Chapter 14
Apportionment

Chapter Outline

Chapter Summary

The apportionment problem is to round a set of fractions so that their sum is maintained at a constant value. This problem occurs in several contexts, among them the apportioning of representative bodies such as the United States House of Representatives. Following the ideal of proportional representation, the number of House seats (quota) to which a state is entitled is unlikely to be a whole number. This fact necessitates rounding these quotas to obtain apportionments whose sum is 435, the current number of seats in the House.

Methods of apportionment differ in how they compute quotas and then obtain apportionments. The Hamilton method determines the quota for a state by determining the fraction of the U.S. population that resides in that state and setting the quota equal to that fraction of 435. Apportionment takes place in two stages. First, each state is assigned a number of seats equal to the integer part of its quota. Second, seats unapportioned are assigned to states, one seat per state, based on the size of the fractional part of the state's quota. The state with the largest fractional part gets the first unassigned seat, the state with the next largest fraction the second seat, and so on until all seats are assigned. Note that some states will fail to get an extra seat in this second stage.

Three other procedures that have been used to apportion the House of Representatives are divisor methods. These methods, proposed by Jefferson, Webster, and Hill-Huntington, divide each state's population by a fixed number d to determine the state's modified quota. One may think of d as the average size of a congressional district. But a choice of d, which may not be unique, is determined by the House size and the manner in which the particular appointment method rounds its fractions.

A trial-and-error method for determining d starts with a value of d obtained by dividing the U.S. population by 435, the current House size. This value of d can then be raised or lowered depending on whether the resulting apportionment exceeds or falls short of 435. A systematic way of determining a decisive value of d is by calculating the critical divisors for each state.

The three methods round fractions in different ways. In the Jefferson method, all of the fractions are rounded down, and if the total number of seats apportioned totals 435, then d is a decisive divisor for this method, and the procedure terminates. In the Webster method, fractions are rounded up or down in the usual way. The Hill-Huntington method, which has been in use since 1940, is slightly more complicated; the cutoff point for rounding up is not

0.5 as in Webster, but depends on the state's modified quota. If the integer part of this quota is *n*, then the cutoff point is $\sqrt{n(n-1)}$, the geometric mean of n and $n-1$. For example, if the modified quota of a state is between 7 and 8, then it will receive eight seats if this quota exceeds $\sqrt{7\times 8}\approx 7.4833$. Otherwise, it receives just seven seats.

There are several conditions that a good apportionment method should satisfy, three of which are pertinent to our discussion: (1) quota condition: requires that a state's apportionment be equal to the integer part of the state's quota, or to the integer part plus 1; (2) house monotone condition: an increase in House size cannot cause a state's apportionment to go down; (3) population monotone condition: if a state's population increases while all other populations remain constant, then the state's apportionment should not go down.

Hamilton's method satisfies (1) but not (2) (Alabama paradox). The work of Balinski and Young establishes that (2) and (3) are satisfied by divisor methods only. Unfortunately, every divisor method will violate (1) under certain circumstances (Webster's method is least likely to). Thus, there is no "perfect" apportionment method. However, each divisor method minimizes some measure of inequity in apportionment and so the choice of apportionment method becomes essentially a political decision.

Skill Objectives

1. State the apportionment problem.

2. Explain the difference between quota and apportionment.

3. State the quota condition and be able to tell which apportionment methods satisfy it and which do not.

4. Do the same for the house monotone and population monotone conditions.

5. Know that some methods have bias in favor of large or small states.

6. Recognize the difference in computing quotas between the Hamilton method and divisor methods.

7. Calculate the apportionment of seats in a representative body when the individual population sizes and number of seats are given, using the methods of Hamilton, Jefferson, Webster, and Hill-Huntington.

8. Be able to give at least three reasons to support the claim that Webster's method is the "best" apportionment method.

9. Calculate the critical divisor for each state.

Teaching Tips

1. The apportionment of the U.S. House of Representatives can be discussed and results given. One might not want to do the whole House in class, but a few states can be treated and the results given. Spreadsheet software can be used to compute apportionments using various methods and various divisors.

2. When using Hamilton's method, the number of seats unapportioned after the first stage is always less than the number of states. This is due to the fact that this number of seats is the sum of fifty fractional parts, each of which is less than one.

3. Emphasize how the value of the divisor *d* is changed based on how the apportionment turns out. If the apportionment is too large, the value of *d* should be increased (to decrease quotas). Similarly, if the apportionment is too small, the value of *d* should be decreased.

4. Students often find methods of rounding other than the standard one unnatural. It may take some time for them to become comfortable with these new rounding rules. Examples can help.

5. One can find support for rounding fractions as in the Jefferson method in the wording of Article 1 of the U.S. Constitution: ". . .The number of Representatives shall not exceed one for every thirty thousand."

6. Emphasize to the students that the meaning of the word "quota" in this chapter is very different from that in Chapter 11. Students have a tendency to confuse these two uses of the same term.

7. In the previous editions of the book, the search for a decisive divisor in the methods of Jefferson, Webster, and Hill-Huntington was carried out by trial-and-error. In the current edition, a systematic calculation is performed using critical divisors for each state. However, this computation is not trivial, and many students may prefer to use the trial-and-error approach.

Research Paper

Investigating the lives and contributions of Michel L. Balinsky and H. Peyton Young may be of interest to students. There is historical background to complement that given in the text. Interpreting an article such as, "The Apportionment of Representation" by M. L. Balinski and H. P. Young (pp. 1–29, *Fair Allocation*, *Proceedings of Symposia in Applied Mathematics*, Vol. 3, American Mathematical Society, Providence, RI, 1985) may be of additional interest.

Collaborative Learning

Section Allocation

1 You are chairman of the Mathematics Department of a small high school. Registrations for the coming year are as follows:

- Algebra 94 students
- Geometry 74 students
- Calculus 32 students

The chairman teaches four sections and the other member of the department teaches five sections. Hence there are a total of nine sections to be allocated among the three subjects. Ask the students to come up with a fair method of deciding how many sections of each subject should be scheduled.

2. After introducing the subject of apportionment and the Hamilton method, but before discussing the various paradoxes, have the students determine the allocation of sections in the previous problem according to Hamilton. After completing this exercise, announce that the chairman has decided to teach a fifth section, so that 10 sections in all are now available. Recompute the Hamilton allocation, and note that the Alabama paradox occurs.

Solutions

Skills Check:

1. c	2. a	3. b	4. a	5. a	6. b	7. c	8. a	9. a	10. b
11. a	12. b	13. c	14. c	15. a	16. b	17. c	18. a	19. c	20. c

Cooperative Learning:

No single answer to Exercise 1. In Exercise 2, with 9 sections the allocation is 4 for Algebra, 3 for Geometry, and 2 for Calculus. With 10 sections, Algebra has 5, Geometry has 4, and Calculus only 1, which is a manifestation of the Alabama paradox.

