Chapter 11
Weighted Voting Systems

Chapter Outline

Chapter Summary

Weighted voting systems are decision-making procedures in which the participants (voters) have varying numbers of votes. Examples of such systems are shareholder elections in which each shareholder of a company votes with a number of votes equal to the number of shares of stock he or she owns. In analyzing such systems, we need some measure of the power of each participant. We will interpret power as the ability to influence decisions. Two common measures of power, the *Banzhaf power index* and the *Shapley-Shubik power index*, are introduced in this chapter.

To begin an analysis of a weighted voting system, we need three pieces of information.

- how many voters there are
- how many votes each voter has
- how many votes it takes to approve an issue (the system quota)

The usual notation for systems provides a list of numbers, the first being the quota and the rest being the number of votes held by the individual voters. An obvious requirement is that the quota be greater than one-half the total number of votes in the system.

A system may have a *dictator*, an individual voter having at least as many votes as the quota. By any measure of power, such a voter has all the power in the system, rendering all other voters powerless. Voters with no power to influence decisions are called *dummy* voters. The existence of a dictator or a dummy is a serious defect in a system.

Voters often group together in an attempt to amass enough votes to meet or exceed the quota. Such groups of voters are called *coalitions*, which can contain any number of voters in the system (from all to none). A coalition is *winning* if its vote total is at least as large as the quota, and it is *blocking* if it has enough votes to defeat a measure by opposing it. (Blocking coalitions are not necessarily winning ones.) Finally, a winning coalition is *minimal* if a defection by any one of its members turns it into a losing coalition. A voter is called *critical* if his or her defection from a coalition causes it to change from winning to losing, or from blocking to nonblocking. Finding the critical voters is simplified by determining extra votes. A voter in a coalition is critical if his weight exceeds the difference between the total weight of the coalition and the quota. In a *minimal winning coalition*, every member of the coalition is critical. A voter's Banzhaf index is equal to the total number of winning and blocking coalitions for which that voter is critical.

A voter's Banzhaf index is equal to the total number of winning coalitions and blocking coalitions for which that voter is a swinger. The same relative power structure can be obtained by taking the index to be the number of losing coalitions that a voter can change to winning coalitions. In the latter case, the index of each voter will be exactly half that obtained in the former.

The Shapley-Shubik index is computed somewhat differently. Coalitions are built one voter at a time from permutations of the voter set. For a given permutation, the voter whose votes change the coalition from losing to winning is called the pivot for that coalition. The Shapley-Shubik index for a voter is the fraction obtained by dividing the number of permutations for which the voter is a pivot by the total number of permutations.

Each index has its uses but there are situations too complicated to be modeled by either scheme. In any event, the indices give us some insight into weighted voting systems. In particular, they show us that the relative numbers of votes held by the voters is not a reliable indicator of their actual power in the system.

Finally, one can compare voting systems by determining if they are *equivalent*. Two voting systems are equivalent is there is a way to exchange all voters from the first system with voters of the second while maintaining the same winning coalitions. A *minimal winning coalition* is one in which each voter is critical to the passage of a measure. Examining minimal winning coalitions allows one to completely describe a voting system.

Skill Objectives

1. Interpret the symbolic notation for a weighted voting system by identifying the quota, number of voters, and the number of votes each voter controls.

2. Identify if a dictator exists in a given weighted voting system.

3. Identify if a dummy exists in a given weighted voting system.

4. Identify if a single voter has veto power in a given weighted voting system.

5. Calculate the number of permutations of voters in a given weighted voting system.

6. List the possible permutations for a three- or four-voter weighted voting system.

7. Given a permutation of voters, identify the pivotal voter.

8. Calculate the Shapley-Shubik index for a three- or four-voter weighted voting system.

9. Identify winning coalitions by analyzing a given weighted voting system.

10. Identify blocking coalitions by analyzing a given weighted voting system.

11. When given a specific winning or blocking coalition from a weighted voting system, determine the critical voters.

12. Determine the extra votes for a winning coalition.

13. Calculate the Banzhaf power index for a given weighted voting system.

14. Determine a specific value of C_k^n by using the combination formula as well as Pascal's triangle.

15. Determine if two voting systems are equivalent and when given a voting system, find an equivalent system.

16. Explain the difference between a winning coalition and a minimal winning coalition.

Teaching Tips

1. When indicating winning and losing voter coalitions, students have a tendency to list the voter's number of votes, rather than the voter's name or other identifying factor. The text lists voters by name (letter) which may help alleviate this problem. Nonetheless, it may be helpful to set up a correspondence at the beginning of a problem between the voter's identification factor and his or her number of votes.
2. Because a yes/no vote constitutes a binomial structure, listing all possible voting combinations (coalitions) for a weighted voting system is structurally similar to constructing a tree diagram or creating a table of values for the experiment of tossing a given number of coins. Students may find it interesting to see the same structure used in different applications.
3. In determining the Banzhaf power index, it helps to consider pairs of complementary coalitions. The index of a voter is the number of complementary pairs in which that voter's vote is critical.
4. Most people think of a coalition as meaning two or more, yet a single voter (a one-member subset) and even no voter (the null set) are both considered coalitions. It may be helpful to mention that to students.
5. After completing the time-consuming and tedious task of listing all combinations for how voters can vote yes or no on a given issue and then analyzing each voter in each situation to determine whether or not he is critical, students often tally the number of times a player is critical. They then report this number in the voter's position in the Banzhaf power index. In so doing, they forget that they have counted both the number of times that the voter changed a winning coalition to a losing one and a losing coalition to a winning one. Thus, the value of the number is double what it should be.
6. Point out that in computing the Banzhaf index, there are cases in which a winning coalition has a single critical voter, other cases with several critical voters, and still others with none at all. On the other hand, each permutation in the computation of the Shapley-Shubik index has *exactly* one pivot.

Research Paper

Have students investigate the U.S Electoral College (Spotlight 11.1) in which the number of votes allotted each state is based upon its population. Students can give a summary of how the process works today, how the Florida vote had an impact on the 2000 Presidental election, and when and why this system was originally proposed. Students may also investigate other systems that were proposed and what made them undesirable. There are many interesting occurrences in the electing of a U.S. President. Other points of interest are:

- In 1800 the Democratic-Republican Electors gave Thomas Jefferson and Aaron Burr an equal number of electoral votes. The House of Representatives settled in Jefferson's favor. The 12th Amendment came about to effectively prevent this from happening again.
- In 1824, with four viable candidates in the Democratic-Republican Party, the provisions of the 12th Amendment allowed the choice of president to be determined by the House of Representatives who selected John Quincy Adams even though Andrew Jackson had more electoral votes.

If you wish to assign a research paper with older historical references, you may choose to ask students to investigate the constitution of the Roman Republic. After 509 B.C., a series of documents were gradually created that together constitute the Roman constitution of today. The Struggles of the Orders was a struggle between the patrician and plebeian classes. Although plebeians gained the right to vote, the patricians could form a block against other groups due to ordering of the votes. Also, as changes occurred, certain groups gained absolute veto power.

Spreadsheet Project

To do this project, go to http://www.whfreeman.com/fapp7e.

This activity examines weighted voting systems with a variety of voters and weights.

Collaborative Learning

Power among Partners

Part I
Discuss the following problem in groups. The first two or three companies should be readily found. The fourth one, however, will lead to a better understanding of the need for a formal definition of power.

Consider the following partnerships, in which each of the partners holds a certain amount of stock. In each case, how is power distributed among the partners? (Here, power means the ability to influence decisions.)

Company 1:	Partner	Stock	Power
	A	51%	
	B	49%	

Company 2:	Partner	Stock	Power
	A	49%	
	B	49%	
	C	2%	

Company 3:	Partner	Stock	Power
	A	35%	
	B	30%	
	C	25%	
	D	10%	

Company 4:	Partner	Stock	Power
	A	40%	
	B	20%	
	C	20%	
	D	20%	

Part II
Analyze the following situation in groups.

A committee consists of a chairman and 3 ordinary members. Each member, including the chairman, has one vote, but in case of a tie the chairman has the power to break the tie.

a. Calling the chairman *A*, and the ordinary members *B*, *C*, and *D*, list all of the winning coalitions.

b. Convert this situation into one involving weighted voting.

