Chapter 12
Electing the President

Chapter Outline

Chapter Summary

The procedure for electing the American president is unique among world democracies. No other country incorporates an institution similar to the Electoral College, a feature that plays a large role in both determining the outcome of our elections, as well as in the strategies of the candidates. The selection of the nominees who compete in a grueling series of primary elections is also unique to the American political process.

To try to make mathematical sense of these features, we make certain assumptions. We assume that the candidates are completely aware of the positions of the voters along the spectrum of political opinion. We also assume that the outcome of an election hinges on a single issue (and not on grounds of personality, ethnicity, and so forth), with each citizen choosing to vote for the candidate whose opinion lies closest to his own. Under these simplifying assumptions, the problem for the candidates is thus reduced to deciding where to situate themselves on the political spectrum in order to maximize their vote totals. In the case of just two candidates, the *optimal position* for each is at the median. This choice is in *equilibrium,* in the sense that if either candidate deviates from this position, then the opponent can exploit this movement to his own benefit.

Multi-candidate elections are both more interesting and more difficult to analyze. Such situations occur regularly in primaries, as well as in general elections with serious third-party candidates, such as Ross Perot in 1992 or Ralph Nader in 2000. While unable to win, such candidates may play the role of spoilers who can affect the outcome of the election. In particular, questions arise as to whether it pays for a third (or fourth) candidate to enter the race and where to position himself if he does. The answers to these questions depend upon the political stances of those who have preceded him. For example, if the first two candidates are centrists who have positioned themselves at the median (the optimal choice for two candidates), then a third entry can benefit by moving slightly to the right or left of the median. Of course, the entry of the third candidate may cause the first two to jockey for more attractive positions. The *1/3-separation obstacle* and the *2/3-separation opportunity* help us understand the dynamics of new candidates entering the race. Moreover, anticipation of the entry of additional candidates may affect the original positions adopted by the earlier candidates.

These dynamics are also affected by those who drop out of the race, either because of poor poll performance or because they lost some of the early primaries. In some situations, we can determine which of the remaining candidates benefit from the reduced field.

Following the 2000 election, many ideas arose for reform of the presidential election process, including the possibility of eliminating the Electoral College or eliminating its winner-take-all feature. Although not aired in the general public, *approval voting* has been discussed in the mathematical community, and it has several positive aspects which would affect multi-candidate elections. It is likely that Al Gore would have won the 2000 election under approval voting, since many of Ralph Nader's supporters would have cast approval votes for Gore as well. A voter is said to have *dichotomous preferences* if he divides the set of candidates into two subsets – a preferred subset and a nonpreferred one – and is indifferent among all candidates in each of these subsets. A *dominant strategy* for a dichotomous voter is to vote for all candidates in his preferred subset and no others. Under approval voting, if all voters have dichotomous preferences and chose their dominant strategy, then a Condorcet winner will always be elected.

A major decision confronting the nominees of the two main parties is the allocation of resources. We assume that the votes in a state for each candidate will be proportional to the amounts spent there. Clearly, the candidates will concentrate their spending in the toss-up states, but within this category it turns out that the lion's share will go to the largest states. The *3/2's rule*, under which parties allocate their resources to the toss-up states according to the $\frac{3}{2}$ power of their electoral votes, maximizes the expected electoral vote of a candidate, provided that his opponent adheres to the same rule. This rule is a *local maximum,* in that it is invulnerable to small deviations by the opponent. But this rule is vulnerable to large deviations, such as when a candidate spends little or nothing in certain states (thereby conceding those states) and uses the money saved there in other battleground states. However, this strategy is dangerous, since if his opponent becomes aware of it at an early stage, he can exploit this knowledge to his benefit. Hence, allocation decisions put us in the realm of game theory.

A robust choice of allocations is the *proportional rule* in which resources are divided according to the sizes of the toss-up states. This rule maximizes the expected total popular vote for a candidate, regardless of whether his opponent adheres to it or not, so that it yields a *global maximum.* However, since the presidential election is determined by winning a majority of the electoral votes, rather than the popular votes, adopting this strategy may lead to a situation similar to 2000, in which the winner of the popular vote lost the election.

