

Chapter 19

Symmetry and Patterns

Chapter Outline

Introduction

Section 19.1 Fibonacci Numbers and the Golden Ratio

Section 19.2 Symmetries Preserve the Pattern

Section 19.3 Rosette, Strip, and Wallpaper Patterns

Section 19.4 Notation for Patterns

Section 19.5 Symmetry Groups

Section 19.6 Fractals Patterns and Chaos

Chapter Summary

Our sense of *symmetry* helps us appreciate patterns. Pleasing patterns exhibit elements of *balance*, *similarity*, and *repetition*. We find symmetry in crystals, the spiral patterns of sunflower seeds and pineapple scales, and the arrangement of petals on a flower. The *Fibonacci numbers* have a mysterious connection with the number of spirals in patterns such as those of sunflower seeds and pineapple scales.

Legend has it that the *golden ratio* $\left(\frac{1+\sqrt{5}}{2}\right)$ was the centerpiece of the Greek ideas of beauty and symmetry. Recent research discounts this theory. Nonetheless, the golden ratio is an interesting number. If we divide a line segment of length l into two pieces of lengths, s and w , so that $\frac{l}{s} = \frac{s}{w}$, then the common value of these fractions is the golden ratio. Further, the golden ratio is the ratio of a diagonal of a pentagon to one of its sides. Finally, the ratio of successive Fibonacci numbers provides ever better approximations to the golden ratio.

A pattern with repetition is classified by the types of *rigid motions* that preserve it, i.e., that move it in such a way that we cannot distinguish the displaced pattern from the original. A rigid motion is a transformation of a pattern such as a *rotation*, *reflection*, or *translation* that does not change the size or shape of the pattern.

This chapter discusses plane patterns. These can be divided into three categories having indefinitely many repetitions in no direction, exactly one direction, or more than one direction. The first type consists of the *rosette patterns*, which are basically patterns preserved by rotations. Essentially, a rosette pattern describes a petal arrangement for a flower. There are two classes of rosettes, *dihedral* and *cyclic*, depending on whether the petals have reflection symmetry or not. These are the only two possibilities (*Leonardo's theorem*).

Strip patterns repeat in one direction and can have *translation*, *reflection*, and *glide reflection* symmetries. They are classified by the combinations of these symmetries. An analysis of the possibilities demonstrates that there are only seven possible strip patterns, which serve as good examples of symmetry groups. The patterns in the final class are referred to as *wallpaper patterns*, and there are only seventeen possibilities.

Methods for classifying patterns, originated with crystallographers. Crystal structures are three-dimensional patterns and analysis has shown that there are exactly 230 possibilities. It should be emphasized that pattern analysis has nothing to do with the individual elements (or motifs) of the pattern design; rather, it concerns how the repetition of the motifs across the plane is structured.

Computers provide a modern way of producing symmetric patterns, obtained by iterating certain polynomial functions a large number of times, and coloring each pixel according to how many times it is visited.

Skill Objectives

1. List the first ten terms of the Fibonacci sequence.
2. Beginning with the number 3, form a ratio of each term in the Fibonacci sequence with its next consecutive term and simplify the ratio; then identify the number that these ratios approximate.
3. List the numerical ratio for the golden section.
4. Name and define the four transformations (rigid motions) in the plane.
5. Analyze a given rosette pattern and determine whether it is dihedral or cyclic.
6. Given a rosette pattern, determine which rotations preserve it.
7. Analyze a given strip pattern by determining which transformations produced it.

Teaching Tips

1. To help explain glide reflection you can point out that if a strip pattern has reflection along a horizontal axis, it will by necessity have glide reflection along that axis as well.
2. Islamic art is filled with designs that demonstrate both line symmetry and rotational symmetry. Much of this art and many other patterns are found in the Alhambra in Spain. The Metropolitan Museum of Art in New York has available Islamic art slides that your class may find interesting.
3. As mentioned in the chapter summary, it is important to emphasize that saying only so many patterns are possible is not saying only so many designs are possible. Pattern analysis concentrates on how the repetition of the motifs is structured.
4. An interesting approach, which presents a different method of classification as well as a computer tutorial, is found in an article by Russell J. Hendel, "A Symmetry Tutor for Introductory Liberal Arts Mathematics Courses," *Collegiate Microcomputer*, 11 (1993), 80–88.