Exercises:

1. Jane's total expenses are $71. The calculation of the percentages is shown in the table.

	Percentage	rounded
Rent	$\frac{31}{71}\times 100\% = 43.66\%$	44%
Food	$\frac{16}{71}\times 100\% = 22.54\%$	23%
Transportation	$\frac{7}{71}\times 100\% = 9.86\%$	10%
Gym	$\frac{12}{71}\times 100\% = 16.90\%$	17%
Miscellaneous	$\frac{5}{71}\times 100\% = 7.04\%$	7%

The percentages add up to 101%.

2. There are 20 teaching assistants and 1125 students: That is 1125÷20 = 56.25 students per teaching assistant. We obtain the quota for each level by dividing that level's enrollment by 56.25. The results are shown in the following table.

Calculus I	500 ÷ 56.25 =	8.89
Calculus II	100 ÷ 56.25 =	1.78
Calculus III	350 ÷ 56.25 =	6.22
Calculus IV	175 ÷ 56.25 =	3.11
Total		20

Round these quotas to obtain the numbers of teaching assistants assigned to each level of the course.

Calculus I	9
Calculus II	2
Calculus III	6
Calculus IV	3
Total	20

3. The new enrollments are obtained by subtracting from the enrollment of each level the number of students who are moving to a lower level, and add to each the number of students who are moving from a higher level. Here are the calculations.

Calculus I	500 + 45 =	545
Calculus II	100 − 45 + 41 =	96
Calculus III	350 − 41 + 12 =	321
Calculus IV	175 − 12 =	163

The total number of students enrolled remains 1125, and the average number of students per teaching assistant is still 56.25. Here are the new quotas.

Calculus I	545 ÷ 56.25 =	9.69
Calculus II	96 ÷ 56.25 =	1.71
Calculus III	321 ÷ 56.25 =	5.71
Calculus IV	163 ÷ 56.25 =	2.90

The new rounded quotas are as follows.

Calculus I	10
Calculus II	2
Calculus III	6
Calculus IV	3
Total	21

This calls for too many teaching assistants, so the numbers must be adjusted. The apportionment methods introduced in this chapter present a variety of approaches to solving this problem.

4. The fractional part of each number is less than 0.50, so all are rounded down as follows.

$$8 + 10 + 12 + 5 + 3 = 38$$

To preserve the sum of 40, we must either add 2 to one of the rounded numbers, or supplement two of the rounded numbers by 1. Apportionment methods provide ways of selecting the amounts to be increased in equitable ways, but each has a different answer to the question, "what is equitable?"

5. Rounding each of the summands down, the sum of the lower quotas is 0 + 1 + 0 + 2 + 2 + 2 = 7. The three numbers with the greatest fractional parts, 0.99, 1.59, and 2.38, receive their upper quotas. The apportioned sum is 0 + 2 + 1 + 2 + 3 + 2 = 10.

6. (a) The populations in this apportionment problem are the amounts invested, the seats are the numbers of pearls awarded, and the states are the three friends. The standard divisor is as follows.

$$\$14{,}900 \div 36 = \$413.89$$

(This is the average price paid for one of the pearls.) Here are the quotas.

Abe	\$5,900 ÷ \$413.89 =	14.255 pearls
Beth	\$7,600 ÷ \$413.89 =	18.363 pearls
Charles	\$1,400 ÷ \$413.89 =	3.383 pearls

The lower quotas, 14, 18, and 3 add up to 35, so one more pearl has to be apportioned. It goes to Charles because his quota has the greatest fractional part. The apportionment is as follows.

Abe	14 pearls
Beth	18 pearls
Charles	4 pearls

Continued on next page

6. continued

(b) The standard divisor (average cost per pearl) has fallen to \$14,900 ÷ 37 = \$402.70. Here are the new quotas.

Abe	\$5,900 ÷ \$402.70 =	14.651 pearls
Beth	\$7,600 ÷ \$402.70 =	18.873 pearls
Charles	\$1,400 ÷ \$402.70 =	3.477 pearls

The lower quotas still sum to 35, so we have to give Abe and Beth, whose quotas have the largest fractional parts, their upper quotas. The new apportionment is as follows.

Abe	15 pearls
Beth	19 pearls
Charles	3 pearls

In effect, the newly found pearl goes to Abe, and Charles has to give one of his pearls to Beth.

(c) This is an instance of the Alabama paradox!

7. The total population is 510,000, so the standard divisor is 510,000 ÷ 102 = 5,000. The quotas are obtained by dividing each state's population by this divisor, obtaining 50.8, 30.6, and 20.6, respectively. The lower quotas add up to 100, so we must increase the apportionment of two states to their upper quotas. The first state, whose quota of 50.8 has the largest fractional part, gets an increase. The fractional parts of the quotas of the remaining two states are both equal to 0.6: they are tied for priority in receiving the last seat. A coin toss is probably the fairest way to settle this dispute.

8. The total population is 97. The number of sections (these are the seats in this apportionment problem) is 5, so the average class size (the standard divisor) is 97 ÷ 5 = 19.4. The quotas are as follows.

Geometry	52 ÷ 19.4 =	2.68 sections
Algebra	33 ÷ 19.4 =	1.70 sections
Calculus	12 ÷ 19.4 =	0.62 sections

The sum of the lower quotas is 3, so we must give two subjects their upper quotas. The subjects with the greatest fractional parts are geometry and algebra. The final apportionment is as follows.

Geometry	3 sections
Algebra	2 sections
Calculus	Cancelled!

9. The total enrollment is 115, and the standard divisor is 23. The quotas are as follows.

Geometry	77 ÷ 23 =	3.35 sections
Algebra	18 ÷ 23 =	0.78 sections
Calculus	20 ÷ 23 =	0.87 sections

The lower quota for geometry is 3, and the other two subjects have 0 lower quotas. Because they have larger fractional parts than geometry, they both receive their upper quotas, 1 each. The apportionment is as follows.

Geometry	3 sections
Algebra	1 section
Calculus	1 section

10. The standard divisor according to the old census is 130,609.62, and with the new census it has increased a little, to 132,517.70. Dividing state populations by these we obtain the following quotas.

State	Old	New
A	42.305	42.693
B	26.569	26.468
C	29.586	29.322
D	1.540	1.517

In each case the lower quotas add up to 98, leaving two seats to be apportioned. In the old census, these go to *B* and *C*, but in the new census they go to *A* and *D*. The apportionments are given in the following table.

State	Old census	New census
A	42	43
B	27	26
C	30	29
D	1	2
Total	100	100

States *B* and *C* had population increases, and decreased apportionments. Although the population of state *D* decreased slightly, its apportionment increased. This is an example of the population paradox.

11. The states in this apportionment problem are the investors, the seats are the 100 coins, and the populations are the individual investments. Thus, the standard divisor is \$10,000 ÷ 100 coins = \$100 per coin. The quotas, which represent the number of coins each investor should receive if fractional coins were possible, are obtained by dividing each investment by this divisor.

