 7, $128\frac{4}{7}°$; 8, $135°$; 9, $140°$; 10, $144°$; 11, $147\frac{3}{11}°$; and 12, $150°$.

5. The usual notation for a vertex figure is to denote a regular *n*-gon by *n*, separate the sizes of polygons by periods, and list the polygons in clockwise order starting from the smallest number of sides, so that, e.g., 3.3.3.3.3.3 denotes six equilateral triangles meeting at a vertex. The possible vertex figures are 3.3.3.3.3.3, 3.3.3.3.6, 3.3.3.4.4, 3.3.4.3.4, 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.6.3.6, 3.4.4.6, 3.4.6.4, 3.12.12, 4.4.4.4, 4.6.12, 4.8.8, 5.5.10, and 6.6.6.

6. 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.4.4.6, and 5.5.10.

7. 3.7.42, 3.9.18, 3.8.24, 3.10.15, and 4.5.20.

8. $30°, 75°, 75°$.

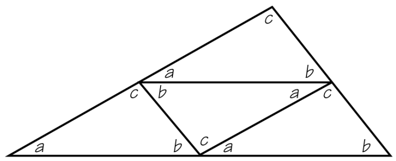
9. At each of the vertices except the center one, six triangles meet, with angles (in clockwise order) of $75°, 75°, 30°, 30°, 75°$, and $75°$.

10. A regular polygon with 12 sides has interior angles of $150°$, and a regular polygon with 8 sides has interior angles of $135°$. No integer combination of these numbers can add up to $360°$.

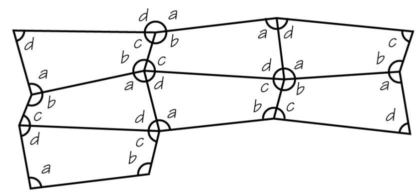
11. Yes, because the half pentagon is a quadrilateral, and any quadrilateral can tile the plane.

12. See figures below.

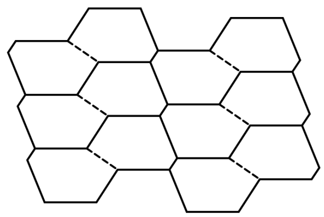
(a)



(b)



(c)



13. (a) No.

(b) No.

(c) No.

14. (d) Yes.

(e) No.

(f) No.

(g) No.

15. Answers will vary.

16. Answers will vary.

17. No.; No.

18. The top of one L must go either under the left square of another (doing that leaves an unfillable square between them) or else under the right square (which works).

19. The only way to tile by translations is to fit the outer "elbow" of one tile into the inner "elbow" of another. Labeling the corners as follows works: the corners on the top *A* and *B*, those on the rightmost side *C* and *D*, the middle of the bottom *E*, and the middle of the leftmost side *F*.

20. Answers will vary.

21. Just label the four corners consecutively *A*, *B*, *C*, and *D*.

22. Solutions may vary. One solution: Label the four lower corners *F*, *E*, *D*, and *C* from left to right, and label the top corners of the middle square in the top row as *A* and *B* from left to right.

23. Place the skew-tetromino on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A = (1,2)$, $B = (3,2)$, $C = (2,0)$, and $D = (0,0)$ works.

24. Solutions may vary. One solution: *A* is the upper leftmost corner, *B* is the upper left corner of the upper square, *C* is the upper right corner, *D* is the midpoint of the righthand side, *E* is one-third of the way along the bottom edge (from the left side), and *F* is the lower left corner.

25. Place the skew-tetromino on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A = (0,1)$, $B = (1,2)$, $C = (3,2)$, $D = (3,1)$, $E = (2,0)$, and $F = (0,0)$ works.

26. If A and B coincide, so must D and E, and vice versa. Other such pairs: $B-C$ and $E-F$, and $C-D$ and $F-A$.

27. (a) Yes.

(b) No.

(c) No.

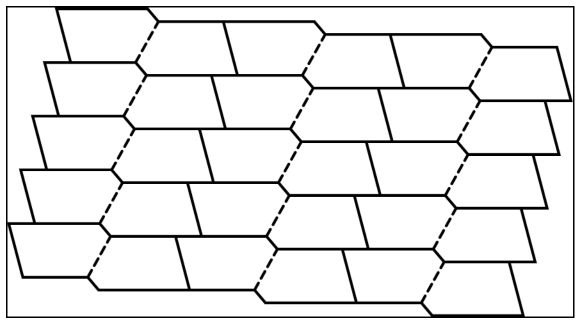
28. (d) Yes.

(e) No.

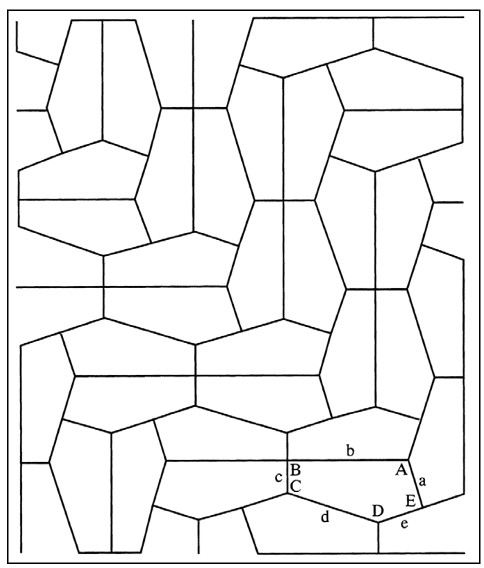
(f) Yes.

