Chapter 15
Game Theory: The Mathematics of Competition

Chapter Outline

Chapter Summary

Game theory is the mathematical discipline that analyzes situations of conflict and cooperation. Modern game theory originated in 1944 with the publication of *The Theory of Games and Economic Behavior*, by John von Neumann and Oskar Morgenstern.

A mathematical game involves two or more players who can choose between various strategies that have payoffs for the players. It is assumed that the players will act in a rational manner.

In games of total conflict, a payoff representing a gain for one player represents a corresponding loss for the other(s). It is natural to assume that players will seek a strategy that will maximize their gain (or minimize their loss). A basic type of total-conflict game is the two-person, zero-sum game. These games are also called matrix games because they can be represented by a matrix.

In a two-person, matrix game, there is one row for each strategy of one player (row player) and one column for each strategy of the other (column player). The entries in the matrix represent the payoffs to the row player (the negatives of these are the payoffs to the column player). If the payoff matrix has a *saddlepoint*, an entry simultaneously the smallest in its row and largest in its column, then neither player can do better than to employ the strategy corresponding to the row (for row player) and column (for column player) containing the saddlepoint. In this case, we say that in the solution to the game (i.e., the specification of each player's optimal strategy), each player has a pure strategy. The value of the game is taken to be the payoff to the row player.

If the payoff matrix does not have a saddlepoint, then the best a player can do is adopt a mixed strategy. Such a strategy involves a player playing each of his or her strategies with a certain probability. The existence of *optimal mixed strategies* for each player is the content of the minimax theorem. In this situation, the value of the game is the expected payoff to the row player if the player plays his or her strategies with the probabilities specified by the optimal strategy.

Two-person games of partial conflict, such as chicken or *prisoner's dilemma*, produce payoff matrices whose entries are ordered pairs representing the respective payoffs to the row and column players. Games of this type are not usually zero-sum. Typically, each player will benefit if the pair cooperates. However, a player can usually obtain a higher payoff, at the expense of the other player, by being selfish. When neither player can benefit by departing unilaterally from his or her strategy associated with an outcome, the strategies of the players constitute a *Nash equilibrium*. Often, if both players are selfish, the result is detrimental to both. Thus, these games mirror a common social paradox. In the context of larger games, the *status-quo paradox* is an interesting example of this phenomenon. It leads to the notion of sophisticated voting, in which the voters anticipate the moves of their opponents, thereby adjusting their own votes.

In classical game theory, the players choose their options independently and then reveal their choices simultaneously. In a recent extension of the classical case, known as the *Theory of Moves* (*TOM*), a dynamic element is introduced, in which the players move sequentially. An example illustrating this theory is a truel, a three-person version of a duel, in which each player can choose to fire or not fire his gun at either of the other two players. Assuming each player has exactly one bullet, and each is a perfect shot, the question is how a player should proceed in order to maximize his chance of survival (his primary objective), preferably with as few survivors as possible (his secondary objective). In classical game theory, where the players must choose simultaneously, it pays for each player to fire at one of his opponents, thereby killing that person. The probability of any player surviving is just 0.25. Applying TOM, however, each player can look ahead at the consequences of this strategy and determine that it is in his best interest not to fire. Thus, all of the players will survive, which achieves the primary objective of each of them.

Skill Objectives

1. Apply the minimax technique to a game matrix to determine if a saddlepoint exists.

2. When a game matrix contains a saddlepoint, list the game's solution by indicating the pure strategies for both row and column players and the playoff.

3. Interpret the rules of a zero-sum game by listing its payoffs as entries in a game matrix.

4. From a zero-sum game matrix whose payoffs are listed for the row player, construct a corresponding game matrix whose payoffs are listed for the column player.

5. If a two-dimensional game matrix has no saddlepoint, write a set of linear probability equations to produce the row player's mixed strategy.

6. If a two-dimensional game matrix has no saddlepoint, write a set of linear probability equations to produce the column player's mixed strategy.