	Quota	Lower quota
Abe	36.190	36
Beth	18.620	18
Charles	22.580	22
David	20.100	20
Esther	2.510	2
Total	100.00	98

Two investors will receive their upper quotas: Beth and Charles, who have the largest fractions. Here are the apportionments, *before the excise tax was paid.*

Abe	36
Beth	19
Charles	23
David	20
Esther	2
Total	100

When the excise tax is added, populations change, and the standard divisor changes as follows.

$$\$10{,}050 \div 100 = \$100.50 \text{ per coin}$$

We have to recalculate the quotas. The revised investments are divided by the new standard divisor as follows.

	Investment	Quota	Lower quota
Abe	\$3,635	36.169	36
Beth	\$1,864	18.547	18
Charles	\$2,259	22.478	22
David	\$2,042	20.318	20
Esther	\$250	2.488	2
Total	\$10,050	100.000	98

Again, two investors will receive their upper quotas: Beth and Aunt Esther. The final apportionments are as follows.

	Before tax	After tax
Abe	36	36
Beth	19	19
Charles	23	22
David	20	20
Esther	2	3
Total	100	100

So, Aunt Esther not only got a dollar back, but Charles had to give her one of his rare coins! At least it's still in the family. The cause of this confusion is, of course, the population paradox.

12. The first census recorded a total population of 230,000, and the second census recorded 232,265. Therefore, the standard divisors (the average number of residents for each of the 100 seats in the legislature) are 2,300 and 2,322.65, respectively. To obtain the quotas, divide each state population by the standard divisor. The result is as follows.

Quotas for two censuses

	Censuses	
	Last year	This year
Standard Divisor	2,300.00	2,322.65
Province		
Ash	40.570	41.095
Beech	15.590	15.468
Chestnut	17.620	17.468
Date	24.720	24.485
The desert	1.500	1.484

Here are the lower quotas, based on each of the two censuses.

Lower quotas for two censuses

	Censuses	
Province	Last year	This year
Ash	40	41
Beech	15	15
Chestnut	17	17
Date	24	24
The desert	1	1
Total	97	98

For last year's census, the three provinces with the largest quotas get their upper quotas, because the lower quotas only fill 97 of the 100 seats. Thus, Date, Chestnut and Beech Provinces receive increased apportionments. In this year's census, only two provinces receive their upper quotas: Date and the desert. The resulting apportionments are as follows.

Apportionment according to two censuses

Province	Last year	This year
Ash	40	41
Beech	16	15
Chestnut	18	17
Date	25	25
The desert	1	2
Total	100	100

There is a paradox: Chestnut province gained population and lost a seat, while the desert lost population and gained a seat. This is an instance of the population paradox.

13. In the following table, the critical divisors and quotas are displayed.

House size	82	83	84	89	90	91
Divisor	220,997	218,334	215,735	203,615	201,353	199,140
	Quotas					
A	25.233	25.540	25.848	27.387	27.694	28.002
B	6.278	6.354	6.431	6.814	6.890	6.967
C	15.087	15.271	15.455	16.375	16.559	16.743
D	33.995	34.410	34.824	36.897	37.312	37.7276
E	1.407	1.424	1.442	1.527	1.544	1.562

The next table displays the lower quotas and their sum for each of the house sizes under consideration.

State	Lower Quotas					
A	25	25	25	27	27	28
B	6	6	6	6	6	6
C	15	15	15	16	16	16
D	33	34	34	36	37	37
E	1	1	1	1	1	1
Total	80	81	81	86	87	88
Shortage	2	2	3	3	3	3

The last row of the above table records the number of seats that still must be apportioned. These seats go to the states whose quotas have the largest fractional parts. The final apportionments are as follows.

State	State Population	Apportionments					
A	5,576,330	25	26	26	27	28	28
B	1,387,342	6	6	6	7	7	7
C	3,334,241	15	15	16	16	17	17
D	7,512,860	34	34	35	37	37	38
E	310,968	2	2	1	2	1	1
Total	18,121,741	82	83	84	89	90	91

The Alabama paradox occurs when the apportionment for the smallest state decreases from 2 to 1 as the house size increases from 83 to 84, and it occurs again as the house size increases from 89 to 90.

14. The Webster method rounds all numbers x with fractional parts ≥ 0.5 up, and numbers x with fractional parts less than 0.5 down. We have to verify that the formula $\lfloor x+0.5 \rfloor$ does the same thing. Let $n=\lfloor x \rfloor$. Thus, x is a number at least n but less than $n+1$. If the fractional part of x is less than 0.5, then $x+0.5$ is still a number between n and $n+1$, and hence $\lfloor x+0.5 \rfloor = n$. If $x \geq n+0.5$, then $x+0.5 \geq n+1$ and hence $\lfloor x+0.5 \rfloor = n+1$. In both cases the formula $\lfloor x+0.5 \rfloor$ agrees with the Webster rounding of x.

15. As with the Hamilton method, we have the following quotas.

Geometry	3.35 sections
Algebra	0.78 sections
Calculus	0.87 sections

The tentative apportionments are geometry, 3; algebra and calculus, 0. The critical divisors are determined by adding 1 to the tentative apportionments and dividing the result into the population of the subject, and are as follows.

Geometry	77 ÷ 4 =	19.25 students
Algebra	18 ÷ 1 =	18 students
Calculus	20 ÷ 1 =	20 students

Calculus has the greatest critical divisor, and its tentative apportionment is now 1. It receives a new critical divisor, 20 ÷ 2 = 10. Now the greatest critical divisor is that of geometry, so its apportionment is 4. The house is full, and the Jefferson apportionment is

Geometry	4 sections
Algebra	cancelled!
Calculus	1 section

16. Let's just round off the quotas (found in Exercise 8): 3, 2, and 1 sections for geometry, algebra, and calculus, respectively. These sum to 6, so we have to figure out critical divisors.

Geometry	$52 \div (3-0.5) =$	20.8 students
Algebra	$33 \div (2-0.5) =$	22 students
Calculus	$12 \div (1-0.5) =$	24 students

Geometry has the least critical divisor, so its apportionment is reduced to 2. The final Webster apportionment is as follows.

Geometry	2 sections
Algebra	2 sections
Calculus	1 section

17. All three divisor methods start with the quotas, which were computed in Exercise 6.

	36 pearls	37 pearls
Abe	14.25	14.65
Beth	18.36	18.87
Charles	3.38	3.48

Jefferson method: The tentative apportionments are, for 36 or 37 pearls, Abe, 14; Beth, 18; and Charles, 3. With 36 pearls, 1 is left to be apportioned; with 37 there are 2 left. Here are the critical divisors.

Abe	\$5,900 ÷ 15 =	\$393.33
Beth	\$7,600 ÷ 19 =	\$400.00
Charles	\$1,400 ÷ 4 =	\$350.00

The 36th pearl goes to Beth. When the 37th pearl is discovered, there is no need to repeat the calculations. Beth's critical divisor (only) has to be recomputed, because she has another pearl now. Now her critical divisor is \$7,600 ÷ 20 = \$380.00. The highest priority for the 37th pearl goes to Abe. Here are the final Jefferson apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	19	19
Charles	3	3

Webster method: The tentative apportionments are obtained by rounding the quotas. With 36 pearls, all the quotas are rounded down, so the tentative apportionments add up to 35. We will have to calculate critical divisors to allocate the 36th pearl.

