Chapter 13
Fair Division

Chapter Outline

Chapter Summary

The *fair-division problem* is to divide a collection of goods among several players in such a way that each player perceives that he or she has received a fair and unbiased share of the goods. Such problems arise often, whether in dividing an estate among heirs or in cutting the cake at a child's birthday party. The fair-division problem is considered for two different cases: the continuous case, in which the goods are finely divisible, and the discrete case, in which the goods are essentially indivisible. The methods discussed involve only the players themselves.

Examples involving cake cutting are particularly apt in the continuous case. For two players, a good method is to have one player cut and the other choose. This method achieves a fair division, provided: (1) either player can cut the cake so that either piece is acceptable to that player; and (2) given any division, the player choosing will find at least one piece acceptable.

A similar method, *lone-divider*, extends "cut-and-choose" reasoning to the three-player case. One player cuts the cake into three pieces and the other two indicate which pieces they find acceptable. A problem arises only if the second and third players agree that just one of the pieces is acceptable. In this case, one of the unacceptable pieces is given to the cutter (who feels that they are *all* equal), the two remaining pieces are reassembled, and we are back to the case of a two-player division.

An alternative multiple player scheme is *last-diminisher*. One player cuts what he or she considers a fair piece. In turn, the other players can pass the piece on or diminish its size. The piece goes to the last player to diminish it. The remaining players repeat the process until all are served.

In 1949, Hugo Steinhaus proved that, for any size group of players, there is a fair-division scheme that is envy-free; that is, each player has a strategy that guarantees him a piece at least as large as any other player's (as each player perceives the allocations), no matter what the other players do. In 1992, Steven Brams and Alan Taylor actually devised such a method for three or more players.

The discrete case is treated in the context of divorce and estate settlement. In the *Adjusted Winner Procedure* each party starts with 100 points and bids on each item.

Initially, each item is allocated to the highest bidder. If each party has received the same total number of points, the process stops. If, however, they differ, then assets (or portions thereof) are transferred in order to equalize the number of points each receives.

For estate settlement, which may involve many heirs and unequal shares, an auction scheme has been devised by *Knaster*. Each heir fixes his estimate of the value of the estate by bidding on the items in it. The fair share of each heir is determined from the sum of his or her bids. Surplus value in the estate is divided among the heirs in accordance with the share to which each is entitled. An interesting feature of this scheme is that each heir receives value greater than the fair share expected, based on his or her valuation of the estate. Modifications of this scheme are discussed in the chapter exercises. Trimming (or diminishing) procedures can be introduced into this scheme if the estate has a large amount of cash or many small objects, thus providing a component of the estate that is essentially finely divisible.

Another method involves taking turns, in which the participants alternately choose items from the estate. While this method is easy to understand and implement, it leads to a considerable amount of strategic planning and insincere choices, based upon each participant's knowledge of the preferences of his opponent. For example, one of the players may initially pass over his first choice if he knows that his opponent does not care for that item and is unlikely to choose it.

Skill Objectives

1. Describe the goal of a fair-division problem.

2. Define the term "player".

3. Define the set-theoretic term "partition" and describe its application to a fair-division problem.

4. List three different categories of fair-division problems.

5. Explain what is meant by a continuous case, fair-division problem and give an example.

6. List two approaches to solving a continuous case, fair-division problem.

7. Explain what is meant by a discrete case, fair-division problem and give an example.

8. Describe a method for solving a discrete case, fair-division problem.

9. Calculate a discrete, fair division for a small number of players and objects when:

   a. each player has an equal share;

   b. the players all have different shares.

Teaching Tips

1. To explain fair division of discrete objects, start with the case of two people and one object using an appropriate chart. Add a second object to the problem and rework it. Finally, go to the case of three people and three objects. The students should be able to do any problem now.

2. As an extra-credit exercise, students can write an original narrative describing a fair allocation problem that has affected their personal relationships, describe how the allocation was determined at the time, and then apply the methods of this chapter to calculate a fair division.