HINT: Assign 1 vote each to *B*, *C*, and *D*. Then find a value of the quota and a weight for *A* that will produce the winning coalitions you found in part a.

Solutions

Skills Check:

1. b	2. c	3. c	4. b	5. b	6. b	7. a	8. a	9. b	10. b
11. c	12. a	13. b	14. c	15. c	16. a	17. a	18. c	19. c	20. b

Cooperative Learning:

Part I: Company 1: A – 100%, B – 0%
Company 2: A, B, and C share power equally.
These two examples show that power is *not* proportional to the amount of stock owned.
Company 3: A, B, and C share power equally while D is a dummy.
Company 4: A, B, C, and D are all equal, but the amount of power depends upon a appropriate definition of power. Hence, this example is an excellent introduction to the need for such a definition.

Part II: a. Winning coalitions are $\{A,B\}$, $\{A,C\}$, $\{A,D\}$, $\{A,B,C\}$, $\{A,B,D\}$, $\{A,C,D\}$, $\{B,C,D\}$, and $\{A,B,C,D\}$.
b. {3: 2, 1, 1, 1}

Exercises:

1. (a) A winning or blocking coalition would be 50 senators plus the vice president, or more than 50 senators.
(b) The vice president will not be able to break a tie. A winning or blocking coalition requires 50 or more senators.
(c) A winning coalition require at least 67 senators. A coalition of 34 or more senators can block.

2. No. Suppose the sum of the voting weights of all of voters is n, and the quota is q. Then a coalition with more than $n-q$ votes is needed to block a measure. Because $q > \frac{1}{2}n$, $n-q < \frac{1}{2}n$, and hence $q > n-q$. It follows that a coalition with q votes can block a measure.

3. (a) No. A dictator needs 9 votes.
(b) The weight-5 and weight-4 voters have veto power, because the coalition of all the voters has only 3 extra votes, less than they have.
(c) The weight-3 voter is a dummy, because the only winning coalition he or she he belongs to is the coalition with all the voters, and it has 3 extra votes.

4. (a) The weight-30 voter. If any voter with less weight had veto power, he or she would, too.
(b) The sum of all the weights is 100. We know that the coalition of all the voters except the weight-30 voter is a losing coalition, so the quota is more than 70. However, the voter with weight 29 does not have veto power, so the coalition that consists of all voters except him or her is winning. That coalition has a total weight of 71. It follows that the quota is 71.
(c) No. To form a winning coalition, it takes the weight-30 voter and two others – it doesn't matter which two. The three voters whose weights are less than 30 have equal voting power.

5. No. If a voter X is pivotal in a permutation, then that voter is a critical voter in the winning coalition consisting of X and every voter that precedes X in the permutation. A dummy voter is not a critical voter in any winning coalition.

6. The last juror in the permutation is the pivotal voter.

7. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [51: 30, 25, 24, 21]$. No voter has veto power; this means that the last voter in a permutation can never be the pivot. No voter is a dictator; thus the first voter in a permutation isn't a pivot either.

(a) The weight-30 voter (Voter *A*) is pivotal in all permutations where he or she occupies position 2 because her weight, combined with any other voter's, is enough to win. *A* is also the pivot in all permutations where he or she occupies position 3, because the two voters ahead of him or her would have a combined weight of at most 49, less than the quota. There are 12 permutations to list:

Permutations
B A C D
B A D C
B C A D
B D A C
C A B D
C A D B
C B A D
C D A B
D A B C
D A C B
D B A C
D C A B

(b) Voters other than *A* will be pivotal if and only if they are second in the permutation and *A* is first, or they are third in the permutation and *A* last. Thus, *B* is pivotal in the following four permutations:

Permutations
A B C D
A B D C
C D B A
D C B A

(c) *A* is pivotal in 12 permutations, and *B*, *C*, and *D* are each pivotal in 4. There are $4! = 24$ permutations in all. The Shapley-Shubik power index of this weighted voting system is therefore $\left(\frac{12}{24}, \frac{4}{24}, \frac{4}{24}, \frac{4}{24}\right) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$.

8. Let's call the voters *A*, *B*, *C*, *D*, and *E*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D, w_E, w_F] = [7: 3, 2, 2, 2, 2, 2]$.

(a) Voter *A* is pivotal if he or she is in position 3, where he or she brings the weight to 7. Voter *A* is also pivotal if he or she is in position 4, where he or she brings the weight from 6 to 9.

(b) There are 5 weight-2 voters, and the weight-3 voter (Voter *A*) doesn't care about their order. There are $2 \times 5! = 2 \times 120 = 240$ permutations in which the weight-3 voter is pivotal.

(c) There are a total of $6! = 720$ permutations. There are $720 - 240 = 480$ permutations in which one of the 5 weight-2 voters is pivotal. That would imply each weight-2 voter is pivotal $480 \div 5 = 96$ times. Thus, the Shapley-Shubik index of the system is as follows.

$$\left(\frac{240}{720}, \frac{96}{720}, \frac{96}{720}, \frac{96}{720}, \frac{96}{720}, \frac{96}{720}\right) = \left(\frac{1}{3}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}\right)$$

9. None of these voting systems have dictators, nor does anyone have veto power. Therefore the pivotal position in each permutation is in position 2 or 3. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [q: 30, 25, 24, 21]$.

(a) $[q: w_A, w_B, w_C, w_D] = [52: 30, 25, 24, 21]$

A is pivot in four permutations where he or she is in position 2, and in all six permutations where she is in position 3: that's 10 in all.

Permutations
$B\,A\,C\,D$
$B\,A\,D\,C$
$C\,A\,B\,D$
$C\,A\,D\,B$

Permutations
$B\,C\,A\,D$
$B\,D\,A\,C$
$C\,B\,A\,D$
$C\,D\,A\,B$
$D\,B\,A\,C$
$D\,C\,A\,B$

B is pivot in two positions where he or she is in position 2 and four permutations where he or she is in position 3. Thus, Voter *B* is a pivot in 6 permutations.

Permutations
$A\,B\,C\,D$
$A\,B\,D\,C$

Permutations
$A\,D\,B\,C$
$D\,A\,B\,C$
$C\,D\,B\,A$
$D\,C\,B\,A$

C has the same power as *B*, and *D* is a pivot in the remaining two permutations.

Permutations
$B\,C\,D\,A$
$C\,B\,D\,A$

The Shapley-Shubik power index of this weighted voting system is therefore the following.

$$\left(\tfrac{10}{24}, \tfrac{6}{24}, \tfrac{6}{24}, \tfrac{2}{24}\right) = \left(\tfrac{5}{12}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{12}\right)$$

(b) $[q: w_A, w_B, w_C, w_D] = [55: 30, 25, 24, 21]$

Now *A* is pivotal in only two permutations where he or she is in position 2. Voter *A* is still pivotal in all permutations when in position 3. Thus, Voter *A* is a pivot in 8 permutations.

Permutations
$B\,A\,C\,D$
$B\,A\,D\,C$

Permutations
$B\,C\,A\,D$
$B\,D\,A\,C$
$C\,B\,A\,D$
$C\,D\,A\,B$
$D\,B\,A\,C$
$D\,C\,A\,B$

B now has the same voting power as *A*. *C* and *D* are also equally powerful. Each is pivot in four permutations in which he or she is in third position and not preceded by *A* and *B*. Thus, the Shapley-Shubik power index of this weighted voting system is the following.

$$\left(\tfrac{8}{24}, \tfrac{8}{24}, \tfrac{4}{24}, \tfrac{4}{24}\right) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}\right)$$

(c) $[q: w_A, w_B, w_C, w_D] = [58: 30, 25, 24, 21]$

Any three voters have enough votes to win, and no two can win. The voters have equal power and the Shapley-Shubik power index of this weighted voting system is therefore $\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)$.

10. Bush's margin in Nevada would be $\dfrac{414,939-1000}{393,372+1000}=1.0496$. This would move Nevada past Ohio in the permutations. Now Ohio's votes bring the total to 269 for Bush-Cheney, less than the quota. Nevada's 5 votes raises the total to 274, more than the quota. Therefore Nevada would be the pivot.

11. (a) We can represent a "yes" with 1, and a "no" with 0. Then the voting combinations are the 16 four-bit binary numbers: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.