Abe	\$5,900 ÷ 14.5 =	\$406.90
Beth	\$7,600 ÷ 18.5 =	\$410.81
Charles	\$1,400 ÷ 3.5 =	\$400.00

Beth, with the greatest critical divisor, gets the 36th pearl. With 37 pearls, Abe's and Beth's quotas are both rounded up, and Charles's is rounded down. These tentative apportionments, 15, 19, and 3, add up to 37. Abe receives the 37th pearl. Here are the final Webster apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	19	19
Charles	3	3

Hill-Huntington method: The rounding point for numbers between 3 and 4 is $\sqrt{3\times 4}=3.464$; for numbers between 14 and 15 it is $\sqrt{210}=14.491$; and for numbers between 18 and 19 it is $\sqrt{342}=18.493$. Rounding *a la* Hill-Huntington, we obtain the following tentative apportionments.

	36 pearls	37 pearls
Abe	14	15
Beth	18	19
Charles	3	4
Total	35	38

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17. continued

When the calculation is done with 36 pearls, only 35 are accounted for by the tentative apportionments, and with 37, the apportionments add up to 38. Let's calculate critical divisors to determine who gets the 36th and 37th pearls.

Abe	$\$5{,}900 \div \sqrt{14\times15}$	=	\$407.14
Beth	$\$7{,}600 \div \sqrt{352}$	=	\$405.08
Charles	$\$1{,}400 \div \sqrt{12}$	=	\$404.15

Abe has priority for the 36th pearl, and once he receives it, his critical divisor is recomputed as $\$5{,}900 \div \sqrt{15\times16} = \$380.84$. The priority for the 37th pearl goes to Beth. Here are the final Webster apportionments.

	36 pearls	37 pearls
Abe	15	15
Beth	18	19
Charles	3	3

With 36 pearls, there is a difference between the Hill-Huntington apportionment and the others, but with 37, the three methods produce the same results. If there is a principle on which to choose a method, it would probably be to choose the method by which the cost per pearl is as close as possible to the same for each of the friends. The cost per pearl is the district size. The method that minimizes relative differences in the cost per pearl is Hill-Huntington method. If the friends would prefer to minimize absolute differences, they would have to use the Dean method, which was not covered in this chapter. Charles might want to study up on it, though, because it allocates the 36th pearl to Beth, and the 37th to him!

18. The average price per diamond is \$1,000, and that is the critical divisor. The quotas are Abe, 15.5; Beth, 10.5; and Charles, 10. With the Webster method, they would have to round Abe's and Beth's apportionments, and since their fractional parts are both equal to 0.5, that can't work out.

Here's a suggestion: If Abe gets 16 diamonds, his cost is \$15,500 ÷ 16 = \$968.75 per diamond. In this case, Beth would get 10 diamonds for \$10,500 ÷ 10 = \$1,050 per diamond. She is paying \$81.25 more per diamond (or 8.39% more) than Abe.
If we gave the 36th diamond to Beth instead of Abe, his cost per diamond would be \$15,500 ÷15 = \$1,033.33 and hers would be \$10,500 ÷ 11 = \$954.55. Now Abe is paying \$78.78 more per diamond (8.25% more). Thus, to make the cost per diamond as close to the same (on an absolute *or* relative basis) for each of the friends, the 36th diamond should be Beth's.

19. The percentages are the quotas.

Hamilton method: Start with the lower quotas, $87 + 10\times1$, whose sum is 97. The three percentages with the greatest fractional parts, 87.85, 1.26, and 1.25, are rounded up to get the upper quotas; the remaining percentages are rounded down. The final apportionment is

$$88 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 100\%.$$

The first three percentages are rounded to upper quotas, and the remaining percentages are rounded to lower quotas. The quota condition is satisfied.

Jefferson method: Tentatively apportion to each percentage its lower quota. The critical divisors are then the unrounded percentage divided by (1 + the tentative apportionment). Thus, the critical divisor belonging to 87.85% is 87.85 ÷ 88 = 0.9983, while the critical divisors belonging to the smaller percentages range from 1.26 ÷ 2 = 0.63 down to 1.17 ÷ 2 = 0.585. The largest critical divisor belongs to 87.85%, so its tentative apportionment is increased to 88 and its new critical divisor is 87.85 ÷ 89 = 0.9871. This is still the largest critical divisor, so the apportionment of 87.85% is increased to 89. The new critical divisor, 87.85 ÷ 90 = 0.9761, is still the largest, so its apportionment is increased to 90. Now the house is full, and the Jefferson apportionment is

$$90 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 100\%.$$

This apportionment rounds 87.85% to 90%, more than the upper quota. The quota condition is violated.

Webster method: The rounded percentages add up to 98, so we need to calculate critical divisors. The critical divisor belonging to 87.85% is 87.85 ÷ 88.5 = 0.9927. Among the smaller percentages, the largest critical divisor is that of 1.26%, which is 1.26 ÷ 1.5 = 0.84. The point goes to 87.85%, whose apportionment increases to 89. This calls for a new critical divisor, 87.85 ÷ 89.5 = 0.9816, which exceeds the critical divisors of the smaller percentages. The apportionment of 87.85% is therefore increased again to 90. The final apportionment is the same as the Jefferson apportionment, so it too violates the quota condition.

20. The percentages are the quotas.

Hamilton method: Start with the lower quotas, $92 + 5\times 1$, whose sum is 97. The three percentages with the greatest fractional parts, 1.59, 1.58 and 1.57 are rounded up to get the upper quotas; the remaining percentages are rounded down. The final apportionment is

$$92 + 2 + 2 + 2 + 1 + 1 = 100\%.$$

Three percentages are rounded to upper quotas, and three are rounded to lower quotas. The quota condition is satisfied.

Jefferson method: Tentatively apportion to each percentage its lower quota. The critical divisors are then the unrounded percentages divided by (1 + the tentative apportionment). Thus, the critical divisor belonging to 92.15% is $92.15 \div 93 = 0.9909$, while the critical divisors belonging to the smaller percentages range from 1.59 ÷ 2 = 0.795 down to 1.55 ÷ 2 = 0.775. The largest critical divisor belongs to 92.15%, so its tentative apportionment is increased to 93 and its new critical divisor is 92.15 ÷ 94 = 0.9803. This is still the largest critical divisor, so the apportionment of 92.15% is increased to 94. The new critical divisor, 92.15 ÷ 95 = 0.97, is still the largest, so its apportionment is increased to 95. Now the house is full, and the Jefferson apportionment is

$$95 + 1 + 1 + 1 + 1 + 1 = 100\%.$$

This apportionment rounds 92.15% to 95%, more than the upper quota. The quota condition is violated.