Research Paper

Steven J. Brams and Alan D. Taylor are two of the authors of your text. Have students investigate their contributions to fair division and cake cutting via research on the Internet. Other mathematical figures, such as D. Marc Kilgour and Francis E. Su, have contributions in this area. An interesting website for students to investigate is http://www.math.hmc.edu/~su/fairdivision. There is a fair division calculator applet and other useful links.

Collaborative Learning

Fair Division—The Cake Problem

Two people wish to divide a cake in such a way that each of them feels that he or she has received at least one-half of the cake. An old method for doing this is for one of the people to cut the cake into what he or she perceives as two equal pieces, and the second person chooses the piece that he or she thinks is larger.

Now suppose that three people wish to divide a cake in such a way that each of them feels that he or she has received at least one-third of the cake. Devise a procedure to accomplish this.

Fair Division—The Inheritance Problem

Eccentric, but *very* rich Aunt Millie died recently, leaving a rather peculiar estate and will. Her liquid assets (approximately $20,000,000) go to a local hospital. Her mansion and surrounding 30 acres are to be turned into a home for animals, and her stretch limousine will be used for taking homeless people to shelters. Aside from clothing (bought in thrift shops) and some worthless costume jewelry, the following are the only remaining items of value:

- Babe Ruth's rookie card, with his signature
- A Ming vase
- The first printed edition of Euclid's *Elements*
- The original score of Beethoven's Ninth Symphony
- Ringo Starr's drums that he played in the first Beatles' concert

Bob, Ann, and Tom will inherit these items. However, there are some catches:

1. Under the terms of the will, *nothing* can be sold.

2. The three of you will inherit equal portions.

3. Since Aunt Millie accumulated her huge estate through frugality, she won't allow you to waste any money on appraisers. In other words, you'll have to devise a method for dividing these five items equally without professional help.

Now, the problem facing you is that you don't agree on the value of the items. These are the values that you attach to them:

	Bob	Ann	Tom
Ruth's card	$6,000	$3,000	$5,000
Ming vase	$5,000	$13,000	$10,000
Euclid	$7,000	$11,000	$13,000
Beethoven	$9,000	$15,000	$12,000
Ringo	$12,000	$6,000	$11,000

Can you devise a procedure that will satisfy everyone? That is, try to find a method of division under which everyone believes that he or she has obtained at least a fair share. (Note: Although nothing may be sold, money may change hands among the heirs.)

Fair Division—The Roommate Problem

Bob and Tom shared an apartment for their four college years, and accumulated a number of items that they want to keep. The problem is that both of them would like to have these items. They devise the following procedure for dividing the items fairly.

The roommates are each given 100 points, and they can place as many of these points as they wish on the individual items.

Item	Bob's points	Tom's points
Encyclopedia	20	25
Easy chair	15	10
Painting	30	25
Rug	20	15
Kitchen set	15	25

It is logical to give each item to the person who values it most. However, if we do that, then the roommates will end up with different point totals. Can you figure out a way to equalize the point totals?

Solutions

Skills Check:

1. a	2. b	3. a	4. b	5. d	6. c	7. b	8. c	9. b	10. c
11. b	12. c	13. a	14. a	15. b	16. c	17. a	18. b	19. b	20. a

Exercises:

1. Donald initially receives the Palm Beach mansion (40 points) and the Trump Tower triplex (38 points) for a total of 78 points. Ivana initially receives the Connecticut estate (38 points) and the Trump Plaza apartment (30 points) for a total of 68 points. Because Ivana has fewer points than Donald, she receives the cash and jewelry (on which they both placed 2 points) bringing her total to 70 points. As Donald still has more points (78 to 70), we begin transferring items from him to her. To determine the order of transfer, we must calculate the point ratios of the items that Donald now has.

The point ratio of the Palm Beach mansion is $\frac{40}{20} = 2.0$.

The point ratio of the Trump Tower triplex is $\frac{38}{10} = 3.8$.