7. When given either the row player's or the column player's strategy probability, calculate the game's payoff.

8. State in your own words the minimax theorem.

9. Apply the principle of dominance to simplify the dimension of a game matrix.

10. Construct a model for an uncomplicated two-person game of partial conflict.

11. Determine when a pair of strategies is in equilibrium.

12. Be able to interpret and construct game trees.

Teaching Tips

1. In applying the minimax technique to a given matrix, students may become confused about the "column maxima" and the "row minima." For these students, a visual approach may be helpful. First, draw a circle around the minimum number in each row. Next, draw a square around the maximum number in each column. If a matrix entry has both a circle and a square drawn around it, that location represents a saddlepoint.

2. Students are sometimes unclear about the form of a game matrix solution. It may be helpful to reinforce the idea that each row of the matrix represents a different strategy option for the row player; and each column, a different strategy option for the column player. Because a saddlepoint exists at the intersection of a row and a column, those corresponding strategies are the pure strategies to be selected by each player. Each player will then fare best by choosing that one strategy all the time. The numerical value in the saddlepoint location is then the payoff to the player in question (i.e., the average value he or she can hope to achieve through repeated plays of the game).

3. Often students find the graphical approach to a mixed-strategy problem a compelling argument; however, the relationship between the probability equations taken from the matrix and the lines themselves is occasionally blurred. The following review work could prove helpful:

a. Construct a formula for expected value;

b. Derive the slope-intercept form for the equation of a line;

c. Develop the equation of each line from its two given points.

4. Using the principle of dominance to reduce the size of a game matrix by deleting rows or columns, whose numerical entries are overpowered by corresponding entries from other rows or columns of the matrix, will sometimes require practice by the student. This somewhat tedious work can be softened if the student understands that he or she is simplifying the matrix so that he or she can restrict the problem to two dimensions and thus produce a graph.

Research Paper

As stated in the opening paragraph of this chapter, conflict has been prevalent throughout human history. The Babylonian Talmud compiles ancient law and traditions for the first five centuries A.D. (0–500 A.D.). Have students research the "marriage contract problem." They can research how the Talmud makes recommendations of estate distribution, depending on the value. They are not proportional to the three wives, depending on the value of the estate. In 1995, this problem was recognized to have anticipated modern theory of cooperative games. Each recommendation for the distribution of the estate corresponds to an appropriately defined game.

Spreadsheet Project

To do this project, go to http://www.whfreeman.com/fapp7e.

The spreadsheet project, *Game Theory*, allows you to analyze mixed strategies for fastballs and curveballs for a pitcher and batter.

Collaborative Learning

The Coin Problem #1

This exercise is to be done in pairs. Designate one of the two members of the pair as Player 1 and the other as Player 2. The players each have two coins: a penny and a nickel. On each round of the game, the players simultaneously put out one of the coins. Note that the coins are not flipped; each of the players chooses which of the coins to put out.

1. The first game is played as follows: If the two coins match, Player 1 wins $1. If they do not match, Player 2 wins $1. Play this game 25 times, keep score of the outcomes, and see if you can develop a good strategy for playing this game. Does the game seem to be fair?

2. In the second game, change the payoffs. Once again, Player 1 wins when both coins match. This time, however, he wins $1 if both coins are pennies, but $5 if both are nickels. When the coins are different, Player 2 wins $3. Play this game 25 times and answer the same questions that you answered for Part 1.

The Coin Problem #2

This exercise is a follow-up to the previous problem. We agreed that the game

$$\begin{array}{cc} & \begin{array}{cc} P & N \end{array} \\ \begin{array}{c} P \\ N \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

was a fair one. Thus, the value of the game is 0. This means that each player, in the long run, can achieve the outcome 0 by optimal play.
Working with your partner, try to develop an optimal strategy that will guarantee you a long-term outcome of 0.

TOM

Ask the students to discuss a recent long-standing political conflict in the context of TOM.
Good examples include the Middle East and Northern Ireland.