(b) $\{\ \}$, $\{D\}$, $\{C\}$, $\{C,D\}$, $\{B\}$, $\{B,D\}$, $\{B,C\}$, $\{B,C,D\}$, $\{A\}$, $\{A,D\}$, $\{A,C\}$, $\{A,C,D\}$, $\{A,B\}$, $\{A,B,D\}$, $\{A,B,C\}$, and $\{A,B,C,D\}$.

(c) If the first bit of a given permutation is 1, then *A* votes "yes". If the second bit is 1, *B* votes "yes" in the corresponding coalition. The third bit tells us how *C* votes, and the fourth indicates the vote of *D*.

(d) i. 1
ii. 4
iii. 6

12. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [51: 30,\ 25,\ 24,\ 21]$.

The winning coalitions are those whose weights sum to 51 or more.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	49	0	0	0	0
{*A*, *B*, *C*}	79	28	1	0	0	0
{*A*, *B*, *D*}	76	25	1	0	0	0
{*A*, *C*, *D*}	75	24	1	0	0	0
{*B*, *C*, *D*}	70	19	0	1	1	1
{*A*, *B*}	55	4	1	1	0	0
{*A*, *C*}	54	3	1	0	1	0
{*A*, *D*}	51	0	1	0	0	1
			6	2	2	2

(a) Any voter that has a weight that exceeds the number of extra votes will be critical to that coalition. The critical voters in a coalition are indicated by a 1 in the table above.

(b) *A* has a critical vote in 6 coalitions; *B*, *C*, and *D* each have critical votes in 2. Doubling to account for blocking coalitions, the Banzhaf power index is $(12,4,4,4)$.

(c) If the combined weight of all voters is *n* voters, then a blocking coalition must have a weight at least $n-q+1$. For this weighted voting system, a blocking coalition must have a weight of at least $(30+25+24+21)-51+1=100-51+1=50$.

Blocking coalition	Weight	Dual winning coalition
{*A*, *C*, *D*}	75	{*A*, *B*}
{*A*, *D*}	51	{*A*, *B*, *C*}
{*A*, *C*}	54	{*A*, *B*, *D*}
{*A*, *B*, *D*}	76	{*A*, *C*}
{*A*, *B*}	55	{*A*, *C*, *D*}
{*A*, *B*, *C*}	79	{*A*, *D*}

13. Let's call the voters *A*, *B*, *C*, and *D*. This weighted voting system can be written as $[q: w_A, w_B, w_C, w_D] = [q: 30,\ 25,\ 24,\ 21]$.

(a) $[q: w_A, w_B, w_C, w_D] = [52: 30,\ 25,\ 24,\ 21]$

We'll copy the table of coalitions we made for Exercise 12, reducing the extra votes of each by 1. The coalition {*A*, *D*} becomes a losing coalition because its weight is only 51. It will be marked losing, and dropped when we increase the quota again.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	48	0	0	0	0
{*A*, *B*, *C*}	79	27	1	0	0	0
{*A*, *B*, *D*}	76	24	1	1	0	0
{*A*, *C*, *D*}	75	23	1	0	1	0
{*B*, *C*, *D*}	70	18	0	1	1	1
{*A*, *B*}	55	3	1	1	0	0
{*A*, *C*}	54	2	1	0	1	0
{*A*, *D*}	51	losing				
			5	3	3	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(10, 6, 6, 2)$.

(b) $[q: w_A, w_B, w_C, w_D] = [55: 30,\ 25,\ 24,\ 21]$

We copy the table from part (a), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	45	0	0	0	0
{*A*, *B*, *C*}	79	24	1	1	0	0
{*A*, *B*, *D*}	76	21	1	1	0	0
{*A*, *C*, *D*}	75	20	1	0	1	1
{*B*, *C*, *D*}	70	15	0	1	1	1
{*A*, *B*}	55	0	1	1	0	0
{*A*, *C*}	54	losing				
			4	4	2	2

Doubling to account for blocking coalitions, the Banzhaf power index is $(8, 8, 4, 4)$.

(c) $[q: w_A, w_B, w_C, w_D] = [58: 30,\ 25,\ 24,\ 21]$

We copy the table from part (b), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	42	0	0	0	0
{*A*, *B*, *C*}	79	21	1	1	1	0
{*A*, *B*, *D*}	76	18	1	1	0	1
{*A*, *C*, *D*}	75	17	1	0	1	1
{*B*, *C*, *D*}	70	12	0	1	1	1
{*A*, *B*}	55	losing				
			3	3	3	3

Doubling to account for blocking coalitions, the Banzhaf power index is $(6, 6, 6, 6)$.

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13. continued

(d) $[q: w_A, w_B, w_C, w_D] = [73: 30,\ 25,\ 24,\ 21]$

We copy the table from part (c), dropping the losing coalition and reducing quotas by 15. One more coalition will lose. *A* acquires veto power.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	27	1	0	0	0
{*A*, *B*, *C*}	79	6	1	1	1	0
{*A*, *B*, *D*}	76	3	1	1	0	1
{*A*, *C*, *D*}	75	2	1	0	1	1
{*B*, *C*, *D*}	70	losing				
			4	2	2	2

Doubling to account for blocking coalitions, the Banzhaf power index is $(8, 4, 4, 4)$.

(e) $[q: w_A, w_B, w_C, w_D] = [76: 30,\ 25,\ 24,\ 21]$

We copy the table from part (d), dropping the losing coalition and reducing quotas by 3. One more coalition will lose.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	24	1	1	0	0
{*A*, *B*, *C*}	79	3	1	1	1	0
{*A*, *B*, *D*}	76	0	1	1	0	1
{*A*, *C*, *D*}	75	losing				
			3	3	1	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(6, 6, 2, 2)$.

(f) $[q: w_A, w_B, w_C, w_D] = [79: 30,\ 25,\ 24,\ 21]$

We copy the table from part (e), dropping the losing coalition and reducing quotas by 3. One more coalition will lose. In this system, *D* is a dummy.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	21	1	1	1	0
{*A*, *B*, *C*}	79	0	1	1	1	0
{*A*, *B*, *D*}	76	losing				
			2	2	2	0

Doubling to account for blocking coalitions, the Banzhaf power index is $(4, 4, 4, 0)$.

(g) $[q: w_A, w_B, w_C, w_D] = [82: 30,\ 25,\ 24,\ 21]$

Only one winning coalition is left, with 18 extra votes. This is less than the weight of each participant. All voters are critical. In this system, a unanimous vote is required to pass a motion.

Winning coalition	Weight	Extra votes	Critical votes *A*	*B*	*C*	*D*
{*A*, *B*, *C*, *D*}	100	18	1	1	1	1
			1	1	1	1

Doubling to account for blocking coalitions, the Banzhaf power index is $(2, 2, 2, 2)$.

14. In each system, the voters will be denoted A, B, ….

(a) $[q: w_A, w_B] = [51: 52,\ 48]$

The winning coalitions are {A} with 1 extra vote and {A, B} with 49 extra votes. A is a dictator, the Banzhaf power index is $(4, 0)$.

(b) $[q: w_A, w_B, w_C] = [3: 2,\ 2, 1]$

The winning coalitions are {A, B, C}, with 2 extra votes, {A, B} with 1 extra vote, and {A, C} and {B, C} both with 0 extra votes. No one is critical in the coalition {A, B, C, D} but all voters are critical in the other three, so each voter casts 2 critical votes in winning coalitions, and another 2 critical votes in blocking coalitions. The Banzhaf power index is $(4, 4, 4)$.

(c) $[q: w_A, w_B, w_C] = [8: 5,\ 4, 3]$

In this system, A has veto power. The winning coalitions are {A, B, C} with 4 extra votes, {A, B} with 1 extra vote, and {A, C} with 0 extra votes. Thus A casts 3 critical votes in winning coalitions, while B and C cast only 1. The Banzhaf power index is $(6, 2, 2)$.

(d) $[q: w_A, w_B, w_C, w_D] = [51: 45,\ 43,\ 8,\ 4]$

Let's notice that any two of A, B, C have at least 51 votes, but D cannot join any other voter to make the quota. Therefore, D is a dummy. The winning coalitions are {A, B}, {A, C}, {B, C}, {A, B, D}, {A, C, D}, and {B, C, D} in which all except D are critical voters; and the following coalitions that contain no critical voters at all: {A, B, C} and {A, B, C, D}. Thus, A, B, and C each have 4 critical votes in winning coalitions, which we double to obtain the Banzhaf power index: $(8, 8, 8, 0)$.