Because 2.0 < 3.8, the first item to be transferred is the Palm Beach mansion. However, if all of it were given to Ivana, her point total would rise to 70 + 20 = 90, and Donald's point total would fall to 78 – 40 = 38. This means that only a fraction of the Palm Beach mansion will be transferred from Donald to Ivana.

Let x be the fraction of the Palm Beach mansion that Donald retains, and let $1-x$ be the fraction of it that is given to Ivana. To equalize point totals, x must satisfy $38+40x=70+20(1-x)$. Thus, using algebra to solve this equation yields the following.

$$\begin{aligned} 38+40x &= 70+20-20x \\ 38+40x &= 90-20x \\ 60x &= 52 \\ x &= \tfrac{52}{60} \\ x &= \tfrac{13}{15} \end{aligned}$$

Thus Donald receives the Trump Tower triplex and $\frac{13}{15}$ (about 87%) ownership of the Palm Beach mansion for a total of about 72.7 of his points, and Ivana gets the rest (for about 72.7 of her points).

2. Calvin initially receives the cannon (10), the sword (15), the cannon ball (5), the wooden leg (2), the flag (10) and the crow's nest (2) for a total of 44 points. Hobbes initially receives the anchor (20), the unopened chest (20), the doubloon (14), and the figurehead (30) for a total of 84 points. To determine the order of transfer, we must calculate the point ratios of the items that Hobbes has.

The point ratio of the anchor is $\frac{20}{10} = 2.0$.

The point ratio of the unopened chest is $\frac{20}{15} = 1.33$.

The point ratio of the doubloon is $\frac{14}{11} = 1.27$.

The point ratio of the figurehead is $\frac{30}{20} = 1.5$.

Continued on next page

2. (continued)

The first item to be transferred is the doubloon because it has the lowest point ratio. Calvin's point total now becomes 44 + 11 = 55, and Hobbes's point total now becomes 84 – 14 = 70. Because the transfer of the unopened chest would result in Calvin having more points than Hobbes, this is the one item they will have to split or share. Let x be the fraction of the unopened chest that Hobbes retains, and let $1-x$ be the fraction of it that goes to Calvin. To equalize point totals, x must satisfy $50+20x=55+15(1-x)$.

Thus, using algebra to solve this equation yields the following.

$$\begin{aligned} 50+20x &= 55+15-15x \\ 50+20x &= 70-15x \\ 35x &= 20 \\ x &= \tfrac{20}{35} \\ x &= \tfrac{4}{7} \end{aligned}$$

Thus Hobbes keeps $\frac{4}{7}$ (about 57%) of the unopened chest, and Calvin gets $\frac{3}{7}$ (about 43%) of it.

All in all, Calvin gets the cannon, the doubloon, the sword, the cannon ball, the wooden leg, the flag, the crow's nest, and 43% of the unopened chest for a total of 61.4 of his points. Hobbes gets the rest.

3. Mike initially gets his way on the room party policy (50), the cleanliness issue (6), and lights-out time (10) for a total of 66 points. Phil initially gets his way on the stereo level issue (22), smoking rights (20), phone time (8), and the visitor policy (5) for a total of 55 points. Because Phil has fewer points that Mike, he gets his way on the alcohol use issue, on which they both placed 15 points, bringing his total to 70. To determine the order of transfer (from Phil to Mike), we must calculate the point ratios of the issues on which Phil got his way.

Point ratio of the stereo level issue is $\frac{22}{4}=5.5$.

Point ratio of the smoking rights issue is $\frac{20}{10}=2.0$.

Point ratio of the alcohol issue is $\frac{15}{15}=1.0$.

Point ratio of the phone time issue is $\frac{8}{1}=8.0$.

Point ratio of the visitor policy issue is $\frac{5}{4}=1.25$.

The first issue to be transferred is the alcohol issue, because it has the lowest point ratio. However, if all of it were given to Mike, his point total would rise to 66 + 15 = 81, and Phil's point total would fall to 70 – 15 = 55. This means that only a fraction of the alcohol issue will be transferred from Phil to Mike.

Let x be the fraction of the alcohol issues that Phil retains, and let $1-x$ be the fraction of it that is given to Mike. To equalize point totals, x must satisfy $55+15x=66+15(1-x)$.