Skill Objectives

1. To determine the equilibrium positions of the candidates.

2. To determine the optimal positions of the candidates.

3. To determine the effect that a new candidate who enters the field has on previous candidates.

4. To determine which candidates benefit when another drops out of the race.

5. To understand the 1/3-separation obstacle and 2/3-separation opportunity.

6. To apply the proportional rule for allocation of resources.

7. To apply the 3/2's rule for the allocation of resources.

Teaching Tips

1. This chapter has a very contemporaneous feel to it, motivated partially by the presidential election of 2000. As we move away from the events of that year, it might be a good idea to refresh the students' memories by reviewing the details of that election.

2. Point out to the students that this chapter incorporates just about all of the topics found in this part of the book: various voting methods, Condorcet winner, approval voting, weighted voting, sincere and insincere voting, and apportionment (which decides the distribution of votes in the Electoral College).

Research Paper

Have students to research one of the following questions. The suggested length for the first writing project is 1 to 2 pages and for the second is 2 to 3 pages.

1. What evidence is there that the median-voter theorem applies in presidential elections with only two serious contenders? In cases where the candidates do not approach the median, is this because they view their main supporters as more extreme and do not want to alienate them?

2. It is generally agreed that presidential nomination races today are "front-loaded" – candidates must devote most of their resources to the early primaries even to stay in contention, much less win. Does this fact tend to help moderates or extremists as the field is narrowed? Would this be true under approval voting? Is it desirable that momentum plays such a large role in presidential primaries?

Spreadsheet Project

To do this project, go to http://www.whfreeman.com/fapp7e.

This activity provides an examination of the popular vote and the Electoral College.

Collaborative Learning

Primary Battles

The following is a major group project to be done mainly out of class. Choose several presidential election years in which there were interesting primary contests, with several candidates initially in the field. Among others, these include Democratic primaries in 1960, 1972, 1976, and 1988, and Republican ones in 1976, 1996, and 2000.

Divide the class into groups and assign to each group one of these interesting primary battles. By consulting contemporary sources, have the students research the development of the nominating races, culminating in the selection of one of the candidates. The students should focus on the positions of the candidates, the order in which they entered the race, the effect that new candidates had, when candidates dropped out, and so forth, all in the light of the concepts developed in this chapter. Then have each group report back to the class.

Solutions

Skills Check:

1. b	2. b	3. c	4. a	5. c	6. c	7. a	8. c	9. c	10. b
11. b	12. b	13. a	14. a	15. b	16. b	17. c	18. a	19. b	20. a

Exercises:

1. Assume a distribution is skewed to the left. The heavier concentration of voters on the right means that fewer voters are farther from the median. Because there are fewer voters "pulling" the mean rightward, it will be to the left of the median. Likewise, a distribution skewed to the right will have a mean to the right of the median.

2. Assume the candidates take a common position at 0.6. Then there are 12 voters to the left and 11 voters to the right of this position. If one of the candidates moves left to 0.5, then the opponent will win by 13 to 12 votes. If one of the candidates moves right to 0.7, then the opponent will win by 14 to 11 votes. Hence, neither candidate has an incentive to depart from position 0.6, making it an equilibrium position.

3. While there is no median position such that half the voters lie to the left and half to the right, there is still a position where the middle voter (if the number of voters is odd) or the two middle voters (if the number of voters is even) are located, starting either from the left or right. In the absence of a median, less than half the voters lie to the left and less than half to the right of this middle voter's (voters') position (positions).

Hence, any departure by a candidate from a position of a middle voter to the position of a non-middle voter on the left or right will result in that candidate's getting less than half the votes – and the opponent's getting more than half. Thus, the middle position (positions) is (are) in equilibrium, making it (them) the extended median.