(e) $[q: w_A, w_B, w_C, w_D] = [51: 45,\ 43,\ 6,\ 6]$

The winning coalitions, with extra vote counts, are as follows.

Winning coalition	Weight	Extra votes	Critical votes			
			A	B	C	D
{A, B, C, D}	100	49	0	0	0	0
{A, B, C}	94	43	1	0	0	0
{A, B, D}	94	43	1	0	0	0
{A, B}	88	37	1	1	0	0
{A, C, D}	57	6	1	0	0	0
{B, C, D}	55	4	0	1	1	1
{A, C}	51	0	1	0	1	0
{A, D}	51	0	1	0	0	1
			6	2	2	2

A casts a critical vote in 6 winning coalitions, while B, C, and D each cast one in 2. Doubling to count the critical votes in blocking coalitions, we find that the Banzhaf power index is $(12, 4, 4, 4)$.

15. Generating powers of 2 is often helpful in such conversions.

n	0	1	2	3	4	5	6	7
2^n	1	2	4	8	16	32	64	128

n	8	9	10	11	12	13	14
2^n	256	512	1024	2048	4096	8192	16,384

(a) Since 2^9 represents the largest power of 2 that doesn't exceed 585, we start there.

$$585-512=585-2^9=73$$
$$73-64=73-2^6=9$$
$$9-8=9-2^3=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are b_9, b_6, b_3 and b_0. The binary expression is 1001001001.

(b) Since 2^{10} represents the largest power of 2 that doesn't exceed 1365, we start there.

$$1365-1024=1365-2^{10}=341$$
$$341-256=341-2^8=85$$
$$85-64=85-2^6=21$$
$$21-16=16-2^4=5$$
$$5-4=5-2^2=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are $b_{10}, b_8, b_6, b_4, b_2,$ and b_0. The binary form is 10101010101.

(c) Since 2^{10} represents the largest power of 2 that doesn't exceed 2005, we start there.

$$2005-1024=2005-2^{10}=981$$
$$981-512=981-2^9=469$$
$$469-256=469-2^8=213$$
$$213-128=213-2^7=85$$
$$85-64=85-2^6=21$$
$$21-16=16-2^4=5$$
$$5-4=5-2^2=1$$
$$1-1=1-2^0=0$$

Thus, the nonzero bits are $b_{10}, b_9, b_8, b_7, b_6, b_4, b_2,$ and b_0. The binary form is 11111010101.

16. (a) $C_3^7=\frac{7!}{3!(7-3)!}=\frac{7!}{3!4!}=\frac{7\times 6\times 5}{3\times 2\times 1}=7\times 5=35.$

(b) We can't use the formula that applied in part (a) because we'd get $\frac{50!}{100!(-50)!}$ and factorials of negative numbers are not defined. But really, the definition is all we need. If there are 50 voters, how many coalitions are there with 100 "yes" votes? NONE. The answer is $C_{100}^{50}=0.$

(c) $C_2^{15}=\frac{15!}{2!(15-2)!}=\frac{15!}{2!13!}=\frac{15\times 14}{2\times 1}=15\times 7=105.$

(d) By the duality formula, $C_{13}^{15}=C_2^{15}=105.$ by the result of part (c).

17. (a) $C_3^6 = \dfrac{6!}{3!(6-3)!} = \dfrac{6!}{3!3!} = \dfrac{6\times5\times4}{3\times2\times1} = 5\times4 = 20.$

(b) $C_2^{100} = \dfrac{100!}{2!(100-2)!} = \dfrac{100!}{2!98!} = \dfrac{100\times99}{2\times1} = 50\times99 = 4950.$

(c) By the duality formula, $C_{98}^{100} = C_2^{100} = 4950.$ by the result of part (b).

(d) $C_5^9 = \dfrac{9!}{5!(9-5)!} = \dfrac{9!}{5!4!} = \dfrac{9\times8\times7\times6}{4\times3\times2\times1} = 9\times2\times7 = 126.$

18. (a) The higher quota gives the smaller communities voting power. They would argue that since they pay taxes, they deserve some influence in county affairs. The larger communities would (and did) argue that the higher quotas (supermajorities) prevent representatives of a majority of the county's voters from getting their way.

(b) In 1958, *B*, *G*, and *L* were dummies. In 1964, *N*, *G* and *L* were dummies. There were no dummies in later years.

(c) In 1958 and 1964, Hempstead would dictate; all of the other supervisors would be dummies. In 1970 and 1976, the Hempstead supervisors need at least one other voter. Thus, there would still be no dummies. In 1982, *G* would be a dummy.

(d) In 1958 and 1964, Hempstead would dictate; all of the other supervisors would be dummies.

The Hempstead supervisors (think as one voter with weight 18 or 62, depending on the year) are critical voters in $C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4$ coalitions. Performing this calculation (or using Pascal's triangle), we have the following.

$$C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4 = 1+4+6+4+1 = 16$$

Thus, the Hempstead supervisors are critical voter in 16 coalitions; doubling this, we find that their Banzhaf power index is 32.

Year	Banzhaf index
1958	$(32,0,0,0,0)$
1964	$(32,0,0,0,0)$

In 1970 and 1976, the Hempstead supervisors need at least one other voter. Thus, there would still be no dummies. The Hempstead supervisors (think as one voter with weight 62 or 70, depending on the year) are critical voters in $C_1^4 + C_2^4 + C_3^4 + C_4^4$ coalitions. Performing this calculation (or using Pascal's triangle), we have the following.

$$C_1^4 + C_2^4 + C_3^4 + C_4^4 = 4+6+4+1 = 15$$

Thus, the Hempstead supervisors are critical voter in 15 coalitions; doubling this, we find that their Banzhaf power index is 30. Each of the remaining voters are only critical once. Thus, each of the other voters have a Banzhaf power index of 2.

Year	Banzhaf index
1970	$(30,2,2,2,2)$
1976	$(30,2,2,2,2)$

Continued on next page

18. (d) continued

In 1982, the Hempstead supervisors need at least 1 of the three voters (*N*, *B*, or *L*) to have a motion to pass. *G* would be a dummy. Since *G* can be there or not, the Hempstead supervisors (think as one voter with weight 58) are critical voters in twice $C_1^3 + C_2^3 + C_3^3$ coalitions. Performing this calculation (or using Pascal's triangle), we have the following.

$$C_1^3 + C_2^3 + C_3^3 = 3 + 3 + 1 = 7$$

Thus, the Hempstead supervisors are critical voters in 14 coalitions; doubling this, we find that their Banzhaf power index is 28. Since *G* can be there or not, *N*, *B*, and *L* are each critical in $2 \times 1 = 2$ coalitions. Thus, *N*, *B*, and *L* each have a Banzhaf power index of 4.

Year	Banzhaf index
1982	$(28, 4, 4, 0, 4)$

(e) $[q : w_H, w_N, w_B, w_G, w_L] = [72 : 58, 15, 22, 6, 7]$

The *G* and *L* supervisors are dummy voters. *N* and *B* have the same power. The Hempstead supervisors need at least 1 of the two voters, *N* or *B* to have a motion to pass. The Hempstead supervisors would be critical in $\left(C_1^2 + C_2^2\right)\left(C_0^2 + C_1^2 + C_2^2\right)$ coalitions.

$C_1^2 + C_2^2$ for either having one or both of *B* or *G* in a winning coalition

$C_0^2 + C_1^2 + C_2^2$ for either having none, one, or both of *G* and *L* in a winning coalition

Since $\left(C_1^2 + C_2^2\right)\left(C_0^2 + C_1^2 + C_2^2\right) = (2+1)(1+2+1) = (3)(4) = 12$ the Hempstead supervisors are critical voters in 12 coalitions; doubling this, we find that their Banzhaf power index is 24.

N and *B* are critical in winning coalitions that contain just one of them, the Hempstead supervisors, and either none, one or both of *G* and *L*. This occurs in $C_0^2 + C_1^2 + C_2^2 = 1 + 2 + 1 = 4$ ways. Thus, *N* and *B* each have a Banzhaf power index of 8. Thus, we have a Banzhaf power index of $(24, 8, 8, 0, 0)$.