Research Paper

In Section 19.1, Wolfgang Amadeus Mozart (1756 – 1791) was mentioned as someone who was fascinated by mathematics. Students can research his life and involvement in mathematics and patterns via his compositions and method of composition. One interesting aspect of his times in composing music was the introduction of *dice music*. Students, particularly those that have an interest in music, may enjoy researching this topic and its connection to randomness (dice) and musical fragments (patterns) from an array of choices.

Spreadsheet Project

To do this project, go to <http://www.whfreeman.com/fapp7e>.

This project guides students to use spreadsheets to generate and investigate Fibonacci and Lucas sequences.

Collaborative Learning

Finding and Seeing Patterns

1. After the basic ideas of this chapter have been introduced, a good, out-of-class assignment is to have pairs of students walk around campus to look for strip patterns, sketch them, and return to class to report on their findings.
2. Although there are pictures in the book illustrating the existence of Fibonacci numbers in spirals, it is difficult to do the actual counting that verifies this fact. If you can get hold of a number of pine cones, you can ask the students to do the verification.

Solutions

Skills Check:

1. a 2. a 3. b 4. c 5. b 6. c 7. b 8. a 9. b 10. b
 11. c 12. b 13. b 14. a 15. c 16. c 17. a 18. b 19. a 20. c

Exercises:

- 5, 8 and 13
- Answers will vary with species of pinecone but will be consecutive Fibonacci numbers.
- Answers will vary but will be Fibonacci numbers.
- (a) At end of each month, just before any births: 1, 2, 3, 5, 8, 13 pairs.
 (b) 1, 1, 2, 3, 4, 6 pairs.
- (a) The digits after the decimal point do not change.
 (b) The digits after the decimal point do not change.
 (c) $\phi^2 = \phi + 1$
 (d) $\frac{1}{\phi} = \phi - 1$
- $(1 - \phi)^2 = 1 - 2\phi + \phi^2 = 1 - 2\phi + (\phi + 1) = (1 - \phi) + 1$
- (a) $\sqrt{3 \times 27} = \sqrt{81} = 9$
 (b) The area of the rectangle is $4 \times 64 = 256$, so the square must have side $\sqrt{256} = 16$.
- (a) $4 \times 9 = 6$
 (b) $2l + 2w = 6$ and $\frac{l}{w} = \phi$, so $l = \phi w$ and $2\phi w + 2w = 6$, hence $w(\phi + 1) = 3$. The rectangle must be $\frac{3}{1 + \phi}$ by $\frac{3\phi}{1 + \phi}$.
- (a) 4, 7, 11, 18, 29, 47, 76, 123.
 (b) 3, 1.333, 1.75, 1.571, 1.636, 1.611, 1.621, 1.617, 1.618; The ratios approach ϕ .
- (a) $n = 1: F_1 = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^1 - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^1 = \frac{1 + \sqrt{5}}{2\sqrt{5}} - \frac{1 - \sqrt{5}}{2\sqrt{5}} = \frac{1 + \sqrt{5} - 1 + \sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$
 $n = 2: F_2 = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^2 = \frac{1}{\sqrt{5}} \left(\frac{1 + 2\sqrt{5} + 5}{4} \right) - \frac{1}{\sqrt{5}} \left(\frac{1 - 2\sqrt{5} + 5}{4} \right)$
 $= \frac{1}{\sqrt{5}} \left(\frac{6 + 2\sqrt{5}}{4} \right) - \frac{1}{\sqrt{5}} \left(\frac{6 - 2\sqrt{5}}{4} \right) = \frac{3 + \sqrt{5}}{2\sqrt{5}} - \frac{3 - \sqrt{5}}{2\sqrt{5}} = \frac{3 + \sqrt{5} - 3 + \sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$
 (b) $n = 5: F_5 = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^5 - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^5 = 5$
 (c) 233