Thus, using algebra to solve this equation yields the following.

$$\begin{aligned} 55+15x &= 66+15-15x \\ 55+15x &= 81-15x \\ 30x &= 26 \\ x &= \tfrac{26}{30} \\ x &= \tfrac{13}{15} \end{aligned}$$

Thus, Phil gets his way on the stereo level issue, the smoking rights issue, the phone time issue, the visitor policy issue, and $\frac{13}{15}$ (about 87%) of his way on the alcohol issue for a total of 68 points. Mike gets his way on the rest.

4. Labor initially gets its way on the benefits issue (35) and the issue of vacation time (15) for a total of 50 points, while management gets its way on the base salary issue (50) and salaries (40) for a total of 90 points. To determine the order of transfer, we must calculate the point ratios of the issues on which management got its way.

The point ratio of the base salary issue is $\frac{50}{30} = 1.67$.

The point ratio of the salary increase issues is $\frac{40}{20} = 2.0$.

The first issue to be transferred is the base salary issue, because it has the lowest point ratio. But if all of it were transferred, labor would then have more points than management. Let x be the fraction of the base salary issue that management retains. To equalize point totals, x must satisfy $40 + 50x = 50 + 30(1 - x)$.

Thus, using algebra to solve this equation yields the following.

$$\begin{aligned} 40 + 50x &= 50 + 30 - 30x \\ 40 + 50x &= 80 - 30x \\ 80x &= 40 \\ x &= \tfrac{40}{80} \\ x &= \tfrac{1}{2} \end{aligned}$$

Thus management gets its way on the salary increase issue and 50% of its way on the base salary issue for a total of 65 points. Labor gets its way on the rest.

5. – 6. Answers will vary.

7. Allocation 1:

(a) Not proportional: Bob gets 10% in his eyes.

(b) Not envy-free: Bob, for example, envies Carol.

(c) Not equitable: Bob thinks he got 10% and Carol thinks she got 40%.

(d) Example: Give Bob X, Carol Y, and Ted Z.

Allocation 2:

(a) Not proportional: Carol gets 30% in her eyes.

(b) Not envy-free: Carol, for example, envies Bob.

(c) Not equitable: Bob thinks he got 50% and Carol thinks she got 30%.

(d) Example: Give Bob Y, Carol X, and Ted Z.

Allocation 3:

(a) Not proportional: Carol and Ted get 0% in their eyes.

(b) Not envy-free: Carol and Ted envy Bob.

(c) Not equitable: Bob thinks he got 100% and Carol thinks she got 0%.

(d) It is Pareto optimal – for Carol or Ted to get anything, Bob will have to get less.

Allocation 4:

(a) Not proportional: Carol gets 30% in her eyes.

(b) Not envy-free: Carol, for example, envies Bob.

(c) Not equitable: Bob thinks he got 50% and Carol thinks she got 30%.

Allocation 5:

(a) It is proportional.

(b) Not envy-free: Bob, for example, envies Carol.

(c) It is equitable.

8. Mary gets the car and places $\frac{32,100}{2} = 16,050$ in a kitty. John takes out $\frac{28,225}{2} = 14,112.50$. The remaining 16,050 – 14,112.50 = 1,937.50 is split equally. The net effect of this is that Mary receives the car and pays John $15,081.25.

9. They handle the car first, as in Exercise #8. Then Mary gets the house and places $\frac{59,100}{2} = 29,550$ in a kitty. John takes out $\frac{55,900}{2} = 27,950$ and they split the remaining $29,550 - 27,950 = 1,600$ equally. Thus, for the house, Mary gets it and gives John $28,750. In total, Mary gets both the car and the house and pays John $\$15,081.25 + \$28,750 = \$43,831.25$.

10. Answers will vary.

11. First, *C* gets the house and places two-thirds of 165,000 (i.e., 110,000) in a kitty. *A* then withdraws one-third of 145,000 (i.e., 48,333) and *B* withdraws one-third of 149,999 (i.e., 50,000). They divide the remaining 11,667 equally among the three of them.