(f) Let's convert Table 11.6 into a table of percentages.

Supervisor from	Population	Percent of votes	Banzhaf power index	
Quota			65	72
Hempstead (Presiding)	56%	28%	29%	28%
Hempstead		26%	25%	24%
North Hempstead	16%	14%	17%	20%
Oyster Bay	23%	20%	21%	20%
Glen Cove	2%	6%	2%	2%
Long Beach	3%	6%	6%	7%

It is apparent that the percentages of power closely match the population. That is the argument that was used to justify this voting system. However, Hempstead has a majority of the population. If every citizen of Hempstead were for a measure, and the rest of the county were opposed, the measure would not pass. Although the Hempstead supervisors have a majority of the votes, they have less than the quota.

19. (a) {A, C, D} and {A, B}

(b) A belongs to each winning coalition, so if A opposes a motion it will not pass. There are no other minimal blocking coalitions that include A, but we may notice that every winning coalition contains either B or C and D. Thus, if B can combine forces with either C or D to defeat a motion, {B, C} and {B, D} are also minimal blocking coalitions.

(c) A has veto power and thus is a critical voter in all 5 of the winning coalitions. B is critical in 3 winning coalitions: {A, B, C}, {A, B, D}, and {A, B}. Finally, C and D are only critical in one coalition: {A, C, D}. The Banzhaf power index is $(10,6,2,2)$.

(d) $[q : w_A, w_B, w_C, w_D] = [5:3,2,1,1]$ is one set of weights that works, but there are many other solutions. One can reason that A, the only voter with veto power, must have the most votes, while B is more powerful than C or D (who are equally powerful).

(e) A will pivot in any permutation in which he or she comes after B or after C and D. He or she automatically pivots in the 6 permutations where he or she is in position 4, and also the 6 permutations where he or she is in position 3 because if B is not last in such a permutation, then he or she comes before A, and if Voter A is last, then C and D come before A. There are two permutations, *BACD* and *BADC* where A pivots in position 2. This adds up to $2+6+6=14$ pivots for A. D pivots in permutations where A and C appear before him or her, and B is last. There are 2 such permutations: *ACDB* and *CADB*. C has the same number of pivots as D. We have accounted for $14+2+2=18$ permutations. The remaining 6 belong to B. The Shapley-Shubik index is $\left(\frac{14}{24},\frac{6}{24},\frac{2}{24},\frac{2}{24}\right)=\left(\frac{7}{12},\frac{1}{4},\frac{1}{12},\frac{1}{12}\right)$.

20. Let's call the chairperson A, and the other members B_1, B_2, B_3, and B_4. The minimal winning coalitions have the form $\{A, B_i\}$, for $i=1,2,3,4$, and $\{B_1, B_2, B_3, B_4\}$ is a winning coalition because the chairperson can't block a motion unless another member joins him or her. With the proposed weights, each of these coalitions has weight 4, and it follows that all winning coalitions have weight at least 4. Losing coalitions (the chair alone, or 3 or fewer of the other members) would have weight less than 4. Thus the voting system is equivalent to the one with the proposed weights.

21. Let's call the chairperson A, and the other members B_1, B_2, B_3, and B_4. A is a critical voter in all winning coalitions that include at least one other voter but not all of them. There are $2^4-2=16-2=14$ such coalitions. Each of the B_i is a critical voter in two coalitions: $\{A, B_i\}$ for $i=1,2,3,4$, and $\{B_1, B_2, B_3, B_4\}$. The Banzhaf power index is $(28,4,4,4,4)$.

22. (a) Give the dean 2 votes, and each faculty member 1 vote. The quota would have to be 4. We would have $\left[q : w_D, w_{F_1}, w_{F_2}, w_{F_3}\right]=[4:2,1,1,1]$.

(b) Give the dean and the provost each 2 votes (they have equal power on the committee), and each faculty member 1 vote. The quota is 6. We would have the following.

$$\left[q : w_D, w_P, w_{F_1}, w_{F_2}, w_{F_3}\right]=[6:2,2,1,1,1]$$

23. Let's look at the minimal winning coalitions. We call the faculty members F_1, F_2, F_3, and F_4. We call the administrators A_1, A_2, and A_3. The following are two winning coalitions: $\{F_1, F_2, F_3, F_4, A_1, A_2\}$, in which the administrators are critical, but the faculty members aren't; and $\{F_1, F_2, F_3, A_1, A_2, A_3\}$ in which the faculty members are critical and the administrators are not. In any weighted voting system, the critical voters in a coalition must have more weight than those who are not critical. The first coalition that we cited indicates that the administrators should have more weight, while the second indicates that the faculty members have more weight. These contradictory requirements cannot be satisfied, so the system is not equivalent to a weighted voting system.

24. An administrator casts a critical vote in any coalition that includes one other administrator and three or four faculty members. There are 2 ways to choose the other administrator, and 5 ways to choose the group of faculty members: 10 winning coalitions in which the administrator is critical.
A faculty member is critical in any coalition that includes exactly 2 other faculty members, and 2 or 3 administrators. There are 3 ways to choose the other faculty members, and 4 ways to assemble 2 or 3 administrators: 12 winning coalitions in which the faculty member is critical. The Banzhaf power index is $(24, 24, 24, 24, 20, 20, 20)$. The administrators probably think they are more powerful, but actually they aren't.

25. There are $7! = 5040$ permutations, so let's not make a list. Consider F_4. He or she will be critical in a permutation when he or she is fifth, followed by another faculty member and an administrator (in either order), or sixth, followed by a faculty member. If F_4 is fifth, there are 3 ways to choose the faculty member, 3 ways to choose the administrator, and 2 ways to put those two in order, The remaining 4 participants, who come before F_4 in the permutation, can be ordered 4! ways. Thus, there are $3 \times 3 \times 2 \times 4! = 432$ permutations where F_4 is a pivot in position 5. If F_4 is in position 6 and another faculty member in position 7, there are 5! ways to order the voters coming before the two faculty members, and 3 ways to choose the last voter in this type of permutation: $5! \times 3 = 360$ permutations in all. The number of permutations in which F_4 is a pivot is thus $432 + 360 = 792$. The other three faculty members are each pivot in 792 permutations, so the faculty members are pivot in a total of $4 \times 792 = 3168$ permutations. That leaves $5040 - 3168 = 1872$ permutations for the administrators, 624 each. The Shapley-Shubik index of this voting system is (in lowest terms)

$$\left(\frac{11}{70}, \frac{11}{70}, \frac{11}{70}, \frac{11}{70}, \frac{13}{105}, \frac{13}{105}, \frac{13}{105}\right)$$

By this measure, each faculty member is more powerful than any administrator. A faculty member has about 15.7% of the power, and an administrator has about 12.4%.

26. A voter has veto power if and only if he or she belongs to every winning coalition. Because each winning coalition contains at least one minimal winning coalition (you can obtain a minimal winning coalition by removing non-critical voters, one at a time, until there are no more), a voter who belongs to all minimal winning coalitions has veto power. If there is only one minimal winning coalition, then every voter in that coalition has veto power, and every voter who does not belong is a dummy. If there are only two minimal winning coalitions, since they overlap, at least one voter belongs to both and thus has veto power.

27. All four-voter systems can be presented as weighted voting systems.

Minimal winning coalitions	Weighted voting systems
$\{A, B, C, D\}$	[4:1,1,1,1]
$\{A, B\}$, $\{A, C, D\}$	[5:3,2,1,1]
$\{A, B, C\}$, $\{A, B, D\}$	[5:2,2,1,1]
$\{A, B\}$, $\{A, C\}$, $\{A, D\}$	[4:3,1,1,1]
$\{A, B\}$, $\{A, C\}$, $\{B, C, D\}$	[5:3,2,2,1]
$\{A, B\}$, $\{A, C, D\}$, $\{B, C, D\}$	[4:2,2,1,1]
$\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$	[4:2,1,1,1]
$\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C, D\}$	[4:3,2,1,1]
$\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, $\{B, C, D\}$	[3:1,1,1,1]

28. (a) Let's call the voters A, B, C, and D. This weighted voting system can be written as $[q : w_A, w_B, w_C, w_D] = [51 : 48, 23, 22, 7]$.