Solutions

Skills Check:

1. c	2. c	3. b	4. c	5. a	6. c	7. a	8. a	9. c	10. a
11. a	12. b	13. c	14. b	15. b	16. b	17. b	18. c	19. b	20. b

Cooperative Learning:

The Coin Problem #1:

1. The game is fair.

2. The game is not fair (it favors Player 2).

The Coin Problem #2:

Choose P and N randomly with probability $\frac{1}{2}$.

Exercises:

1.

$$\begin{bmatrix} 6 & 5 \\ 4 & 2 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ \boxed{5} \\ 2 \end{matrix}$$

$$\text{Column Maxima} \quad 6 \quad \boxed{5}$$

(a) - (b) Saddlepoint at row 1 (maximin strategy), column 2 (minimax strategy), giving value 5.

(c) Row 2 and column 1.

2.

$$\begin{bmatrix} 0 & 3 \\ -5 & 1 \\ 1 & 6 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ 0 \\ -5 \\ \boxed{1} \end{matrix}$$

$$\text{Column Maxima} \quad \boxed{1} \quad 6$$

(a) - (b) Saddlepoint at row 3 (maximin strategy), column 1 (minimax strategy), giving value 1.

(c) Rows 1 and 2, column 2.

3.

$$\begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ \boxed{-2} \\ \boxed{-2} \end{matrix}$$

$$\text{Column Maxima} \quad \boxed{1} \quad 3$$

(a) No saddlepoint.

(b) Rows 1 and 2 are both maximin strategies; column 1 is the minimax strategy.

(c) None.

4.

$$\begin{bmatrix} 13 & 11 \\ 12 & 14 \\ 10 & 11 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ 11 \\ \boxed{12} \\ 10 \end{matrix}$$

$$\text{Column Maxima} \quad \boxed{13} \quad 14$$

(a) No saddlepoint.

(b) Row 2 is the maximin strategy; column 1 is the minimax strategy.

(c) Row 3.

5.

$$\begin{bmatrix} -10 & -17 & -30 \\ -15 & -15 & -25 \\ -20 & -20 & -20 \end{bmatrix} \quad \begin{matrix} \text{Row Minima} \\ -30 \\ -25 \\ \boxed{-20} \end{matrix}$$

$$\text{Column Maxima} \quad -10 \quad -15 \quad \boxed{-20}$$

(a) - (b) Saddlepoint at row 3 (maximin strategy), column 3 (minimax strategy), giving value –20.

(c) Column 3 dominates columns 1 and 2, so column player should avoid strategies from columns 1 and 2.

6.

		Pitcher Fastball	*Pitcher* Curve	Row Minima
Batter	Fastball	0.300	0.200	[0.200]
Batter	Curve	0.100	0.400	0.100
	Column Maxima	[0.300]	0.400	

There is no saddlepoint.

		Pitcher Fastball	*Pitcher* Curve	
Batter	Fastball	0.300	0.200	q
Batter	Curve	0.100	0.400	$1-q$
		p	$1-p$	

Batter:

$$E_F = 0.3q + 0.1(1-q) = 0.3q + 0.1 - 0.1q = 0.1 + 0.2q$$
$$E_C = 0.2q + 0.4(1-q) = 0.2q + 0.4 - 0.4q = 0.4 - 0.2q$$

$$E_F = E_C$$
$$0.1 + 0.2q = 0.4 - 0.2q$$
$$0.4q = 0.3$$
$$q = \tfrac{0.3}{0.4} = \tfrac{3}{4}$$
$$1-q = 1 - \tfrac{3}{4} = \tfrac{1}{4}$$

The batter's optimal mixed strategy is $(q, 1-q) = \left(\tfrac{3}{4}, \tfrac{1}{4}\right)$.