Second, *A* gets the farm and places two-thirds of 135,000 (i.e., 90,000) in a kitty. *B* then withdraws one-third of 130,001 (i.e., 43,334) and *C* withdraws one-third of 128,000 (i.e., 42,667). They divide the remaining 3,999 equally among the three of them.

Third, *C* gets the sculpture and deposits two-thirds of 127,000 (i.e., 84,667) in a kitty. *A* then withdraws one-third of 110,000 (i.e., 36,667) and *B* withdraws one-third of 80,000 (i.e., 26,667). They divide the remaining 21,333 equally among them.

Thus, *A* gets the farm and receives $52,222 + $43,778 and pays $44,667 + $44,000, so *A*, in total, receives the farm plus $7,333. Similarly, *B* receives $132,334 and *C* receives both the house and the sculpture, while paying $139,667.

12. First, *E* receives the Duesenberg and deposits $12,000, and then *F* and G each withdraw $5,000. They divide the remaining $2,000 equally among the three of them.

Second, *F* receives the Bentley and deposits $16,000, and then *E* withdraws $6,000 and *G* withdraws $6,667. They divide the remaining $3,333 equally among the three of them.

Third, *G* receives the Ferrari and deposits $11,000, and then *E* withdraws $5,333 and *F* withdraws $4,000. They divide the remaining $1,667 equally among the three of them.

Fourth, *F* receives the Pierce-Arrow and deposits $10,000, and then *E* withdraws $4,667 and *G* withdraws $4,500. They divide the remaining $833 equally among the three of them.

Fifth, *E* receives the Cord and deposits $16,000, and then *F* withdraws $6,000 and *G* withdraws $7,333. They divide the remaining $2,667 equally among the three of them.

For the final resolution, *E* receives the Duesenberg and Cord and pays $8,500, *F* receives the Bentley and Pierce-Arrow and pays $7,500, and *G* receives the Ferrari plus $16,000.

13. The bottom-up strategy fills in the blanks as follows:

Bob:	investments		car		CD player	
Carol:		boat		television		washer-dryer

Thus, Bob first chooses the investments, and the final allocation has him also receiving the car and the CD player.

14. The bottom-up strategy fills in the blanks as follows:

Carol:	investments		boat		washer-dryer	
Bob:		car		television		CD player

Thus, Carol first chooses the investments, and the final allocation has her also receiving the boat and the washer-dryer.

15. The bottom-up strategy fills in the blanks as follows:

Mark:	tractor		truck		tools	
Fred:		boat		car		motorcycle

Thus, Mark first chooses the tractor, and the final allocation has him also receiving the truck and the tools.

16. The bottom-up strategy fills in the blanks as follows:

Fred:	boat		car		motorcycle	
Mark:		tractor		truck		tools

Thus, Fred first chooses the boat, and the final allocation has him also receiving the car and the motorcycle.

17. The bottom-up strategy fills in the blanks as follows (CT stands for Connecticut):

Donald:	mansion		triplex		cash and jewelry
Ivana:		CT estate		apartment	

Thus, Donald first chooses the Palm Beach mansion, and the final allocation has him also receiving the Trump Tower triplex and the cash and jewelry.

18. The bottom-up strategy fills in the blanks as follows (CT stands for Connecticut):

Ivana:	CT estate		apartment		cash and jewelry
Donald:		mansion		triplex	

Thus, Ivana first chooses the Connecticut estate, and the final allocation has her also receiving the Trump Plaza apartment and the cash and jewelry.

19. The chooser. As divider, I'd get exactly 50% (or risk getting less). As chooser, I have a guarantee of getting at least 50% and the possibility (depending on the division) of getting more than 50%.

20. One way is to have Bob divide the cake into four pieces and to let Carol choose any three. Another is to have Bob divide the cake into two pieces and then let Carol choose one. Then they can do divide-and-choose on the piece that Carol did not choose.