Winning coalition	Weight	Extra votes	Losing coalition	Weight	Votes needed
{A, B, C, D}	100	49	{A}	48	3
{A, B, C}	93	42	{B, C}	45	6
{A, B, D}	78	27	{B, D}	30	21
{A, C, D}	77	26	{C, D}	29	22
{A, B}	71	20	{B}	23	28
{A, C}	70	19	{C}	22	29
{A, D}	55	4	{D}	7	44
{B, C, D}	52	1	{ }	0	51

(b) A cannot sell more than 4 shares to B, because that is all the extra votes of {A, D}. The right column of the table in part (a) indicates that all of the losing coalitions involving B need more than 4 votes, so no additional winning coalitions would be created.

(c) When selling to D, the extra votes of {A, D} are unaffected. A can sell 19 shares to D. without changing any winning coalitions. The strongest losing coalition involving D needs 21 votes, and its status would not be affected by the sale.

(d) A is again limited to the extra votes of {A, D}, 4 shares. Before starting, 8 new winning coalitions should be created, by combining E and each of the winning coalitions, and 8 losing coalitions would be created, combining E with the previous losing coalitions. No vote counts would change, because E has 0 shares at the outset. The sale would not affect the losing coalition E; it would still need 3 shares to win. The first losing coalition that would gain shares is {B, C, E}, which would still need 2 more shares to win.

(e) D can either sell 4 shares to B or C (here the limitation is to the coalition {A, D} because {B, C, D} would not be affected); or D can sell 1 share to A or E: now the limitation is the extra votes of {B, C, D}.

(f) D can sell 2 shares to A: if she sells more than that, {A} would become a winning coalition and the rest of the shareholders (D included) would be dummies. She can sell 5 shares to B or C, more than that would cause {A, D} to lose and {B, C} to win, leaving no critical votes for D. She could sell 5 shares to E: that would cause {B, C, D} and {A, D} to lose, but it would create new winning coalitions, including {B, C, D, E}, which would have 1 extra vote, and D, who still has 2 shares, would be a critical voter.

(g) The limiting coalitions are {A, B}, with 20 extra votes, and {C, D}, which is 22 shares short of winning. B can sell 20 shares to C.

29. The minimal winning coalitions are {A, B} {A, C}, {A, D}, and {B, C, D}. Thus, A is more powerful than the others, and B, C, and D have equal power, even though their voting weights are different. Let's go through the list and see which gives the same minimal winning coalitions.

(a) Each minimal winning coalition has 3 voters. Eliminated.

(b) The minimal winning coalitions are A combined with another voter, or {B, C, D}. This matches our system.

(c) The minimal winning coalitions are A combined with 2 other voters. Eliminated.

(d) {A, C} and {A, D} are not winning coalitions. Eliminated.

The answer: (b).

30. To determine the Banzhaf index, refer to the table in answer 28(a). *A*, with 48 votes, is critical in 6 winning coalitions; *B*, *C*, and *D* are each critical in only 2 winning coalitions: when joined by *A* and no one else, or in a coalition without *A*. Doubling to include blocking coalitions, the Banzhaf power index is $(12, 4, 4, 4)$. To determine the Shapley-Shubik power index, consider the permutations in which *D* is pivot. These are *ADBC*, *ADCB* (if *A* is before *D*, then *B* and *C* must be after *D*, or one of them would be the pivot), *BCDA*, and *CBDA*. Similarly, *B* and *C* pivot in 4 permutations. There are $4! = 24$ permutations in all and *B*, *C*, and *D* pivot in 12. In the remaining 12, *A* pivots. The Shapley-Shubik power index is the following.

$$\left(\frac{12}{24}, \frac{4}{24}, \frac{4}{24}, \frac{4}{24}\right) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

31. (a) The ordinary members are equally powerful, so each gets 1 vote. The quota is 8, to make the coalition of all ordinary members winning, but 7 members losing. The chair gets 6 votes, enough to combine with 2 ordinary members and win. In our notation, the weighted voting system is $\left[q : w_C, w_{O_1}, w_{O_2}, w_{O_3}, w_{O_4}, w_{O_5}, w_{O_6}, w_{O_7}, w_{O_8}\right] = [8 : 6, 1, 1, 1, 1, 1, 1, 1, 1]$.

(b) The chairperson is critical in all winning coalition she belongs to, except the one in which the committee is unanimous. The number of these coalitions is $2^8 - C_0^8 - C_1^8 - C_8^8 = 256 - 1 - 8 - 1 = 246$, because there are 2^8 coalitions of ordinary members in all, of which we must eliminate $C_0^8 + C_1^8$ because they consist of 0 or 1 members, who cannot form a winning coalition with the chairperson, and C_8^8, because when all 8 ordinary members join the chairperson, the chairperson isn't critical. An ordinary member is critical in 8 winning coalitions: when joined by the rest of the ordinary members, and when joined by the chairperson and one of the other 7 ordinary members. Counting an equal number of blocking coalitions, the Banzhaf power index of this system is $(492, 16, 16, 16, 16, 16, 16, 16, 16)$.

(c) Divide the permutations into 9 groups, according to the location of the chairperson. She is pivot in groups 3, 4, 5, 6, 7, and 8. Therefore his or her Shapley-Shubik power index is $\frac{6}{9} = \frac{2}{3}$. Each ordinary member has $\frac{1}{8}$ of the remaining $1 - \frac{2}{3} = \frac{1}{3}$ of the power; hence the Shapley-Shubik power index of this system is as follows.

$$\left(\frac{2}{3}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}, \frac{1}{24}\right)$$

(d) In this system, the chairperson is 30.75 times as powerful as an ordinary member according to the Banzhaf index, but only 16 times as powerful by the Shapley-Shubik power index.

32. (a) A coalition with exactly 6 votes is a minimal winning coalition. If there are 7 or more votes, the coalition isn't minimal, because it must include a borough president, who, with 1 vote, would not be critical. Thus, the minimal winning coalitions would consist of one of the following:

i. 3 city officials,

ii. 2 city officials and 2 borough presidents, or

iii. 1 city official and 4 borough presidents.

Continued on next page

32. continued

(b) The city officials are critical in all coalitions of the types listed in part (a). They are also critical in each coalition formed by one of these, and an additional borough president. The mayor (and the comptroller and the city council president) is therefore a critical voter in 6 winning coalitions of type (i). For type (ii) coalitions, the mayor could be joined by another city official in $C_1^2 = 2$ ways, and 2 borough presidents in $C_2^5 = 10$ ways − that's 20 coalitions. The mayor could also be joined by another city official and 3 borough presidents, another 20 winning coalitions in which the mayor is critical. Turning to type (iii) coalitions, the mayor would be critical in any winning coalition where he or she is joined by 4 or all 5 borough presidents: another 6 coalitions. This makes 52 coalitions in which the mayor is a critical voter. The mayor's Banzhaf power index (and that of the comptroller and the city council president) is therefore 104 (as usual, we have to double to account for the blocking coalitions).

There are $C_2^3 \times C_1^4 = 3 \times 4 = 12$ ways to choose the city officials and the other borough president to form a winning coalition of type (ii) in which the Manhattan Borough president is critical. There are $C_1^3 \times C_3^4 = 3 \times 4 = 12$ ways to assemble a winning coalition of type (iii) in which the Manhattan Borough president is critical. Taking the 24 critical votes of the Manhattan Borough president in winning coalitions, and another 24 critical votes in blocking coalitions, we find that his or her Banzhaf power index is 48. The other borough presidents have the same index.

33. Let's determine the minimal winning coalitions. They would be of the following types:

(a) 3 city officials

(b) 2 city officials and 1 borough president

(c) 1 city official and all of the borough presidents.

Thus, the city officials all have the same power, and the borough presidents, although weaker than the city officials, also have equal power. We will assign a voting weight of 1 to each borough president. Let C denote the voting weight of a city official and let q be the quota. To make the coalition of type (i) win, and 2 city officials lose, we have

$$2C < q \le 3C.$$

To make coalitions of type (ii) win, we require $2C + 1 \ge q$. Combining these inequalities, we see that (if C is an integer), $q = 2C + 1$. The 5 borough presidents plus one city official can win, but 4 borough presidents plus a city official is a losing coalition: therefore

$$C + 4 < q \le C + 5$$

and hence $q = C + 5$. We now have two expressions for q, $2C + 1$ and $C + 5$. Equating them, $2C + 1 = C + 5$, which we can solve for C to obtain $C = 4$, and hence $q = 9$. Finally, the 5 borough presidents form a losing coalition, but win if joined by a city official: this will hold provided

$$5 < q \le C + 5$$

This is also valid for $q = 9$ and $C = 4$.