Continued on next page

20. continued

Webster method: The rounded percentages add up to 102, so we need to calculate critical divisors. These are equal to the unrounded percentages, divided by (tentative apportionment − 0.5). The tentative apportionment of the percentage with the least critical divisor is reduced. The critical divisor belonging to 92.15% is 92.15 ÷ 91.5 = 1.0071. Among the smaller percentages, the critical divisors range from 1.55 ÷ 1.5 = 1.0333 to 1.59 ÷ 1.5 = 1.06. These smaller percentages have the largest critical divisors; thus the apportionment of 92.15% is reduced to 91. The new critical divisor is $92.15 \div 90.5 = 1.0182$, still the smallest. Therefore the apportionment of 92.15% is reduced to 90, and the final apportionment is

$$90 + 2 + 2 + 2 + 2 + 2 = 100.$$

This apportionment gives 92.15% less than its lower quota, and violates the quota condition.

21. (a) $\sqrt{0 \times 1} = 0$

(b) $\sqrt{1 \times 2} = 1.4142$

(c) $\sqrt{2 \times 3} = 2.4495$

(d) $\sqrt{3 \times 4} = 3.4641$

22. The average section will have $(56 + 28 + 7) \div 5 = 18.2$ students; that is the standard divisor. The quotas are as follows.

Algebra	56 ÷ 18.2 =	3.077 sections
Geometry	28 ÷ 18.2 =	1.538 sections
Calculus	7 ÷ 18.2 =	0.385 sections

The Hill-Huntington method always rounds numbers between 0 and 1 up to 1. Numbers between 1 and 2 are rounded up if they are greater than $\sqrt{2} = 1.4142$, and numbers between 3 and 4 are rounded up if they are greater than $\sqrt{12} = 3.4641$. Thus, the rounded quotas are 3, 2, and 1, respectively. This makes 6 sections, so we have to change one of these apportionments. The critical divisors are given by the formula $d = p \div \sqrt{a(a-1)}$, where p is the population of the students enrolled for the subject, and a is the tentative number of sections apportioned to the subject. The section with the least critical divisor will have its tentative apportionment reduced. Here are the critical divisors.

Algebra	$56 \div \sqrt{6}$ =	22.86 students
Geometry	$28 \div \sqrt{2}$ =	19.7990 students
Calculus	7 ÷ 0 =	∞ students

Geometry has the least critical divisor, and its apportionment is reduced. The final apportionment is as follows.

Algebra	3 sections
Geometry	1 section
Calculus	1 section

23. The standard divisor is $(36+61+3)\div 5 = 20$ students. The quotas are as follows.

Algebra	36 ÷ 20 =	1.8 sections
Geometry	61 ÷ 20 =	3.05 sections
Calculus	3 ÷ 20 =	0.15 sections

Webster would round the quotas to 2, 3, and 0, respectively. These tentative apportionments add up to 5, the house size, and are the final Webster apportionments. Because Hill-Huntington rounds all numbers between 0 and 1 to 1, its tentative apportionment would be 2, 3, and 1. This would exceed the house size by 1, so we have to reduce one of the tentative apportionments. This requires critical divisors. They are as follows.

Algebra	$36 \div \sqrt{2\times 1}$	=	25.456 students
Geometry	$62 \div \sqrt{3\times 2}$	=	24.903 students
Calculus	$7 \div \sqrt{1\times 0}$	=	∞ students

The least critical divisor belongs to Geometry, so its apportionment is decreased to 2. In summary, here are the apportionments.

	Webster	Hill-Huntington
Algebra	2	2
Geometry	3	2
Calculus	cancelled!	1

It's likely that the principal would prefer the Webster method, because classes as small as the calculus class, with 3 students, should be cancelled. Notice that the Hill-Huntington apportionment gives Geometry less than its lower quota in order to accommodate Calculus.

24. (a) To see that the triangle is a right triangle, we use the converse of Pythagoras's theorem.

$$\left(\sqrt{AB}\right)^2+\left(\frac{A-B}{2}\right)^2=\frac{4AB}{4}+\frac{A^2-2AB+B^2}{4}=\frac{A^2+2AB+B^2}{4}=\left(\frac{A+B}{2}\right)^2$$

The hypotenuse of this right triangle is the arithmetic mean of A and B, and the base is the geometric mean. Because the hypotenuse is the longest side in any right triangle, the arithmetic mean of two numbers A and B is greater than the geometric mean, unless $A = B$ (when the altitude of the triangle is 0 and the two means are equal).

(b) Let $n=\lfloor q\rfloor$. Provided that q is not an integer, $\lceil q\rceil=n+1$. Let n^* be the geometric mean of n and $n + 1$: $n^*=\sqrt{n(n+1)}$. The arithmetic mean of n and $n+1$ is as follows.

$$\frac{n+(n+1)}{2}=\frac{2n+1}{2}=n+\tfrac{1}{2}$$

By part (a), $n^*<n+\tfrac{1}{2}$. If $q<n^*$, then Webster and Hill-Huntington both round q down to n. If $q\geq n+\tfrac{1}{2}$, then both round q to $n + 1$. The methods differ when $n^*\leq q<n+\tfrac{1}{2}$ because then Hill-Huntington rounds q to $n + 1$ and Webster rounds q to n.

Continued on next page

24. continued

(c) The sum of the rounded quotas under the Webster method is less than or equal to the sum of the Hill-Huntington rounded quotas, because part (b) tells us that each individual quota, rounded *a la* Webster, is less than or equal to the same quota, rounded *a la* Hill-Huntington. Therefore, if the sum of the rounded quotas is greater than the house size under Webster, so that Webster must use a divisor larger than the standard divisor, Hill-Huntington will certainly have to do the same, and will use a divisor even larger than Webster's. This favors small states. If the Webster rounded quotas add up to the house size, so that Webster is neutral, Hill-Huntington rounded quotas may still add up to more than the house size, making an increased divisor necessary and thus also favoring small states. Finally, if Webster requires a decreased divisor, because the sum of the rounded quotas is less than the house size (this would favor large states), Hill-Huntington will use a larger divisor, which will give less benefit to the larger states.

25. Let's start by taking a seat from California, putting it in play. This leaves 52 seats for California, and California's priority for getting the extra seat is measured by its critical divisor,

$$\frac{\text{Population of California}}{\sqrt{52\times 53}} = 646{,}330.227.$$

To secure the seat in play, Utah's population has to increase enough so that its critical divisor,

$$\frac{\text{Revised population of Utah}}{\sqrt{3\times 4}},$$

surpasses California's. Thus, Utah needs a population of more than the following.

$$646{,}330.227\times\sqrt{12} = 2{,}238{,}954$$

The 2000 census recorded Utah's population as 2,236,714, so an additional 2241 residents would be needed.