4. No, because if a candidate moved to 0.5, he would get 8 votes and his opponent at 0.6 would get 11 votes, whereas if he stayed at 0.6 with his opponent, he would split the 18 votes, leading to a tie. Similarly, if one candidate moved to 0.7, she would get 8 votes, but her opponent at 0.6 would get 10 votes. Hence, 0.6 remains an equilibrium position if only the voter at 0.1 might not vote. On the other hand, if the three voters at 0.2 might not vote, then it would pay for one candidate to move to 0.7, getting 8 votes, because her opponent at 0.6 would get only 7 votes.

5. When the four voters on the left refuse to vote for a candidate at 0.6, his opponent can do better by moving to 0.7, which is worse for the dropouts.

6. The extended median would remain 0.6 if there were the two dropouts at 0, and similarly if there was also the dropout at 0.9. The extended median may change if there are more dropouts, however, when voters increasingly refuse to vote for candidates who are not close to them. In the extreme case, if voters do not vote for a candidate unless he or she takes exactly their position, then 0.8, where there are the most votes (6), would be the extended median.

7. The voters are spread from 0.1 to 0.9, so it is a position at 0.5 that minimizes the maximum distance (0.4) a candidate is from a voter. If the candidates are at the median of 0.6, the voter at 0.1 would be a distance of 0.5 from them. In this sense, the median is worse than the mean of 0.56, which would bring the candidates closer to the farthest-away voter and, arguably, be a better reflection of the views of the electorate.

8. If neither candidate has an incentive to depart from an equilibrium position, but in fact departs, he or she would want to return to that position. But this is to say that if one candidate is at that position, the other candidate cannot do better than also take that position.

9. The middle peak will be in equilibrium when it is the median or the extended median. Yes, it is possible that, say, the peak on the left is in equilibrium, as illustrated by the following discrete-distribution example, in which the median is 0.2:

Position i	1	2	3	4	5	6	7
Location (l_i) of position i	0.1	0.2	0.3	0.5	0.6	0.8	0.9
Number of voters (n_i) at position i	7	8	1	2	1	2	1

10. Assume *B*'s position is to the right of the median. Clearly, *A* maximizes his vote total by being just to *B*'s left and therefore closer to the median. Analogously, if *B*'s position is to the left of the median, *A* does best being just to *B*'s right. If *B* is at the median, then from the median-voter theorem, *A* can do no better than be at the median, too.

11. If the population is not uniformly distributed and, say, 80% live between $\frac{3}{8}$ and $\frac{5}{8}$ and only 10% live to the left of $\frac{3}{8}$ and 10% to right of $\frac{5}{8}$, then the bulk of the population will be well served by two stores at $\frac{1}{2}$. In fact, stores at $\frac{1}{4}$ and $\frac{3}{4}$ will be farther away for 80% of the population, so it can be argued that the two stores at $\frac{1}{2}$ provide a social optimum.

12. Probably most people would argue that members of the city council should represent people at different positions so, for example, a 20% minority would have one representative on the council. But if only one candidate is to be elected, he or she presumably, having to represent everybody, should be someone whose position is a centrist one.

13. Presumably, the cost of travel would have to be weighed against how much lower more competitive prices are.

14. Assume the first district comprises the seven voters at positions 1 – 3, the second district the seven voters at positions 4 – 6, and the third district the seven voters at positions 7 – 9:

Position i	1	2	3	4	5	6	7	8	9
Location (l_i) of position i	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Number of voters (n_i) at position i	2	3	1	3	1	2	2	3	1

Their extended medians are 0.2, 0.4 or 0.5, and 0.8, respectively, whereas the extended median for the major is 0.4 or 0.5. If the candidate in the district at positions 4 – 6 chooses the extended median 0.4, and the mayor the extended median 0.5, then none of the district extended medians match the mayor's median. Whether such differences are the basis of mayor-council disagreements would have to be investigated.

15. Since the districts are of equal size, the mayor's median or extended median must be between the leftmost and rightmost medians or extended medians; otherwise, at least $\frac{2}{3}$ of the voters would be on one side of the mayor's position, which would preclude it from being the median or extended median. This is not true of the mean, however, if, say, the left-district positions are much farther away from the mayor's median or extended median than the right-district positions. In such a case, the mayor's mean would be in the interval of the left-district positions.