The weighted voting system is $\left[q : w_M, w_C, w_{CCP}, w_{P_1}, w_{P_2}, w_{P_3}, w_{P_4}, w_{P_5}\right] = [9 : 4, 4, 4, 1, 1, 1, 1, 1]$.

34. (a) A minimal winning coalition consists of all 5 permanent members and 4 other members. If one permanent member joins the opposition, and the 6 other members join, a losing coalition results. Therefore each permanent member must have more weight than 6 ordinary members. We give each of the permanent members 7 votes and each ordinary member 1 vote. The minimal winning coalition then has $5\times 7+4=39$ votes, so the quota is 39.

(b) Each permanent member has the following Banzhaf index.

$$2\times\left(C_4^{10}+C_5^{10}+C_6^{10}+C_7^{10}+C_8^{10}+C_9^{10}+C_{10}^{10}\right)$$

$$\begin{array}{c}
1\\
1\quad 1\\
1\quad 2\quad 1\\
1\quad 3\quad 3\quad 1\\
1\quad 4\quad 6\quad 4\quad 1\\
1\quad 5\quad 10\quad 10\quad 5\quad 1\\
1\quad 6\quad 15\quad 20\quad 15\quad 6\quad 1\\
1\quad 7\quad 21\quad 35\quad 35\quad 21\quad 7\quad 1\\
1\quad 8\quad 28\quad 56\quad 70\quad 56\quad 28\quad 8\quad 1\\
1\quad 9\quad 36\quad 84\quad 126\quad 126\quad 84\quad 36\quad 9\quad 1\\
1\quad 10\quad 45\quad 120\quad 210\quad 252\quad 210\quad 120\quad 45\quad 10\quad 1
\end{array}$$

Using Pascal's triangle, we have $2\times(210+252+210+120+45+10+1)=2\times 848=1696.$

An ordinary member has an index of $2\times C_3^9=2\times 84=168.$ By this index, a permanent member is about 10 times as powerful as an ordinary member.

(c) A minimal winning coalition has 6 votes, including all 5 permanent members. If n is the number of votes assigned to each permanent member, then the quota will be $q=5n+1$ (for 5 permanent members and one non-permanent member).

The coalition that includes the entire Council except for one permanent member (who vetoes) has weight $4n+6$ and is losing. Therefore, $4n+6<5n+1.$ Subtract $4n+1$ from each side of this inequality to obtain $5<n.$ Hence we can put $n=6$ and $q=31.$ The weighted voting system is $[31:6,6,6,6,6,1,1,1,1,1,1]$.

To determine the Banzhaf power index, note that all winning coalitions can be formed by combining the coalition of all the permanent members with any coalition of non-permanent members except the empty coalition, $\{\ \}$. There are $2^6=64$ coalitions of the 6 non-permanent members; excluding $\{\ \}$ we have 63 winning coalitions in all. Each permanent member, with veto power, is critical in all of them. Doubling to account for the blocking coalitions, we find that the Banzhaf power index of each permanent member is 126.

A non-permanent member casts a critical vote in only one winning coalition – when it is the only non-permanent member to join the permanent members in voting "yes." Doubling, to account for the blocking coalition consisting of all the non-permanent members, the Banzhaf power index of each non-permanent member is 2. The Banzhaf power index of this voting system is therefore $(126,126,126,126,126,2,2,2,2,2,2)$.

In this system, each permanent member is 63 times as powerful as each non-permanent member. We have seen that in the current system, the corresponding ratio of power is 10 to 1. Thus the permanent members did relinquish some power by including four more non-permanent members in the Security Council.

35. The three weight-3 voters, or 2 weight-3 voters and one weight-1 voter form minimal winning coalitions.

A weight-3 voter, *A*, is critical in any winning coalition with 7, 8, or 9 votes. There are 6 weight-7 coalitions that include *A*, because they are formed by assembling one of the other 2 weight-3 voters, and one of the 3 weight-1 voters. There are also 6 weight-8 coalitions with *A*: they also need one of the other 2 weight-3 voters and 2 of the 3 weight-1 voters (the number of ways to choose 2 weight-1 voters is $C_2^3 = 4$). Finally, there are 3 coalitions of weight 9 to which *A* belongs: all 3 weight-3 voters is one of them; the other 2 consist of *A* and one of the other 2 weight-3 voters, and all of the weight-1 voters. Thus *A* is critical in a total of 15 winning coalitions, and *A*'s Banzhaf power index is 30.

A weight-1 voter, *D*, is critical in 3 winning coalitions, formed by assembling *D* with 2 of the 3 weight-3 voters. Doubling, we see that the Banzhaf power index of *D* is 6.

The Banzhaf power index of this system is $(30, 30, 30, 6, 6, 6)$.

36. The minimal winning coalitions found in the solution of Exercise 35 are still minimal winning coalitions. There is another type of weight-7 minimal winning coalition as well: one weight-3 voter and all of the weight-1 voters.
Banzhaf power index of a weight-3 voter (as usual, only winning coalitions are counted):

Weight 7 The voter is critical in $C_1^2 \times C_1^4 = 2 \times 4 = 8$ coalitions in which he or she is joined by one of the 2 other weight-3 voters and one of the 4 weight-1 voters. He or she is also critical in 1 coalition where she is joined by all of the weight-4 voters and no other weight-3 voters. **Total: 9 critical votes**.

Weight 8 The voter is critical in $C_1^2 \times C_2^4 = 2 \times 6 = 12$ coalitions where he or she is joined by 1 other weight-3 voter and 2 weight-1 voters. **Total: 12 critical votes**.

Weight 9 The voter is critical in $C_1^2 \times C_3^4 = 2 \times 4 = 8$ coalitions involving the voter, one other weight-3 voter, and 3 weight-1 voters. He or she is also critical in one winning coalition consisting of the 3 weight-3 voters. **Total: 9 critical votes**.

The weight-3 voter is thus critical in 30 winning coalitions, and her Banzhaf power index is 60.

The weight-1 voter has the opportunity to be a critical voter only in weight-7 coalitions. There are $C_2^3 = 3$ such coalitions in which he or she is joined by two weight-3 voters, and $C_1^3 = 3$ coalitions in which he or she is joined by 1 weight-3 voter, and the other 3 weight-1 voters. This is a total of 6 coalitions, so the weight-1 voter's Banzhaf power index is 12.

Each voter's Banzhaf power index has doubled as a result of including the additional voter, and the weight-1 voter still has $\frac{1}{5}$ of the power of the weight-3 voter. However, an individual weight-1 voter now has $\frac{1}{19}$ of the voting power $\left(\frac{12 \text{ critical votes}}{4 \times 12 + 3 \times 60}\right)$, and before the new voter joined, he or she had $\frac{1}{18}$. His or her share of the power has diminished slightly; so has that of each weight-3 voter.

37. Let's start with a weight-1 voter, *A*.

Case I: 3 weight-1 voters

A will be pivot in permutations where he or she is in position 3, and positions 1 and 2 are occupied by weight-3 voters. There are $C_2^3 = 3$ ways to choose the weight-3 voters who come first, 2 ways to put them in order, and 3! ways to put the voters following *A* in order. Thus the Shapley-Shubik power index of *A* is $\frac{3\times 2\times 3!}{6\times 5\times 4\times 3!} = \frac{3\times 2}{6\times 5\times 4} = \frac{1}{5\times 4} = \frac{1}{20}$.

The other weight-1 voters have the same power, and the weight-3 voters share the remaining $1 - 3\times\frac{1}{20} = 1 - \frac{3}{20} = \frac{17}{20}$ of the power.

Case II: 4 weight-1 voters

Now *A* will be pivot in permutations where he or she is in position 3, and positions 1 and 2 are occupied by weight-3 voters. There are still 3 ways to select the 2 weight-3 voters and 2 ways to put them in order, but now there are 4! ways to arrange the voters who follow *A* in the permutation. This gives $6\times 4!$ permutations in which *A* is pivot.