26. Yes, and here is an example involving just two states, with populations

$$p_1 = 1{,}000{,}000 \text{ and } p_2 = 6{,}000{,}000,$$

and with a house size $h = 10$. The standard divisor is $d = 700{,}000$, so the quotas are as follows.

$$q_1 = 1{,}000{,}000 \div 700{,}000 = 1.4286, \text{ and } q_2 = 6{,}000{,}000 \div 700{,}000 = 8.5714$$

With the Hill-Huntington method, a number q between 1 and 2 is rounded to 2 if $q \geq \sqrt{2} = 1.4142$. Therefore, the tentative apportionments are $n_1 = 2$, $n_2 = 9$, for a total of 11 seats. The critical divisors are as follows.

$$d_1 = \frac{1{,}000{,}000}{\sqrt{2\times 1}} = \frac{1{,}000{,}000}{\sqrt{2}} = 500{,}000\times\sqrt{2}$$

and

$$d_2 = \frac{6{,}000{,}000}{\sqrt{9\times 8}} = \frac{6{,}000{,}000}{6\sqrt{2}} = 500{,}000\times\sqrt{2}$$

Because their critical divisors are exactly equal, the states are tied when it comes to relinquishing a seat.

27. Before the excise tax was included, the quotas, calculated as in Exercise 11, are rounded to obtain a tentative apportionment.

	Quota	Rounded quota
Abe	36.19	36
Beth	18.62	19
Charles	22.58	23
David	20.10	20
Esther	2.51	3
Total	100.00	101

One quota must be reduced, so we calculate critical divisors as follows.

Abe	$\$3619 \div (36 - 0.5) =$	\$101.94
Beth	$\$1862 \div (19 - 0.5) =$	\$100.65
Charles	$\$2258 \div (23 - 0.5) =$	\$100.36
David	$\$2010 \div (20 - 0.5) =$	\$103.08
Esther	$\$251 \div (3 - 0.5) =$	\$100.40

The least critical divisor is Charles's, so his apportionment is 22. After the tax is added, new rounded quotas are calculated.

	Quota	Rounded quota
Abe	36.17	36
Beth	18.55	19
Charles	22.48	22
David	20.32	20
Esther	2.49	2
Total	100.01	99

Now one of the tentative apportionments must increase, so we must again compute critical divisors.

Abe	$\$3635 \div (36 + 0.5) =$	\$99.589
Beth	$\$1864 \div (19 + 0.5) =$	\$95.590
Charles	$\$2259 \div (22 + 0.5) =$	\$100.400
David	$\$2042 \div (20 + 0.5) =$	\$99.610
Esther	$\$250 \div (2 + 0.5) =$	\$100.000

Charles has the largest critical divisor, so his apportionment is increased to 23. The final apportionments are as follows.

	Before tax	After tax
Abe	36	36
Beth	19	19
Charles	22	23
David	20	20
Esther	3	2
Total	100	100

Esther must give one of her three rare coins to her nephew.

28. The quotas were determined in the solution of Exercise 12. They are shown in the columns labeled q in the table below. The columns q^* display $\sqrt{\lfloor q \rfloor \times \lceil q \rceil}$. If $q \geq q^*$, then the Hill-Huntington method assigns a tentative apportionment $a = \lceil q \rceil$; and if $q < q^*$ the tentative apportionment is $a = \lfloor q \rfloor$.

Tentative apportionment, Hill-Huntington Method

Apportionment according to two censuses

	Last year			This year		
Province	q	q^*	a	q	q^*	a
Ash	40.570	40.497	41	41.095	41.497	41
Beech	15.590	15.492	16	15.468	15.492	15
Chestnut	17.620	17.493	18	17.468	17.493	17
Date	24.720	24.495	25	24.485	24.495	24
The desert	1.500	1.414	2	1.484	1.414	2
Total	100.000	—	102	100.000	—	99

The sum of the tentative apportionments from this year's census must be increased by 1. The critical divisors are shown in the following table.

Province	Population	Tentative	Critical divisor
Ash	95,450	41	$95,450 \div \sqrt{41 \times 42} = 2300.17$
Beech	35,926	15	$35,926 \div \sqrt{15 \times 16} = 2319.01$
Chestnut	40,572	17	$40,572 \div \sqrt{17 \times 18} = 2319.35$
Date	56,870	24	$56,870 \div \sqrt{24 \times 25} = 2321.71$
The desert	3,447	2	$3,447 \div \sqrt{2 \times 3} = 1407.23$

The largest critical divisor belongs to Date, and its apportionment is therefore 25. The other provinces receive their original tentative apportionments. The sum of the tentative apportionments for last year must be reduced by 2. We calculate the critical divisors, as shown in the following table.

Province	Population	Tentative	Critical divisor
Ash	93,311	41	$93,311 \div \sqrt{41 \times 40} = 2304.15$
Beech	35,857	16	$35,857 \div \sqrt{16 \times 15} = 2314.56$
Chestnut	40,526	18	$40,526 \div \sqrt{18 \times 17} = 2316.72$
Date	56,856	25	$56,856 \div \sqrt{25 \times 24} = 2321.14$
The desert	3,450	2	$3,450 \div \sqrt{2 \times 1} = 2439.52$

Continued on next page

28. continued

Ash has the least critical divisor, and we apportion to it 40 seats. Its new critical divisor is $93{,}311 \div \sqrt{40 \times 39} = 2362.49$. Now the least critical divisor belongs to Beech, and its tentative apportionment is reduced to 15. The final apportionments are as follows.

Hill-Huntington Apportionment according to two censuses

Province	Last year	This year
Ash	40	41
Beech	15	15
Chestnut	18	17
Date	25	25
The desert	2	2
Total	100	100

Again, there is a loser (Chestnut), and a gainer (Ash). However, the population of Ash increased over the year, from 93,311 to 95,450. Chestnut's population increased less, from 40,526 to 40,572. This is not a paradox.

29. The quota for the Liberals is $99 \times 49\% = 48.51$, and the Tories' quota is 50.49. With the Hamilton method, the lower quotas add up to 98, and the additional seat goes to the party whose quota has the largest fractional part. This gives the Liberals 49 votes, and the Tories have 50. The Webster method yields the same result because it would round the Liberals' quota up, and the Tories' down.

The Jefferson starts by giving each party its lower quota, 48 for the Liberals and 50 for the Tories. The last seat is given to the party with the largest critical divisor. The formula for critical divisors is (percent of vote received) ÷ (1 + tentative apportionment). Thus the critical divisor for the Liberals is 49 ÷ (1 + 48) = 1, and the critical divisor for the Tories is 51 ÷ (1 + 50) = 1. There is a tie for the 99th seat.

30. (a) One quota will be rounded up, and the other down to obtain the Webster apportionment. The quota that is rounded up will have fractional part greater than 0.5, and will be greater than the fractional part of the quota that is rounded down. The Hamilton method will give the party whose quota has the larger fractional part an additional seat. Thus the apportionments will be identical.

(b) These paradoxes never occur with the Webster method, which gives the same apportionment in this case.

(c) The Hamilton method, which always satisfies the quota condition, gives the same apportionment.