16. *C* should take a position just to the left or right of *A/B*, on the side closer to the median. Thereby *C* will receive a majority of the votes, and *A* and *B* will split the remainder.

17. If, say, *A* takes a position at *M* and *B* takes a position to the right of *M*, *C* should take a position just to the left of *M* that is closer to *M* than *B*'s position, giving *C* essentially half the votes and enabling him or her to win the election. If neither *A* nor *B* takes a position at *M*, *C* should take a position next to the player closer to *M*; the position that *C* takes to maximize his or her vote may be either closer to *M* (if the candidates are far apart) or farther from *M* (if the players are closer together), but this position may not be winning. For example, assume the voters are uniformly distributed over [0,1]. If $\frac{3}{16}$ of the voters lie between *A* (to the left of *M*) and *M*, and $\frac{3}{16}$ of the voters lie between *M* and *B* (to the right of *M*), then *C* does best taking a position just to the left of *A* or just to the right of *B*, obtaining essentially $\frac{5}{16}$ of the vote. To be specific, assume *C* moves just to the left of *A*. Then *A* will obtain $\frac{3}{16}$ of the vote, but *B* will win with $\frac{1}{2}$ of the vote (that to the right of *M*), so *C*'s maximizing position will not always be sufficient to win.

18. If the distribution is not symmetric but the areas separating *A* and *B* from *M* are the same, then there is still a 1/3-separation obstacle. If the areas are different − say, *A* is closer to *M* than *B* is − then *C* does better moving just to the left of *A* rather than just to the right of *B*. Now if *A* is very close to *M*, then *C* will get almost half the vote, whereas *A* and *B* will split the remainder, enabling *C* to win.

19. Following the hint, *C* will obtain $\frac{1}{3}$ of the vote by taking a position at *M*, as will *A* and *B*, so there will be a three-way tie among the candidates. Because a non-unimodal distribution can be bimodal, with the two modes close to *M*, *C* can win if he or she picks up most of the vote near the two modes, enabling *C* to win with more than $\frac{1}{3}$ of vote.

20. Following the hint, *B* will win when *C* enters to the left of *A*, *A* will win when *C* enters to the right of *B*, and *A* or *B* will win or at least tie when *C* enters in between. For example, if *C* enters at $\frac{1}{2}$, *A* and *B* will tie with $\frac{3}{8}$ of the vote each to *C*'s $\frac{1}{4}$ of the vote.

21. *B* should enter just to the right of $\frac{3}{4}$, making it advantageous for *C* to enter just to the left of *A*, giving *C* essentially $\frac{1}{4}$ of the vote. With *C* and *A* almost splitting the vote to the left of *M* and a little beyond, *B* would win almost all the vote to the right of *M*. (If *C* entered at $\frac{1}{2}$, he or she would get slightly more than $\frac{1}{4}$ of the vote but lose to *A*, who would get $\frac{3}{8}$.)

22. Assume the distribution is triangular, so $y = 4x$ for $0 \le x \le \frac{1}{2}$ and $y = 4 - 4x$ for $\frac{1}{2} < x \le 1$. It is not difficult to show that $\frac{1}{2}$ the area lies between x = 0.354 and x = 0.646. If *C* takes a position at $\frac{1}{2}$, he or she will win the votes of all voters between x = .427 and x = .573 when *A* and *B* take positions at .354 and 0.646, respectively. In this case, *C* will obtain 27.1% of the vote, whereas *A* and *B* will tie with 36.5% each, so this distribution does not rise steeply enough in the middle to enable *C* to win. Clearly, a distribution that rises very sharply near $x = \frac{1}{2}$ is needed to give *C* more than 33.3% of the vote and, thereby, enough to beat *A* and *B*.