(g) Yes.

29. See figure below.



30. See figure below.



31. Answers will vary.

32. Answers will vary.

33. N, Z, W, P, y, I, L, V, X. See www.srcf.ucam.org/~jsm28/tiling/5-omino-trans.ps.gz.

34. Solutions may vary. One solution: Label the middle of the top side A, the upper right corner B, the far lower right corner D, the far left corner F, and C and E each one unit from D.

35. Place the U on a coordinate system with unit length for the side of a square and with the lower left corner at $(0,0)$. Then $A=(2,2)$, $B=(3,2)$, $C=(3,0)$, $D=(1,0)$, $E=(0,0)$, and $F=(0,1)$ works. See www.srcf.ucam.org/~jsm28/tiling/5-omino-rot.ps.gz .

36. Solutions may vary. One solution: Label the top left corner of the T *F* and the top right corner *A*. The corner below *A* is *B*, *C* and *D* are the corners at the foot of the T, and *E* is one unit up from *D*.

37. Answers will vary.

38. Solutions may vary. One solution: Label the top left corner *A*, the top right corner *B* = *C*, the bottom right corner *D*, and the bottom left corner *E* = *F*.

39. *ABAABABA.*

40. Two *B*'s in a row would indicate a baby pair that had a baby pair, which is impossible.

41. The two leftmost *A*'s would have had to come from two *B*'s in a row in the preceding month.

42. Verify that the sequence for the fourth month, *ABA*, follows the rule. Then, assuming that the rule holds for all previous months, consider last month and what happens as it transforms into the current month's sequence. Last month's sequence consists of a first part that is the sequence from two months ago; this first part, we know from our assumption, transforms into last month's sequence. The second part of last month's sequence is the sequence from three months ago, which we know from our assumption transforms into the sequence from two months ago. So the current month's sequence consists of last month's sequence followed by the sequence from two months ago, as claimed.

43. Let S_n, A_n, and B_n be the total number of symbols, the number of *A*'s, and the number of *B*'s at the *n*th stage. We note that the only *B*'s at the *n*th stage must have come from *A*'s in the previous stage, so $B_n = A_{n-1}$. Similarly, the *A*'s at the nth stage come from both *A*'s and *B*'s in the previous stage, so $A_n = A_{n-1} + B_{n-1}$. Using both of these facts together, we have $A_n = A_{n-1} + A_{n-2}$. We note that $A_1 = 0, A_2 = 1, A_3 = 1, A_4 = 2, \ldots$. The A_n sequence obeys the same recurrence rule as the Fibonacci sequence and starts with the same values one step later; in fact, it is always just one step behind the Fibonacci sequence: $A_n = F_{n-1}$. Consequently, $B_n = A_{n-1} = F_{n-2}$, and $S_n = A_n + B_n = F_{n-1} + F_{n-2} = F_n$.

44. In an inflated sequence, the only way a *B* enters is as a *B* preceded by an *A*, so two *B*'s in a row cannot occur. If there were three *A*'s in a row, the first two cannot have been produced by the rule that replaces an *A* by *AB*, so they must have come from two *B*'s in a row, which we just showed is impossible.

45. If a sequence ends in *AA*, its deflation ends in *BB*, which is impossible for a musical sequence. Similarly, if a sequence ends in *ABAB*, its deflation ends in *AA*, which we just showed to be impossible.

46. Suppose that the k^{th} musical sequence is the first not to be an initial subsequence of its inflation. Deflate the k^{th} musical sequence once and its successor twice; these deflations must be the same (according to construction of musical sequences) but also must be different (two different musical sequences cannot deflate to the same sequence), which is a contradiction.

47. The first is, the second is not, part of a musical sequence: *ABAABABAAB* $\rightarrow$ *ABAABA* $\rightarrow$ *ABAB* $\rightarrow$ *AA* $\rightarrow$ (2nd special rule) *BA* $\rightarrow$ (1st special rule) *ABA* $\rightarrow$ *AB* $\rightarrow$ *A*.

ABAABABABA $\rightarrow$ *ABAAAB*, which has three *A*'s.

48. By Exercise 43, the numbers of *A*'s and *B*'s are consecutive Fibonacci numbers, whose ratio tends toward ϕ (Chapter 19, p. 711).

49. If the sequence were periodic, the limiting ratio of A's to B's would be the same as the ratio in the repeating part, which would be a rational number, contrary to the result of Exercise 48.

50. If the sequence were periodic, the limiting ratio of A's to B's would be the same as the ratio in the repeating part, which would be a fixed rational number. But by Exercise 48, the ratio tends toward ϕ, which is not rational.

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