11. Answers will vary

12. (a) In a single day, Joe can do $\frac{1}{3}$ of the ditch and Sam can do $\frac{1}{4}$, so together they can do $(\frac{1}{3} + \frac{1}{4})$ in a day. To do the whole job (1 ditch), it will take $\frac{1}{\frac{1}{3} + \frac{1}{4}} = \frac{12}{7} \approx 1.7$ days

(b) Suppose that $\frac{2}{\frac{1}{x} + \frac{1}{y}} > \sqrt{xy}$, or $\frac{xy}{\frac{x+y}{2}} > (xy)^{1/2}$, or $(xy)^{1/2} > \frac{x+y}{2}$. But this is a contradiction to the result of Exercise 11, that the geometric mean is always less than or equal to the arithmetic mean. Hence our initial supposition must have been false, and we conclude that the harmonic mean is always less than or equal to the geometric mean.

(c) Using H , G , and A for the harmonic, geometric, and arithmetic means, we have

$$H_{x,y} = \frac{1}{\frac{1}{A_{A,B}}} \geq \frac{1}{G_{A,B}} = G_{x,y},$$

where the inequality comes from the fact that the geometric mean is always less than or equal to the arithmetic mean.

(d) $\sqrt[3]{xyz}$; $\sqrt[n]{x_1 x_2 \cdots x_n}$.

(e) $\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{y}}$; $\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$.

13. $m + n + (m + n) + (m + 2n) + (2m + 3n) + (3m + 5n) + (5m + 8n) + (8m + 13n) + (13m + 21n) + (21m + 34n) = 55m + 88n,$

eleven times the seventh number $5m + 8n$.

14. (a) To express a positive integer as a sum of Fibonacci numbers, repeatedly take out the largest Fibonacci number possible; if the integer is a Fibonacci number, it is represented by just itself. The winning strategy is to remove from the pile of counters the last (smallest) number in the Fibonacci representation of the number of counters. Piles with a Fibonacci number of counters are unsafe – no matter what you do, your opponent can win. The strategy works because the Fibonacci representation never uses two consecutive Fibonacci numbers, so that each Fibonacci number in a representation is about $\phi^2 \approx 2.6$ times as large as the next smallest one; the opponent may take at most twice as many counters as you remove, hence cannot reduce the pile to a Fibonacci number of counters. For example, $49 = 34 + 13 + 2$ (so that the intermediate Fibonacci numbers 21, 8, 5, and 3 are not used). To win, remove 2 counters; the opponent can then take at most 4. Say the opponent takes 4; then the number remaining is $43 = 34 + 8 + 1$, so on the next turn you take 1.

(b) For this game, the unsafe positions are piles with a number of counters that is a Fibonacci number plus 1. The winning strategy is to write (the number in the pile minus one) in a Fibonacci representation (as in part (a)) and remove the smallest Fibonacci number used in the representation.

15. (a) 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1,365

(b) $B_n = B_{n-1} + 2B_{n-2}$

(c) 1, 3, 1.667, 2.2, 1.909, 2.048, 1.977, 2.012, 1.994, 2.003, 1.999

(d) $x = 2, -1$; we discard the -1 root.