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6. continued

Pitcher:

$$E_F = 0.3p + 0.2(1-p) = 0.3p + 0.2 - 0.2p = 0.2 + 0.1p$$
$$E_C = 0.1p + 0.4(1-p) = 0.1p + 0.4 - 0.4p = 0.4 - 0.3p$$
$$E_F = E_C$$
$$0.2 + 0.1p = 0.4 - 0.3p$$
$$0.4p = 0.2$$
$$p = \tfrac{0.2}{0.4} = \tfrac{1}{2}$$
$$1 - p = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

The pitcher's optimal mixed strategy is $(p, 1-p) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$, giving value as follows.

$$E_F = E_C = E = 0.2 + 0.1\left(\tfrac{1}{2}\right) = 0.2 + 0.05 = 0.250$$

7.

		Pitcher Fastball	*Pitcher* Knuckleball	Row Minima
Batter	Fastball	0.500	0.200	0.200
Batter	Knuckleball	0.200	0.300	0.200
	Column Maxima	0.500	0.300	

There is no saddlepoint.

		Pitcher Fastball	*Pitcher* Knuckleball	
Batter	Fastball	0.500	0.200	q
Batter	Knuckleball	0.200	0.300	$1-q$
		p	$1-p$	

Batter:

$$E_F = 0.5q + 0.2(1-q) = 0.5q + 0.2 - 0.2q = 0.2 + 0.3q$$
$$E_K = 0.2q + 0.3(1-q) = 0.2q + 0.3 - 0.3q = 0.3 - 0.1q$$
$$E_F = E_K$$
$$0.2 + 0.3q = 0.3 - 0.1q$$
$$0.4q = 0.1$$
$$q = \tfrac{0.1}{0.4} = \tfrac{1}{4}$$
$$1 - q = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

The batter's optimal mixed strategy is $(q, 1-q) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right)$.

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7. continued

Pitcher:

$$E_F = 0.5p + 0.2(1-p) = 0.5p + 0.2 - 0.2p = 0.2 + 0.3p$$

$$E_K = 0.2p + 0.3(1-p) = 0.2p + 0.3 - 0.3p = 0.3 - 0.1p$$

$$\begin{aligned} E_F &= E_K \\ 0.2 + 0.3p &= 0.3 - 0.1p \\ 0.4p &= 0.1 \\ p &= \tfrac{0.1}{0.4} = \tfrac{1}{4} \\ 1-p &= 1 - \tfrac{1}{4} = \tfrac{3}{4} \end{aligned}$$

The pitcher's optimal mixed strategy is $(p, 1-p) = \left(\tfrac{1}{4}, \tfrac{3}{4}\right)$, giving value as follows.

$$E_F = E_K = E = 0.2 + 0.3\left(\tfrac{1}{4}\right) = 0.2 + 0.075 = 0.275$$

8.

		Pitcher		Row Minima
		Blooperball	Knuckleball	
Batter	Blooperball	0.400	0.200	0.200
	Knuckleball	0.250	0.250	0.250
	Column Maxima	0.400	0.250	

Saddlepoint at knuckleball for each player, giving value 0.250.

9. The following table represents the gain or loss for the businessman.

		Tax Agency		Row Minima
		Not Audit	Audit	
Businessman	Not Cheating	\$100	–\$100	–\$100
	Cheating	\$1000	–\$3000	–\$3000
	Column Maxima	\$1000	–\$100	

Saddlepoint is "not cheat" and "audit," giving value –\$100.

10.

		Defense Run	Defense Pass	Row Minima
Offense	Run	0.5	0.8	[0.5]
	Pass	0.7	0.2	0.2
	Column Maxima	[0.7]	0.8	

There is no saddlepoint.