(d) No. Assume that parliament has 100 seats. If one party gets only 0.6% of the vote, and the other party gets 99.4%, the Jefferson critical divisor for the former party will be $\frac{0.6}{1}$, and the latter party will have a critical divisor of $\frac{99.4}{100}$. Jefferson would therefore apportion all 100 seats to the second party, since its critical divisor is the larger. Hamilton would apportion one seat to the first party. On the other hand, Hill-Huntington will give at least one seat to any party that receives at least one vote. Thus, their apportionment would differ from Hamilton's if the vote were to split 0.4% – 99.6%.

31. The following table displays the quotas and tentative apportionment due to the Webster method.

State	Population	Quota	Tentative apportionment
Virginia	630,560	18.310	18
Massachusetts	475,327	13.803	14
Pennsylvania	432,879	12.570	13
North Carolina	353,523	10.266	10
New York	331,589	9.629	10
Maryland	278,514	8.088	8
Connecticut	236,841	6.877	7
South Carolina	206,236	5.989	6
New Jersey	179,570	5.214	5
New Hampshire	141,822	4.118	4
Vermont	85,533	2.484	2
Georgia	70,835	2.057	2
Kentucky	68,705	1.995	2
Rhode Island	68,446	1.988	2
Delaware	55,540	1.613	2
Totals	3,615,920	105	105

Because the tentative apportionment results in the assignment of 105 seats, there is no need for critical divisors: it is the final apportionment. In effect, a seat that had been assigned to Vermont moves to Pennsylvania.

32. The relative difference is the absolute difference, 7 − 5, divided by the lesser of the two numbers, 5. The quotient is 0.4 = 40%.

33. Jim is 7 inches taller than Alice. The relative difference of their heights is 7 inches divided by Alice's height, 65 inches: $\frac{7}{65}$ = 10.77%.

34. (a) North Carolina, because its congressional districts are smaller in population.

(b) The Montana district population is 284,726 larger than the North Carolina district population. That is 45.88% of the North Carolina district population.

35. (a) California, $33{,}930{,}798 \div 53 = 640{,}204$; Utah, $2{,}236{,}714 \div 3 = 745{,}571$.

(b) Absolute difference, $745{,}571 - 640{,}204 = 105{,}367$,

Relative difference, $105{,}367 \div 640{,}204 = 16.46\%$

(c) The district size for California would be 33,930,798 ÷ 52 = 652,515, and the district size for Utah would be 2,236,714 ÷ 4 = 559,178.5. The absolute difference is 652,515 − 559,178 = 93,336.5. The relative difference is 93,336.5 ÷ 559,178.5 = 16.69%

(d) The absolute difference in district populations would be less if California had 52 seats, and Utah had 4. With that revised apportionment, the relative differences would be greater. Thus, the Hill-Huntington method, which was used in apportioning Congress after the 2000 census, did not minimize absolute differences in district population. It minimized relative differences.

36. The absolute difference in district populations is as follows.

$$\frac{p_A}{a_A} - \frac{p_B}{a_B} = \frac{a_B p_A}{a_B a_A} - \frac{a_A p_B}{a_A a_B} = \frac{a_B p_A - a_A p_B}{a_A a_B}$$

To obtain the relative difference, divide this absolute difference by the smaller district population, $\frac{p_B}{a_B}$, and express the resulting fraction as a percent.

$$\frac{a_B p_A - a_A p_B}{a_A a_B} \div \frac{p_B}{a_B} = \frac{a_B p_A - a_A p_B}{a_A a_B} \times \frac{a_B}{p_B} = \frac{a_B p_A - a_A p_B}{a_A p_B}$$

(a_B has been cancelled from the numerator and denominator.)

The relative difference in district population is $\frac{a_B p_A - a_A p_B}{a_A p_B} \times 100\%$.

The state with the smaller representative share is *A*, so the relative difference is as follows.

$$\left(\frac{a_B}{p_B} - \frac{a_A}{p_A}\right) \div \left(\frac{a_A}{p_A}\right) \times 100\%$$

$$\left(\frac{a_B p_A}{p_B p_A} - \frac{a_A p_B}{p_A p_B}\right) \times \left(\frac{p_A}{a_A}\right) \times 100\% = \left(\frac{a_B p_A - a_A p_B}{p_A p_B}\right) \times \left(\frac{p_A}{a_A}\right) \times 100\%$$

$$\frac{a_B p_A - a_A p_B}{a_A p_B} \times 100\%$$

Because this formula is identical to the formula for relative difference in district population, the two measures on inequity yield the same result.

37. With 10 seats for Massachusetts, and 6 for Oklahoma, the representative shares (per million population) for these states are 10 ÷ 6.029051 = 1.6586 seats per million for Massachusetts, and 6 ÷ 3.145585 = 1.9074 for Oklahoma. The inequity in representative share is in favor of Oklahoma, by 0.2488 seats per million population. If Massachusetts had 11 seats, and Oklahoma 5, the respective representative shares would be 1.8245 and 1.5895. The inequity, in favor of Massachusetts, is 0.235 seats per million population. Therefore, the Webster apportionment would give Massachusetts the seat.

38. (a) The sum of the quotas is *h* and unless each quota is a whole number, some state will receive more than its quota. Therefore the total number of seats apportioned will be more than *h*.

(b) Let *p* be the total population and p_i be the population of state *i*. Then the standard divisor is $d = \frac{p}{h}$, and $a_i = \left\lceil \frac{p_i}{d} \right\rceil$ is the apportionment for state *i*. Also let h' be the actual house size resulting from this apportionment. In part (a), we saw that $h' > h$. California is a populous state. If the apportionment had been done by the Hill-Huntington method, it is possible that California would have received its upper quota anyway. If not, it would have received at least its lower quota (it's very unlikely that the Hill-Huntington method will violate the quota condition). The amount that California has to gain is relatively small, but some other states will surely gain, and California's share of the house seats will be less than it would have been if the Hill-Huntington method had been used.

39. (a) Lowndes favors small states, because in computing the relative difference, the fractional part of the quota will be divided by the lower quota. If a large state had a quota of 20.9, the Lowndes relative difference works out to be 0.045. A state with a quota of 1.05 would have priority for the next seat.

(b) Yes, because like the Hamilton method, the Lowndes method presents a way to decide, for each state, if the lower or upper quota should be awarded.

(c) Yes. Since the method is not a divisor method, the population paradox is inevitable.

(d) Let r_i denote the relative difference between the quota and lower quota for state *i*. The following table displays the numbers r_i for each state. Because the lower quotas add up to 97, the 8 states with the largest values in the r_i column will receive their upper quotas.