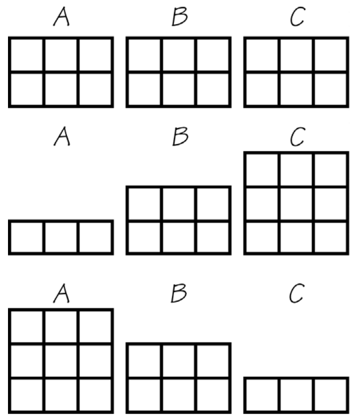
21. (a) Bob gets a piece whose value to him is 9 units (assuming that Bob is the divider), and Carol gets a piece whose value to her is 12 units.

(b) Carol gets a piece whose value to her is 9 units (assuming that Carol is the divider), and Ted gets a piece whose value to him is 15 units.

22. (a) Bob should be the divider. That way, he can get 12 units of value instead of 9 units of value.

(b) Here, Bob knows the preferences of the other party. In Exercise 19, we assumed that the divider didn't know the preferences of the other party.

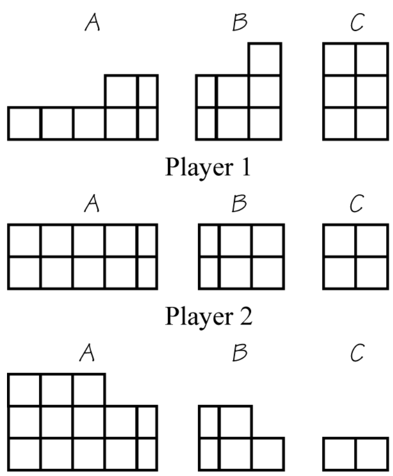
23. (a) See figures below.



(b) Player 2 finds *B* acceptable (6 square units) and *C* acceptable (9 square units). Player 3 finds *A* acceptable (9 square units) and *B* acceptable (6 square units).

(c) Player 3 chooses *A* (9 square units). Player 2 chooses *C* (9 square units). Player 1 chooses *B* (6 square units). Yes, there is another order. Player 2 chooses *C* (9 square units). Player 3 chooses *A* (9 square units). Player 1 chooses *B* (6 square units).

24. (a) See figures below.



(b) Player 2 finds *A* acceptable (9 square units), but not *B* (5 square units) or *C* (4 square units). Player 3 finds *A* acceptable (12 square units), but not *B* (4 square units) or *C* (2 square units).

(c) Players 2 and 3 both find *B* and *C* unacceptable. (*C* is on the right.)

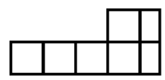
Continued on next page

24. (continued)

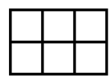
(d) (i) Assume C is given to Player 1. If Player 2 cuts the rest, he will make each piece 7 square units. Player 3 will choose the leftmost piece, which he thinks is 10 square units. Thus, Player 1 gets a piece he thinks is 6 square units. Player 2 gets a piece he thinks is 7 square units, and Player 3 gets a piece he thinks is 10 square units.

(ii) If Player 3 cuts the rest, she will make each piece 8 square units. (This requires a vertical cut two-thirds of the way across the third triple of squares.) Player 2 will choose the rightmost piece, which she thinks is $8\frac{2}{3}$ square units. Thus, Player 1 gets a piece she thinks is 6 square units, Player 2 gets a piece she thinks is $8\frac{2}{3}$ square units, and Player 3 gets a piece she thinks is 8 square units.

25. (a) See figure below.

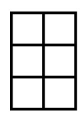


(b) Player 2 will further trim the piece:



in Player 2’s eyes.

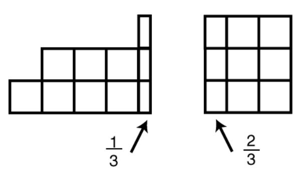
(c) Player 3 will further trim the piece:



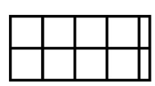
in Player 3’s eyes.

(d) Player 3 receives it, and thinks it is 6 units of value. The one leaving with the first piece always thinks it is one-nth of the value with n players.