		Defense Run (R)	Pass (P)	
Offense	Run (R)	0.5	0.8	q
	Pass (P)	0.7	0.2	$1-q$
		p	$1-p$	

Offense:

$$E_R = 0.5q + 0.7(1-q) = 0.5q + 0.7 - 0.7q = 0.7 - 0.2q$$
$$E_P = 0.8q + 0.2(1-q) = 0.8q + 0.2 - 0.2q = 0.2 + 0.6q$$

$$E_R = E_P$$
$$0.7 - 0.2q = 0.2 + 0.6q$$
$$0.5 = 0.8q$$
$$q = \tfrac{0.5}{0.8} = \tfrac{5}{8}$$
$$1 - q = 1 - \tfrac{5}{8} = \tfrac{3}{8}$$

The offense's optimal mixed strategy is $(q, 1-q) = \left(\tfrac{5}{8}, \tfrac{3}{8}\right)$.

Defense:

$$E_R = 0.5p + 0.8(1-p) = 0.5p + 0.8 - 0.8p = 0.8 - 0.3p$$
$$E_P = 0.7p + 0.2(1-p) = 0.7p + 0.2 - 0.2p = 0.2 + 0.5p$$

$$E_R = E_P$$
$$0.8 - 0.3p = 0.2 + 0.5p$$
$$0.6 = 0.8p$$
$$p = \tfrac{0.6}{0.8} = \tfrac{3}{4}$$
$$1 - p = 1 - \tfrac{3}{4} = \tfrac{1}{4}$$

The defense optimal mixed strategy is $(p, 1-p) = \left(\tfrac{3}{4}, \tfrac{1}{4}\right)$, giving value as follows.

$$E_R = E_P = E = 0.2 + 0.5\left(\tfrac{3}{4}\right) = 0.2 + 0.375 = 0.575$$

11. (a)

	Officer does not patrol	Officer patrols
You park in street	0	–\$40
You park in lot	–\$32	–\$16

(b)

	Officer does not patrol (NP)	Officer patrols (P)	
You park in street (S)	0	–\$40	q
You park in lot (L)	–\$32	–\$16	$1-q$
	p	$1-p$	

You:

$$E_P = (0)q + (-32)(1-q) = 0 - 32 + 32q = -32 + 32q$$

$$E_{NP} = -40q + (-16)(1-q) = -40q - 16 + 16q = -16 - 24q$$

$$E_P = E_{NP}$$
$$-32 + 32q = -16 - 24q$$
$$56q = 16$$
$$q = \tfrac{16}{56} = \tfrac{2}{7}$$
$$1 - q = 1 - \tfrac{2}{7} = \tfrac{5}{7}$$

Your optimal mixed strategy is $(q, 1-q) = \left(\tfrac{2}{7}, \tfrac{5}{7}\right)$.

Officer:

$$E_S = (0)p + (-40)(1-p) = 0 - 40 + 40p = -40 + 40p$$

$$E_L = -32p + (-16)(1-p) = -32p - 16 + 16p = -16 - 16p$$

$$E_S = E_L$$
$$-40 + 40p = -16 - 16p$$
$$56p = 24$$
$$p = \tfrac{24}{56} = \tfrac{3}{7}$$
$$1 - p = 1 - \tfrac{3}{7} = \tfrac{4}{7}$$

The officer's optimal mixed strategy is $(p, 1-p) = \left(\tfrac{3}{7}, \tfrac{4}{7}\right)$, giving the following.

$$E_S = E_L = E = -16 - 16\left(\tfrac{3}{7}\right) \approx -16 - 6.86 = -22.86$$

The value is $-\$22.86$.

(c) It is unlikely that the officer's payoffs are the opposite of yours—that she always benefits when you do not.

(d) Use some random device, such as a die with seven sides.

12. A pure strategy is one in which a player *always* chooses the same course of action, which is to say that he or she chooses it with probability 1 and all other possible courses of action with probability 0. Thus, a pure strategy is a mixed strategy in which all the probability is concentrated on one course of action.

13. (a) Move first to the center box; if your opponent moves next to a corner box or to a side box, move to a corner box in the same row or column. There are now six more boxes to fill, and you have up to three more moves (if you or your opponent does not win before this point), but the rest of your strategy becomes quite complicated, involving choices like "move to block the completion of a row/column/diagonal by your opponent."