State	p_i	q_i	$\lfloor q_i \rfloor$	r_i	rank	a_i
Virginia	630,560	18.310	18	1.7%	14	18
Massachusetts	475,327	13.803	13	6.2%	8	14
Pennsylvania	432,879	12.570	12	4.8%	9	12
North Carolina	353,523	10.266	10	2.7%	13	10
New York	331,589	9.629	9	7.0%	7	10
Maryland	278,514	8.088	8	1.1%	15	8
Connecticut	236,841	6.877	6	14.6%	6	7
South Carolina	206,236	5.989	5	19.8%	5	6
New Jersey	179,570	5.214	5	4.3%	10	5
New Hampshire	141,822	4.118	4	3.0%	11	4
Vermont	85,533	2.484	2	24.2%	4	3
Georgia	70,835	2.057	2	2.9%	12	2
Kentucky	68,705	1.995	1	99.5%	1	2
Rhode Island	68,446	1.988	1	98.8%	2	2
Delaware	55,540	1.613	1	61.3%	3	2
Totals	3,615,920	105	97	–	–	105

40. (a) No. Unless a state's quota is a whole number, its tentative apportionment will be more than its quota. Of course, in the unlikely event that a state's quota is a whole number, its tentative apportionment would equal its quota. The sum of all the quotas is equal to the house size, so the tentative apportionments will add up to more than the house size. Critical divisors will have to be used to decide which states should receive reduced tentative apportionments.

(b) Let state *i* have population p_i. Its critical divisor d_i will be the greatest divisor that will reduce its tentative apportionment n_i to $n_i - 1$. Thus, $n_i - 1 = \dfrac{p_i}{d_i}$ and hence $d_i = \dfrac{p_i}{n_i - 1}$.

The state with the least critical divisor receives the reduced tentative apportionment, and then its critical divisor is recomputed. The process is complete when the sum of the tentative apportionments has been reduced to the house size.

(c) The method favors small states, because it can never increase any state's tentative apportionment. The apportionments are calculated by multiplying each quota by an adjustment factor that is *less* than 1, and rounding up. A populous state's quota will be reduced more than a small state's. Also, a state will never receive more than its upper quota, but can receive less that its lower quota.

(d) If a state's tentative apportionment is 1, then its critical divisor is ∞. Its tentative apportionment will not be reduced. Of course, this method does not work if the number of states is more than the house size!

41. (a) Let $n = \lfloor q \rfloor$. If q is between n and $n+0.4$, then the Condorcet rounding of q is equal to n. Since $q+0.6 < n+1$ in this case, it is also true that $\lfloor q+0.6 \rfloor = n$. On the other hand, if $n+0.4 \le q < n+1$, then the Condorcet rounding of q is $n+1$, and also $n+1 \le q+0.6 < n+1.6$, so $\lfloor q+0.6 \rfloor = n+1$.

(b) The method favors small states, since numbers will be rounded up more often than down; and this makes it more likely that the quotas will be adjusted downward.

(c) If the sum of the tentative apportionments is less than the house size, the critical divisor for state *i*, with population p_i, is the greatest divisor d_i that would apportion another seat to the state. Thus, if the tentative apportionment is n_i, then $n_i + 0.4 = \frac{p_i}{d_i}$, and hence $d_i = \frac{p_i}{n_i + 0.4}$. The state with the largest critical divisor gets the next seat, and then its critical divisor is recomputed. The process stops when the house is full.

If the total apportionment is more than the house size, then the critical divisor for state *i* is the least divisor that would cause the state's tentative apportionment to decrease. Thus $n_i - 1 + 0.4 = \frac{p_i}{d_i}$, so $d_i = \frac{p_i}{n_i - 0.6}$. The state with the least critical divisor of all has its tentative apportionment decreased by 1. Its critical divisor is then recomputed. The process stops when enough seats have been removed so that the number of seats apportioned is equal to the house size.

42. (a) Hill-Huntington rounds all numbers between 0 and 1 to 1. Thus, no finite divisor would cause a tentative apportionment of 0. The critical divisor to reduce a tentative apportionment of 1 to 0 by the Hill-Huntington method is infinite.

The Hamilton, Jefferson, and Webster methods can all produce zero apportionments. For example, suppose that there are 10 seats to be apportioned, state 1 has a population of 97, and state 2 has a population of 3. The standard divisor is 10, and the quotas are 9.7 and 0.3, respectively. Hamilton would give both states their lower quota, 9 and 0, and then state 1, with the larger fraction, would get the 10th seat as well. Webster would round the quota for the states to 10 and 0, respectively. Jefferson would give 9 to state 1, 0 to state 2, and they would compete for the last seat on the basis of critical divisors: $97 \div 10 = 9.7$ for state 1, and $3 \div 1 = 3$ for state 2. State 1 would get the last seat, leaving state 1 with 0.

(b) Yes, for the reason noted in the solution of Exercise 40.

(c) Let n_i be the apportionment of state *i* and p_i be its population. If $n_i = 0$ then the district population for this state is infinite. When we compare this with the finite district population of any state with more than one seat, we see that a transfer of a seat to state *i* would create a situation in which both states have finite district populations. Thus, instead of an infinite difference in district populations, we have a finite difference. Thus a Dean apportionment cannot have two states, one with zero apportionment while the other has two or more seats.

43. Let $f_i = q_i - \lfloor q_i \rfloor$ denote the fractional part of the quota for state *i*. Since the Hamilton method assigns to each state either its lower or its upper quota, each absolute deviation is equal to either f_i (if state *i* received its lower quota) or 1 − f_i (if it received its upper quota). For convenience, let's assume that the states are ordered so that the fractions are decreasing, with f_1 the largest and f_n the smallest. If the lower quotas add up to $h - k$, where *h* is the house size, then states 1 through *k* will receive their upper quotas. The maximum absolute deviation will be the larger of $1 - f_k$ and f_{k+1}.

The maximum absolute deviation for the Hamilton method is less than 1, because each fractional part f_i and its complement, $1 - f_i$, is less than 1. If a particular apportionment fails to satisfy the quota condition, then for al least one state, the absolute deviation exceeds 1, and hence the maximum absolute deviation is greater than that of the Hamilton apportionment.

If an apportionment satisfies the quota condition then — as with the Hamilton method — *k* states receive their upper quotas and $n - k$ states receive their lower quotas.

If a state *j*, where $j \le k$, receives its lower quota, then — to compensate — a state *l*, where *l* > *k*, must get its upper quota. The absolute deviations for these states would be f_j and $1 - f_l$, respectively. Because of the way the fractions have been ordered, we have $1 - f_l \ge 1 - f_k$. Therefore, the absolute deviation for one of states *j* and *l* will be equal to or exceed the maximum absolute deviation of the Hamilton apportionment. We conclude that no apportionment is better than Hamilton's, if what we mean by "better" is "smaller maximum absolute deviation."

44. (a) Use the Jefferson method. If *d* is the divisor used in that apportionment, subject *i* gets $\lfloor p_i \div d \rfloor$ sections (where p_i is the number of students enrolled to take the course in subject *i*). The minimum class size is *d*.

(b) Use the method of John Quincy Adams described in Exercise 40. Now if *d* is the divisor, the apportionment to subject *i* is $\lceil p_i \div d \rceil$ sections. The maximum section size is *d*.

(c) The Webster method minimizes differences in representative share.

(d) The Hill-Huntington method minimizes relative differences in district population.

(e) The Adams and Hill-Huntington methods will apportion one class to any course that has an enrollment of at least one, so they should be avoided.

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