23. If the distribution is uniform, these positions are $\frac{1}{6}$, $\frac{5}{6}$, and $\frac{1}{2}$ for *A*, *B*, and *C*, respectively, making *D* indifferent between entering just to the left of *A*, just to the right of *B*, or in between *A* and *C* at $\frac{1}{3}$ or between *C* and *B* at $\frac{2}{3}$, which would give *D* $\frac{1}{6}$ of the vote in any case.

24. If all Buchanan supporters switched to Bush, that would give Bush a 49% to 48% lead over Gore in the popular vote, and so presumably the election. However, if only half of Buchanan supporters switched to Bush, then Bush would tie with Gore, 48% to 48%. But if Gore picks up a little of Buchanan's support, the tie would be broken in favor of Gore.

25. No, because Gore would get 49%, the same as Bush, so instead of winning Gore would tie with Bush.

26. There is no evidence that such deals were offered, much less made. The fear of discovery and serving jail time is certainly a major factor in deterring bribes or other shady deals. In addition, some politicians find such behavior inherently unethical and so steer clear of it for that reason.

27. It seems far too complicated a "solution" for avoiding effects caused by the Electoral College. Why not just abolish the Electoral College?

28. Because the votes for all other candidates, including the Condorcet winner, shift completely to the top two candidates identified by the poll, only one of these two candidates can win. If the poll identifies the top three candidates, and the Condorcet candidate is not among them, then the Condorcet winner will lose for the same reason − he or she will receive no votes, according to the Poll Assumption.

29. By definition, more voters prefer the Condorcet winner to any other candidate. Thus, if the poll identifies the Condorcet winner as one of the top two candidates, he or she will receive more votes when voters respond to the poll by voting for one or the other of these candidates. The possibility that the Condorcet winner might not be first in the poll, but win after the poll is announced, shows that the plurality winner may not be the Condorcet winner. Some argue that the Condorcet winner is always the "proper" winner, but others counter that a non-Condorcet winner who is, say, everybody's second-most-preferred candidate is a better social choice than a 51%-Condorcet winner who is ranked last by the other 49%.

30. *C* will defeat *A* by 7 votes to 5 votes. This is a desirable result if one believes that the Condorcet winner is always a desirable social choice. But because the 2 class IV voters rank *C* last, and *A* is not ranked last by any voters, *A* may be considered a more desirable "compromise" choice.

31. *D* is the Condorcet winner. It is strange in the sense that a poll that identifies either the top two or the top three candidates would not include *D*.

32. The poll will identify *A* and *B* as the top two candidates. Because the class III and class IV voters prefer *B* to *A*, *B* will defeat *A* by 6 votes to 4 votes.

33. *A* would win with 4 votes to 3 votes for *B* and 3 votes for *C*. It is strange that the number of top contenders identified by a poll can result in opposite outcomes (*A* in this exercise, whereas *B* defeats *A* when only two top contenders are identified by a poll, as in Exercise 32).

34. *A* defeats *B* by 6 votes to 3, *B* defeats *C* by 7 votes to 2, but *C* defeats *A* by 5 votes to 4, so there is Condorcet winner. *A* (4 votes) and *B* (3 votes) are distinguished by the poll; because the 2 class III voters prefer *A* to *B*, *A* will defeat *B* by 6 votes to 3 after the poll is announced. The fact that class II and III voters prefer *C* to *A* might suggest that the choice of *A* is unfair. On the other hand, *whichever* candidate is selected, there is a majority of voters who prefer another candidate because of the absence of a Condorcet winner. When majorities "cycle" in this manner, every outcome might be considered unfair—there is no candidate who is the indisputable majority choice.

35. Assume a voter votes for just a second choice. It is evident that voting for a first choice, too, can never result in a worse outcome and may sometimes result in a better outcome (if the voter's vote for a first choice causes that candidate to be elected).

36. Assume that the voting of all other voters creates a tie for first place between your two most-preferred candidates, with your least-preferred candidate "out of the running." Clearly, voting for your single most-preferred candidate is better than voting for your two most-preferred candidates, because your vote for your single most-preferred candidate would break the tie and elect him or her.