(e) $B_n = \frac{2^n - (-1)^n}{3}$

16. (a) 1, 1, 4, 7, 19, 40, 97, . . . ; $G_n = G_{n-1} + 3G_{n-2}$; 1, 4, 1.75, 2.714, 2.105, 2.425, . . . ; we solve $x^2 = x + 3$, whose positive solution is $x = \frac{1 + \sqrt{13}}{2} \approx 2.30277$.
- (b) 1, 1, $1 + q$, $1 + 2q$, $1 + 3q + q^2$, $1 + 4q + 3q^2$, . . . ; $Q_n = Q_{n-1} + qQ_{n-2}$; 1, $1 + q$, - subsequent ratios are complicated; we solve $x^2 = x + q$, whose positive solution is $x = \frac{1 + \sqrt{1 + 4q}}{2}$, which is approximately \sqrt{q} for large q .
17. Silver mean: $1 \pm \sqrt{2} \approx 2.414$; bronze mean: $\frac{1}{2}(3 \pm \sqrt{13}) \approx 3.303$;
copper mean: $2 \pm \sqrt{5} \approx 4.236$; nickel mean: $\frac{1}{2}(5 \pm \sqrt{29}) \approx 5.193$.
General expression: $\frac{1}{2}(m + \sqrt{m^2 + 4})$.
18. (a) Answers will vary.
- (b) x solves $x = p + q\left(\frac{1}{x}\right)$, or $x^2 - px - q = 0$, so $x = \frac{1}{2}(p \pm \sqrt{p^2 + 4q})$.
- (c) If p and q have opposite signs, then after the first few terms, the signs of the terms of the sequence may alternate between positive and negative. Other behaviors are possible; for instance, try $p = 1$ with $q = -1$.
19. All are true.
20. a, b, and d are true.
21. (a) B, C, D, E, H, I, K, O, X
(b) A, H, I, M, O, T, U, V, W, X, Y
(c) H, I, N, O, S, X, Z
22. (a) c, l (in some fonts), o, x
(b) i, l (in some fonts), o, v, w, x
(c) l (in some fonts), o, s, x, z
23. (a) MOM, WOW; MUd and bUM reflect into each other, as do MOM and WOW.
(b) pod rotates into itself; MOM and WOW rotate into each other.
(c) Here are some possibilities: NOW NO; SWIMS; ON MON; CHECK BOOK BOX; OX HIDE.
24. Answers will vary.
25. For all parts, translations.
- (a) Reflection in vertical lines through the centers of the A's or between them.
(b) Reflection in the horizontal midline, glide reflections.
(c) Reflection in the horizontal midline, reflections in vertical lines through the centers of the O's or between them, 180° rotation around the centers of the O's or the midpoints between them, glide reflections.
(d) None other than translations.

26. For all parts, translations.

- (a) 180° rotations around the centers of the N's or the midpoints between them.
- (b) Reflection in vertical lines between b and d or between d and b.
- (c) Reflection in vertical lines between d and b or between p and q.

27. (a) $c5$

(b) $c12$

(c) $c22$

28. (d) $c2$

(e) $d3$

(f) $d16$

29. (a) $c6$

(b) $d2$ (CBS)

(c) $d1$ (Dodge Ram)

30. (d) $d3$

(e) $c2$

(f) $d3$

31. (a) $c4$

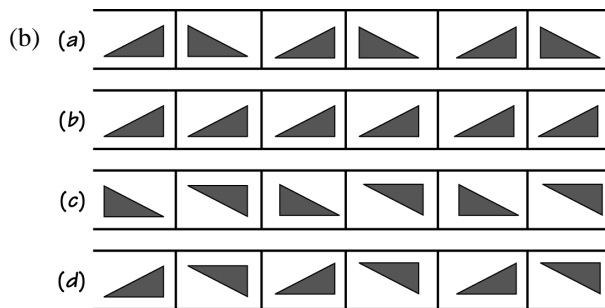
(b) $d2$

32. (a) $c4$

(b) $c4$

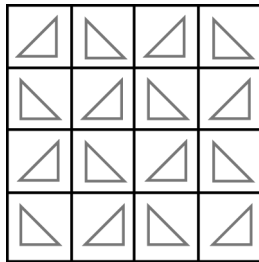
33. $p111, p1a1, p112, pm11, p1m1, pma2, pmm2$

34. (a) Reflection in the vertical line between them; translation; half-turn (rotation by 180°); glide reflection.