A will also be pivot in any permutation where he or she is in position 5 and the final two positions are occupied by weight-3 voters. There are the same number of these permutations. The Shapley-Shubik power index for *A* is therefore $\frac{2\times 6\times 4!}{7\times 6\times 5\times 4!} = \frac{2\times 6}{7\times 6\times 5} = \frac{2}{7\times 5} = \frac{2}{35}$. The other 3 weight-1 voters have the same power, and the remaining $1 - 4\times\frac{2}{35} = 1 - \frac{8}{35} = \frac{27}{35}$ of the power belongs to the weight-3 voters.

Although each voter's share of power decreased proportionally in the Banzhaf model when a new voter joined the system, in this particular situation, each weight-1 voter's power, measured by the Shapley-Shubik model, increased when the new voter was included, because $\frac{2}{35} > \frac{1}{20}$.

38. We will try to assign weights to make this a weighted voting system. Give each recent graduate (RG) a weight of 1, and let *X* be the weight of each of the rich alumni (RA). The quota will be denoted *Q*.

Let's consider the minimal winning coalitions. There are 15 members of the committee, so a majority is 8. The minimal winning coalitions comprise either 3 rich alumni RA and 5 (RG), with $5 + 3X$ votes, or 2 RA and 6 RG, with $6 + 2X$. The weights of these coalitions are at least the quota. Therefore, the following two inequalities hold.

$$5 + 3X \geq Q$$
$$6 + 2X \geq Q$$

Because it is not a majority, 4 RG and 3 RA form a losing coalition; thus $4 + 3X < Q$. Comparing this with the second of the above inequalities, we have $4 + 3X < 6 + 2X$.

Subtract $4 + 2X$ from each side of this inequality to obtain $X < 2$. The weight of one RA has to be less than that of 2 RG.

A coalition comprising 1 RA and all 12 RG is a losing coalition, because the votes of at least 2 RA are needed to pass a motion. Therefore, $12 + X < Q$. Comparing this with the inequality $6 + 2X \geq Q$ yields $12 + X < 6 + 2X$. Subtract $6 + X$ from each side of this inequality to obtain $6 < X$. Each RA must have more voting weight than 6 RG.

It's impossible for each RA to have less weight than 2 RG, and at the same time to have more weight than 6 RG. Therefore, this voting system is not equivalent to a weighted voting system.

39. Consider Maine, which has 2 congressional districts. Ignoring the rest of the country, Maine would be a 3-voter system, in which the state has 2 votes, and each congressional district has 1. There are 3! permutations of voters, but only 2 are possible: 1M2 and 2M1, where the congressional districts are identified by numerals, and the state is M. The reason is as follows. Each entity (district or statewide) is given a score, which is the number of votes recorded for the Bush-Cheney ticket, divided by the number of votes cast in that entity for the Kerry-Edwards ticket. The entity's position in the permutation is determined by that ratio.

Let $r_1, r_2,$ and r_M denote the ratios for the two districts and the state as a whole, respectively, and let $y_1, y_2,$ and y_M be the number of votes cast for the Kerry-Edwards ticket in each entity. The reason that some electoral permutations are impossible is that $y_M = y_1 + y_2$. Each vote is actually counted twice: once for the elector representing the voter's congressional district, and once for the two statewide electors. The number of votes for the Bush-Cheney ticket in the three entities were $r_1 y_1,\ r_2 y_2,$ and $r_M\left(y_1 + y_2\right)$. Because the number of statewide votes for the Bush-Cheney ticket can also be determined by adding the votes in the two districts,

$$r_M\left(y_1 + y_2\right) = r_1 y_1 + r_2 y_2.$$

Dividing by $\left(y_1 + y_2\right)$ we can obtain a formula for r_M :

$$r_M = \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2}.$$

Suppose that $r_1 > r_2$. Then $r_1 y_2 > r_2 y_2,$ and thus

$$r_1 = \frac{r_1\left(y_1 + y_2\right)}{y_1 + y_2} = \frac{r_1 y_1 + r_1 y_2}{y_1 + y_2} > \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2} = r_M$$

Also, $r_1 y_1 > r_2 y_1,$ so

$$r_M = \frac{r_1 y_1 + r_2 y_2}{y_1 + y_2} > \frac{r_2 y_1 + r_2 y_2}{y_1 + y_2} = \frac{r_2\left(y_1 + y_2\right)}{y_1 + y_2} = r_2.$$

These inequalities, taken together show that $r_1 > r_M > r_2,$ if $r_1 > r_2$. Of course, if $r_1 > r_2,$ the same argument would show $r_2 > r_M > r_1$.

Any permutation of the states in which the statewide electors for Maine do not fall between the electors for the two congressional districts is therefore impossible. The same is true for Nebraska: at least one district elector must come before the statewide electors, and one must come after them.

The Shapley-Shubik power index should be computed by using only the possible permutations. A given entity's Shapley-Shubik power index would be the number of possible permutations in which it is pivot, divided by the total number of possible permutations.

Word Search Solution

N	O	I	T	A	T	U	M	R	E	P	F	F	T	A	T	A	R	G	I	X	N	P	G	A
R	E	W	O	P	O	T	E	V	Z	S	A	R	I	E	J	L	E	B	D	C	O	O	N	E
E	L	G	N	A	I	R	T	S	L	A	C	S	A	P	V	F	X	X	I	E	I	E	A	W
Y	E	N	C	I	V	N	S	W	O	S	T	O	D	I	E	Z	E	E	C	C	T	S	D	X
E	X	E	N	I	V	N	Y	P	V	E	O	D	A	V	T	G	D	E	T	L	I	K	D	S
V	T	L	O	E	B	O	S	S	D	T	R	A	T	O	U	Q	N	K	A	E	L	I	I	R
H	S	I	I	L	I	I	G	G	M	O	I	N	B	T	K	M	I	F	T	P	A	T	T	G
C	B	Z	T	P	N	T	N	X	R	V	A	V	L	A	A	L	R	C	O	O	O	V	I	S
R	A	E	I	I	A	I	I	C	B	A	L	T	O	L	J	S	E	U	R	B	C	S	O	B
I	A	N	L	C	R	L	T	O	I	R	S	N	C	V	H	R	W	L	P	A	G	S	N	A
T	D	T	A	N	Y	A	O	M	A	T	Z	E	K	O	E	J	O	L	F	N	N	L	F	T
I	D	P	O	I	N	O	V	V	L	X	T	H	I	T	M	H	P	W	W	Z	I	R	O	W
C	F	A	C	R	U	C	D	I	U	E	P	H	N	E	A	H	K	E	G	H	N	F	R	R
A	T	K	G	P	M	G	E	N	M	U	B	R	G	R	E	P	I	N	C	A	N	E	M	E
L	I	E	N	S	B	N	T	A	R	T	F	G	C	I	O	G	B	R	O	F	I	S	U	J
V	O	P	I	E	E	I	H	T	O	R	M	O	O	W	H	N	U	T	A	P	W	E	L	R
O	N	H	N	T	R	S	G	I	F	G	K	O	A	T	E	W	H	V	L	O	L	B	A	L
T	Y	C	N	O	X	O	I	O	Y	Y	E	N	L	S	S	I	S	E	I	W	A	A	O	E
E	Q	U	I	V	A	L	E	N	T	V	O	T	I	N	G	S	Y	S	T	E	M	S	V	I
R	K	A	W	A	Q	C	W	D	I	N	L	S	T	J	F	Y	E	H	I	R	I	I	S	N
W	M	E	J	R	B	S	V	A	L	O	S	P	I	T	C	T	L	E	O	I	N	D	K	L
C	U	R	Z	T	N	Y	D	D	A	E	B	M	O	I	C	A	P	R	N	N	I	A	D	W
T	L	H	E	X	Z	I	P	D	U	M	M	Y	N	L	B	P	A	N	E	D	M	F	S	P
E	A	V	T	E	E	P	C	P	D	N	A	T	D	R	T	E	H	Q	S	E	X	E	Y	G
A	D	R	N	C	X	E	D	N	I	R	E	W	O	P	W	J	S	O	H	X	F	R	E	E