37. Following the hint, the voter's vote for a first and third choice would elect either *A* or *C*. If the voter also voted for *B*, then it is possible that if *A* and *B* are tied for first place, then *B* might be elected when the tie is broken, whereas voting for just *A* and *C* in this situation would elect *A*.

38. No. A vote for a worst choice may help in electing this candidate, which is never something one would want to do.

39. No. Voting for a first choice can never hurt this candidate and may help elect him or her.

40. Because a Condorcet winner is preferred by a majority to every other candidate, he or she must get more approval votes than any other candidate when voters choose their dominant strategies of voting for all candidates in their preferred subsets. Since the Condorcet candidate is in more preferred subsets, the approval-vote winner must be more preferred than any other candidate.

41. No. If class I and II voters vote for all candidates in their preferred subsets, they create a three-way tie among *A*, *B*, and *C*. To break this tie, it would be rational for the class III voter to vote for both *D* and *C* and so elect *C*, whom this voter prefers to both *A* and *B*. But now class I voters will be unhappy, because *C* is a worst choice. However, these voters cannot bring about a preferred outcome by voting for candidates different from *A* and B.

42. After the class III voter switches, the new poll will indicate *A* and *C* to be the top two candidates, with 3 approval votes each. But no voters will have any incentive to switch from voting for *A* or *C*, so this result will hold after all subsequent polls. In fact, neither of these candidates is preferred by a majority to the other, so both might be considered desirable social choices.

43. Without polling, *A* in case (i), *D* in case (ii), and *B* and *D* in case (iii); with polling, *B* in case (i), *D* in case (ii), and *D* in case (iii).

44. Approval voting seems more likely to find Condorcet winners without polling, but with polling it is difficult to say.

45. Exactly half the votes, or 9.5 votes each.

46. If the Democrat ignores the smallest state and makes proportional allocations to the two largest states, the allocations will be 43.75 units to the 7-voter state and 56.25 units to the 9-voter state. The Republican's optimal response will also be 0 units to the 3-voter state and proportional allocations to the other two states.

47. Substitute into the formula for r_i, in Exercise 46, $d_i = (n_i / N)D$ and $D = R$. The proportional rule is "strategy-proof" in the sense that if one player follows it, the other player can do no better than to follow it. Hence, knowing that an opponent is following the proportional rule does not help a player optimize against it by doing anything except also following it.

48. No, because the formula for r_i, given in Exercise 46, says that proportionality is still optimal when $R \neq D$. However, it is optimal to concentrate one's resources on the two smaller states, which have a majority, in order to maximize one's probability of winning a majority of votes in these two states. In particular, if the Democrat has only half the resources of the Republican, the Democrat can almost match the Republican's expenditures in these two states when the Republican allocates proportionally over all three states.

49. To win in states with more than half the votes, any two states will do. Thus, there is no state to which a candidate should not consider allocating resources. In the absence of information about what one's opponent is doing, all states that receive allocations should receive equal allocations since all states are equally valuable for winning.

50. In the absence of information about what one's opponent is doing, all states that receive allocations should receive equal allocations since all states are equally valuable for winning. But only two of the three states should be targeted for allocations; which two should be determined randomly. Then minimal-winning majorities of voters in each targeted state should be chosen randomly; the state allocations should be divided equally among all members of the minimal-winning majority selected.

51. The Democrat can win the election by winning in any two states or in all three. The first three expressions in the formula for PWE_D give the probabilities of winning in the three possible pairs of states, whereas the final expression gives the probability of winning in all three states.

52. For candidates who desire to win, PWE_D is better, but its maximization does not give a formula as simple as the 3/2's rule. To maximize PWE_D, it is best to choose states with a minimal-winning majority of electoral votes randomly and then allocate resources disproportionally to the larger states in this set, as in the case of the 3/2's rule. However, this strategy, if chosen by both candidates, yields only a local maximum.

53. Yes, but in a complicated way. Intuitively, the large states that are more pivotal, and whose citizens therefore have more voting power (as shown in Chapter 11), are more deserving of greater resources (as shown in this chapter).

Word Search Solution