(c) $pm11; p111; p112; p1a1$

35. (a) $pmm2$
 (b) $p1a1$
 (c) $pma2$
 (d) $p112$
 (e) $pmm2$ (perhaps)
 (f) $p1m1$
 (g) $pma2$
 (h) $p111$
36. (a) $p111$
 (b) $p1m1$
 (c) $pm11$
 (d) $p112$
 (e) $p1a1$
 (f) $pma2$
 (g) $pmm2$
37. (a) Half-turns are preferred on MesaVerde pottery, while reflections predominate on Begho smoking pipes.
 (b) Neither culture completely excludes any strip type. Begho designs are heavily concentrated (almost 90%) in $p1m1$, $p112$, or $pmm2$, while Mesa Verde designs are more evenly distributed over the seven patterns.
 (c) (a) $pm11$ or $pma2$: Mesa Verde. (b) $p112$: Mesa Verde. (c) $pmm2$: Begho. (d) $pm11$: Begho. (e) $p1m1$: Difficult to say. (f) $pmm2$: Begho. (g) $pmm2$: Begho. (h) $pma2$: Mesa Verde. (i) $p1a1$: Mesa Verde.
38. (a) $p1m1$
 (b) $p1mg$
39. (c) Smallest rotation is 90° , there are reflections, there are reflections in lines that intersect at 45° : $p4m$.
 (d) Smallest rotation is 90° , there are no reflections: $p4$.
40. (a) Reflection in a vertical line
 (b) Glide reflection
 (c) See figure below.



- (d) $clg1$

41. Certainly none of the patterns with rotations at 60° or 120° can arise from this method, which eliminates the five patterns in the two bottom branches of the identification flowchart. For the vertical motion, four orientations are possible for the second triangle in the first column, corresponding to rotations clockwise by 0, 1, 2, or 3 times a rotation of 90° of the triangle in the upper left square around the center of its square. Similarly, for the horizontal motion, there are four choices, which we number in the same fashion. Then there are 16 combinations, which we can denote (with vertical motion first) as 00, 01, 02, 03, 04, 10, . . . , 44. However, some horizontal motions are not compatible with some vertical motions, and some of the patterns are reflections of others along the diagonal from upper left to lower right. In fact, the patterns $p4g$ and $p4$ cannot be realized because the original triangle has symmetry within itself, patterns cm and cmm cannot be realized because every square contains a triangle. The remaining eight can all be formed by the technique.
42. $p111$
43. There are no knights facing up or down (so no rotations of 90°), no knights who are upside down (so no rotations of 180° , and no knights at 60° or 120° angles. So the smallest rotation is 360° , there are no reflections, but there is a glide reflection that takes yellow knights into brown knights. So the pattern is pg , provided color is disregarded; if not, then $p1$.
44. $p111$, if different-colored camels are regarded as not the same; $p211$ if, for example, we may rotate a white camel into a red one.
45. The block of three pentagons shaded in gray is repeated at rotations of 60° around the center of the “snowflake” outlined in red. There are no reflections, so the pattern is $p6$.
46. $p6mm$
47. The intersection of the dashed lines are 120° rotation centers, but there are no reflection lines, so the pattern is $p3$
48. - 49. Answers will vary.
50. (a) Ye.
(b) No; 0 does not have an inverse.
51. (a) $d2$.
(b) Any two of: R (180° rotation around the center), V (reflection in vertical line through its center), H (reflection in horizontal line through its center).
(c) $\{I, R, V, H\}$
52. (a) $d4$
(b) 90° rotation R , plus any one of reflection H in horizontal midline, reflection V in vertical midline, reflection D_1 in the diagonal from upper left to lower right, or reflection D_2 in the diagonal from upper right to lower left.
(c) $\{I, R, R^2, R^3, H, V, D_1 = R \circ H, D_2 = R \circ H\}$
53. Let R denote rotation counterclockwise by 90° , V reflection in vertical line through its center and H reflection in horizontal line through its center. Then the group can be written as $\{I, R, R^2, R^3, H, V, RH = VR, RV = HR\}$, where the last two elements are reflections across the diagonals.