(b) Showing that your strategy is optimal involves showing that it guarantees at least a tie, no matter what choices your opponent makes.

14. Player I will choose H $\frac{3}{4}$ of the time and T $\frac{1}{4}$ of the time.

For player II, $E_T = \frac{3}{4}(1)+\frac{1}{4}(-1)=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$ and $E_H = \frac{3}{4}(-1)+\frac{1}{4}(1)=-\frac{3}{4}+\frac{1}{4}=-\frac{2}{4}=-\frac{1}{2}$.

Thus, player II should always play T, winning $\frac{1}{2}$ on average.

15. Player II will choose H $\frac{1}{2}$ of the time and T $\frac{1}{2}$ of the time.

For player I, $E_H = 8\left(\frac{1}{2}\right)-3=4-3=1$ and $E_T = -4\left(\frac{1}{2}\right)+1=-2+1=-1$.

Thus, player I should always play H, winning \$1 on average.

16.

		Economy		
		Poor (P)	Good (G)	
Quantity	Small	\$500,000	\$300,000	q
	Large	\$100,000	\$900,000	$1-q$
		p	$1-p$	

$$E_P = 500{,}000q+100{,}000(1-q)=500{,}000q+100{,}000-100{,}000q=100{,}000+400{,}000q$$

$$E_G = 300{,}000q+900{,}000(1-q)=300{,}000q+900{,}000-900{,}000q=900{,}000-600{,}000q$$

$$\begin{aligned} E_P &= E_G \\ 100{,}000+400{,}000q &= 900{,}000-600{,}000q \\ 1{,}000{,}000q &= 800{,}000 \\ q &= \tfrac{800{,}000}{1{,}000{,}000}=\tfrac{4}{5} \\ 1-q &= 1-\tfrac{4}{5}=\tfrac{1}{5} \end{aligned}$$

Your optimal mixed strategy is $(q,1-q)=\left(\frac{4}{5},\frac{1}{5}\right)$, giving the following.

$$E_P = E_G = E = 100{,}000+400{,}000\left(\tfrac{4}{5}\right)=100{,}000+320{,}000=420{,}000$$

Thus, the value is value \$420,000.

The main alternative is that the economy will perform according to whatever forecast you believe to be most accurate. For example, if you believe the best forecast is that it will be poor with probability $\frac{1}{4}$ and good with probability $\frac{3}{4}$, then choosing "small" will give you an expected payoff of \$350,000, and choosing "large" will give you an expected payoff of \$700,000, so you would choose large. On the other hand, if you wish to maximize your minimum payoff (maximin), then you do better choosing "small," which guarantees you at least \$300,000, whatever the economy does.

17. Rewriting the matrix using abbreviations we have the following.

		Player II		
		F	*C*	*R*
	F	−.25	0	.25
Player I	*BF*	0	0	−.25
	BC	−.25	−.25	0

(a) Player I should avoid "Bet, then call" because it is dominated by "fold" (all entries in *F* row are bigger than corresponding entries in *BC*). Player II should avoid "call" because "fold" dominates it (all entries in *F* column are smaller than corresponding entries in *C*).

(b) Player I will never use "Bet, then call", and Player II will never use "Calls". Removing these, we are left with the following.

		Player II		
		F	R	
Player I	*F*	−.25	.25	q
	BF	0	−.25	$1-q$
		p	$1-p$	

$E_F = -.25q + 0(1-q) = -.25q$

$E_R = .25q + (-.25)(1-q) = .50q - .25$

$$\begin{aligned} E_F &= E_R \\ -.25q &= .50q - .25 \\ -.75q &= -.25 \\ q &= \tfrac{-.25}{-.75} = \tfrac{1}{3} \\ 1-q &= 1-\tfrac{1}{3} = \tfrac{2}{3} \end{aligned}$$

Player I's strategy for (F, BF, BC) is $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

$E_F = -.25p + .25(1-p) = -.25p + .25 - .25p = .25 - .50p$

$E_{BF} = (0)p + (-.25)(1-p) = -.25 + .25p$

$$\begin{aligned} E_F &= E_{BF} \\ .25 - .50p &= -.25 + .25p \\ -.75p &= -.50 \\ p &= \tfrac{-.50}{-.75} = \tfrac{2}{3} \\ 1-p &= 1-\tfrac{2}{3} = \tfrac{1}{3} \end{aligned}$$

Player II's strategy for (F, C, R) is $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$.