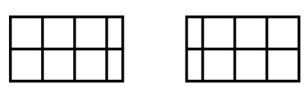
(e) Assume Player 1 is the divider. He sees it as 16 units of value, and he divides it as follows:



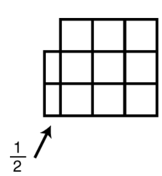
Player 2 chooses the piece on the left, which he sees as follows:



(f) Assume Player 2 is the divider. He sees it as 14 units of value, and he divides it as follows:



Player 1 chooses the piece on the right, which he sees as follows:



26. (a) Ted thinks he is getting at least one-third of the piece that Bob initially received and at least one-third of the piece that Carol initially received. Thus, Ted thinks he is getting at least one-third of part of the cake (Bob's piece) plus one-third of the rest of the cake (Carol's piece).

(b) Bob gets to keep exactly two-thirds (in his own view) of the piece that he initially received and thought was at least of size one-half. Two-thirds times one-half equals one-third.

(c) If, for example, Ted thinks the "half" Carol initially gets is worthless, then Ted may wind up thinking that he (Ted) has only slightly more than one-third of the cake, while Bob has (in Ted's view) almost two-thirds of the cake. In such a case, Ted will envy Bob.

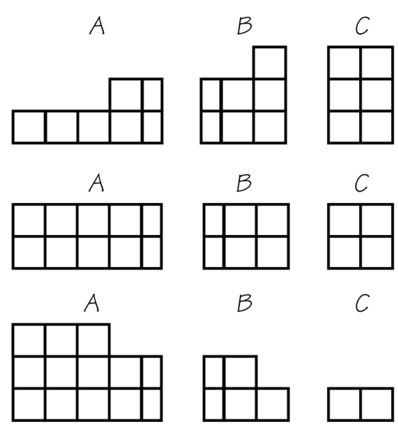
27. Bob, Carol, and Ted each divide the piece he or she has in four parts (equal in his or her own estimation). Alice then chooses one of Bob's four pieces, one of Carol's four pieces, and one of Ted's four pieces.

28. (a) If a player follows the suggested strategy, then clearly he or she will receive a piece of size exactly one-fourth *if* he or she does, in fact, call cut at some point. How could a player (Bob, for example) fail to call cut when using this strategy? Only if each of the other three players "preempted" Bob by calling cut before he did each time the knife was set in motion. But this means that each of the other three is left with a piece that Bob considered to be of size less than one-fourth. Hence, when the other three players have left with their shares, there is, in Bob's view, over one-fourth of the cake left for him.

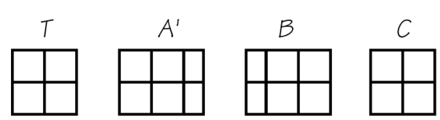
(b) If you call cut first − and thus exit the game with a piece of size exactly one-fourth in your estimation − you will envy the next player to receive a piece *if* no one calls cut until the next piece is larger than one-fourth in your estimation.

(c) If there are four players and the first player has exited with his or her piece, then you could wait to call cut until the knife reaches the point where one-half of the original cake is left. Alternatively, you could wait until the knife passed over one-third of what was left.

29. (a) See figures below.



(b) See figures below.



(c) Player 3 will choose A (which he thinks is of size 6 square units). Player 2 will choose B (which he thinks is of size 5 square units). Player 1 will receive C (which he thinks is of size 6 square units). The proviso does not come into Play (since Player 3 took the trimmed piece).

30. (a) The knife on the left would be at the point where the other knife started. (Thus, the portion between the knives would be the complement of the piece *A*.)

(b) If, for example, Carol thinks the portion between the knives at the beginning (i.e., piece *A*) is of size less than one-half, then she definitely will think the portion between the two knives at the end (i.e., the complement of piece *A*) is of size greater than one-half. Because this portion of cake between the two knives goes from being of size less than one-half in her estimation to being of size greater than one-half in her estimation, there must be a point where it is of size exactly one-half in her estimation. An analogous argument applies if Carol thinks that *A* is of size greater than one-half.

Word Search Solution