54. (a) $d3$
 (b) dI
 (c) cI

55. Answers will vary. Here is one: $0 = (3 - 2) - 1 \neq 3 - (2 - 1) = 2$.

56. (a) $\langle T, R \mid R^2 = I, R \circ T = T^{-1} \circ R \rangle = \{ \dots, T^{-1}, I, T^1, \dots; \dots, T^{-1} \circ R, R, T^1 \circ R, \dots \}$.

(b) $\langle T, R, H \mid R^2 = H^2 = I, T \circ H = H \circ T, R \circ H = H \circ R, (T \circ R)^2 = I \rangle =$
 $\{ \dots, T^{-1}, I, T, \dots; \dots, T^{-1} \circ R, R, T^1 \circ R, \dots; \dots, T^{-1} \circ H, H, T^1 \circ H,$
 $\dots; \dots, T^{-1} \circ H \circ R, H \circ R, T^1 \circ H \circ R, \dots \}$

57. As in Example 5, number fixed positions, label with letters copies of the pattern elements in the positions, and pick a fixed position about which to make a half-turn R .

(a) $\langle T, R \mid R^2 = I, T \circ R = R \circ T^{-1} \rangle = \{ \dots, T^{-1}, I, T^1, \dots; \dots, R \circ T^{-1}, R, R \circ T^1, \dots \}$.

(b) $\langle T, R, H \mid R^2 = H^2 = I, T \circ H = H \circ T, R \circ H = H \circ R, (R \circ T)^2 = I \rangle =$
 $\{ \dots, T^{-1}, I, T, \dots; \dots, R \circ T^{-1}, R, R \circ T, \dots; \dots, H \circ T^{-1}, H, H \circ T,$
 $\dots; \dots, R \circ H \circ T^{-1}, R \circ H, R \circ H \circ T, \dots \}$

58. $\langle R, H \mid R^4 = I, H^2 = I, H \circ R = R^{-1} \circ H \rangle = \{ I, R, R^2, R^3, H, R \circ H, R^2 \circ H, R^3 \circ H \}$, where R is a rotation by 90° and H is a reflection across a line of symmetry.

59. $\langle R \mid R^8 = I \rangle = \{ I, R, R^2, R^3, R^4, R^5, R^6, R^7 \}$, where R is a rotation by 45° .

60. There are 24 elements in the group, including the identity, rotations around axes through centers of opposite faces, rotations around diagonals between opposite corners, and rotations around axes through centers of opposite edges, for a total of 24. See

<http://nohung.math.uh.edu/mike/hti/handouts/handout4/node4.html> and

<http://www.maths.uwa.edu.au/schultz/3P5.2000/3P5.2,3SquareCube.html>.

61. Get an empty cardboard box as a visual and tactile aid. The situation is analogous to the rectangle of Example 3. Place the solid at the origin of a three-dimensional coordinate system with the axes aligned through its center. The solid can be rotated by multiples of 180° about any of the three axes; denote these rotations as $R_x, R_y,$ and $R_z,$ with $R_x^2 = R_y^2 = R_z^2 = I$ and $R_z = R_x R_y = R_y R_x, R_y = R_x R_z = R_z R_x,$ and $R_x = R_y R_z = R_z R_y$. Similarly, there are reflection symmetries across planes in each axis direction; call them $H, V,$ and $Z,$ with $H^2 = V^2 = Z^2 = I$. Because the dimensions of the solid are all unequal, there are no symmetries across diagonals through it. However, the composition HVZ produces one final symmetry, an “inversion” of the solid through its center: Each point goes to a point the same distance and the opposite direction from the center.

62. Answers will vary.

63. The carved head is reproduced in the same shape at different scales.

64. – 68. Answers will vary.

Word Search Solution