$E_F = E_{BF} = E = -.25 + .25\left(\frac{2}{3}\right) = -\frac{1}{4} + \frac{1}{4}\left(\frac{2}{3}\right) = -\frac{3}{12} + \frac{2}{12} = -\frac{1}{12}$

The value is value $-\frac{1}{12}$.

(c) Player II. Since the value is negative, player II's average earnings are positive and player I's are negative.

(d) Yes. Player I bets first while holding *L* with probability $\frac{2}{3}$. Player II raises while holding *L* with probability $\frac{1}{3}$, so sometimes player II raises while holding *L*.

18. (a) Whatever box the first player chose, choose a box as close as possible to that box. If there are several equally close boxes (e.g., that are all adjacent to the box the first player chose), choose one of these closest boxes at random.

(b) No.

19. (a) Leave umbrella at home if there is a 50% chance of rain; carry umbrella if there is a 75% chance of rain.

(b) Carry umbrella in case it rains.

(c) Saddlepoint at "carry umbrella" and "rain," giving value –2.

(d) Leave umbrella at home.

20. The Nash equilibrium is $(4,4)$, but neither player has a dominant strategy. Notice that the players' second strategies guarantee each at least a payoff of 2, whereas their first strategies could result in either $(1,3)$ or $(3,1)$—as well as $(4,4)$—which means that a player could end up with a payoff of only 1 by choosing his or her first strategy. In this sense, the players' first strategies, although associated with the mutually best outcome and Nash equilibrium of $(4,4)$, are "riskier."

21. The Nash equilibrium outcomes are $(4,3)$ and $(3,4)$. [It would be better if the players could flip a coin to decide between $(4,3)$ and $(3,4)$.]

22. The Nash equilibrium is $(4,2)$, and Player I's first strategy is dominant. Notice that Player I ranks the outcomes as if he were playing Prisoners' Dilemma, in which his first strategy is cooperative ("disarm" in the text), whereas Player II ranks the outcomes as if she were playing Chicken, in which her second strategy is cooperative ("swerve" in the text). Like Prisoners' Dilemma and Chicken, the $(3,3)$ cooperative/compromise outcome is not a Nash equilibrium. In this game, it turns out, it is the Chicken player that does worse, at least in terms of comparative rankings, than the Prisoners' Dilemma player.

23. The Nash equilibrium outcome is $(2,4)$, which is the product of dominant strategies by both players.

24. Player II's first strategy is dominant; $(3,4)$ is a Nash equilibrium.

25. The players would have no incentive to lie about the value of their own weapons unless they were sure about the preferences of their opponents and could manipulate them to their advantage. But if they do not have such information, lying could cause them to lose more than 10% of their weapons, as they value them, in any year.

26. These choices give *x* as an outcome. *X* certainly would not want to depart from a strategy that yields a best outcome; furthermore, neither *Y*'s departure to another outcome in the first column, nor *Z*'s departure to another outcome in the second row, can improve on *x* for these players. It seems strange, however, that *Z* would choose *x* over *z*, since *z* is sincere and dominates *x*. Thus, there seem few if any circumstances in which this Nash equilibrium would be chosen.

27. The sophisticated outcome, *x*, is found as follows: *Y*'s strategy of *y* is dominated; with this strategy of *Y* eliminated, *X*'s strategy of *x* is dominated; with this strategy of *X* eliminated, *Z*'s strategy of *z* is dominated, which is eliminated. This leaves *X* voting for *xy* (both *x* and *y*), *Y* voting for *yz*, and *Z* voting for *zx*, creating a three-way tie for *x*, *y*, and *z*, which *X* will break in favor of *x*.

28. Consider the 7-person voting game in which 3 voters have preference *xyz* (one of whom is chair), 2 voters have preference *zxy*, and 2 voters have preference for *zyx*. Then for the 3 *xyz* voters, voting for both *x* and *y* dominates voting for only *x*; and for the 2 *zyx* voters, voting for only *z* dominates voting for both *z* and *y*. With the dominated strategies of *x* and *zy* eliminated, in the second-reduction matrix *z* dominates *zy* for the 2 *zyx* voters, yielding the sophisticated outcome *z*, which is the chair's worst outcome.

29. The payoff matrix is as follows:

		Even		
		2	**4**	**6**
	1	$(2,1)$	$(2,1)$	$(2,1)$
Odd	**2**	$(2,4)$	$(6,3)$	$(6,3)$
	3	$(2,4)$	$(4,8)$	$(10,5)$

Odd will eliminate strategy 1, and Even will eliminate strategy 6, because they are dominated. In the reduced 2×2 game, Odd will eliminate strategy 5. In the reduced 1×2 game, Even will eliminate strategy 4. The resulting outcome will be $(2,4)$, in which Odd chooses strategy 3 and Even chooses strategy 2. The outcome $(2,1)$, in which Odd chooses strategy 1 and Even chooses strategy 2, is also in equilibrium.

30. In the following 3×3 two-person zero-sum game, the saddlepoint—associated with the second strategies of each player—is 2:

4	1	0
3	2	3
0	1	4

Because the three strategies of each player are undominated, however, none can be eliminated through the successive elimination of dominated strategies.

31. If the first player shoots in the air, he will be no threat to the two other players, who will then be in a duel and shoot each other. If a second player fires in the air, then the third player will shoot one of these two, so the two who fire in the air will each have a 50–50 chance of survival. Clearly, the third player, who will definitely survive and eliminate one of her opponents, is in the best position.

32. (a) If A hates B, B hates C, and C hates A, A does not shoot B, lest he be shot by C. B shoots C, putting her in the best position, because she shoots her antagonist, though A also survives.

(b) If B hates A rather than C, A will shoot B, lest he be shot by B. C will then shoot A, so C is in the best position (she alone survives) and A the next-best position (he eliminates his antagonist). By comparison, if A did not shoot B in this case, B would shoot A and survive, because C would not shoot B—and this is worse for A than not shooting since his antagonist survives. These conclusions apply if the players all have only one turn, but if subsequent rounds occur, then in (a) A will shoot B on the second round. So B should not shoot C on the first round; C will not shoot A, because B will then shoot C. Hence, nobody will shoot, because the cycle will repeat itself, and shooting by any player will mean the shooter's death. In (b), subsequent rounds would not change incentives—C, the player nobody hates, would be the sole survivor, because nobody would shoot her on subsequent rounds.

33. In a duel, each player has incentive to fire – preferably first – because he or she does better whether the other player fires (leaving no survivors, which is better than being the sole victim) or does not fire (you are the sole survivor, which is better than surviving with the other player). In a truel, if you fire first, then the player not shot will kill you in turn, so nobody wants to fire first. In a four-person shoot-out, if you fire first, then you leave two survivors, who will not worry about you because you have no more bullets, leading them to duel. Thus, the incentive in a four-person shoot-out—to fire first—is the same as that in a duel.

34. *B* will shoot *C*, because it leads to $(3,3,1)$, which is better for *B* than $(2,2,2)$. Because $(3,3,1)$ is also better for *A* than either $(1,1,4)$ or $(1,4,1)$—the survivors of the other branches that *A* can choose—*A* will not shoot initially, and *B* will shoot *C*.

35. Nobody will shoot.

36. *B* will be indifferent between shooting or not shooting *C*, because whatever *B* does, he or she will be shot in the end by *A*.

37. The possibility of retaliation deters earlier shooting.

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