Chapter 9
Social Choice: The Impossible Dream

Chapter Outline

Chapter Summary

Social-choice theory developed from the attempt to explain voting and other group decision-making processes. Groups face the problem of turning individual preferences for different outcomes into a single choice for the group as a whole. How groups can best solve this problem is one of the basic questions in social choice.

Majority rule is perhaps the simplest voting method. The candidate receiving a majority (over 50%) of the votes cast is elected. This method works fine if there are only two candidates. However, as the number of candidates grows, it becomes increasingly unlikely that any candidate will obtain a majority of the votes cast. If it is desired that the winner should have an absolute majority, there could be a provision for a runoff election. This could be between the top two vote-getters, as in French presidential elections. Runoffs are just one example of *sequential voting* schemes. Another option, more common in American elections, would be to dispense with majority rule and adopt the *plurality method* in which the candidate with the highest vote total, majority or not, wins.

Voters generally have a most-preferred candidate in an election, and it is these top choices alone that influence the outcome in majority rule or the plurality method. If other methods are used, say a sequential method, then each voter's ranking of the entire list of candidates is important because the voter's top choice may not be in the running at a particular point in the election. These rankings are called preference list ballots.

Borda count, the method used in sports polls, asks the voter to rank the candidates in order from most preferred to least preferred and then assigns a point value to each position. The points are totaled for each candidate, and the candidate with the most points wins.

The *Hare system* also asks the voter to rank the candidates in order from most preferred to least preferred. In this system a winner is determined by repeatedly deleting the "least preferred" candidate. In this system, the winner will proportionally be the choice of the majority of the voters.

Condorcet's method requires that each candidate go head-to-head with each of the other candidates in a plurality election. To be a winner, a candidate must defeat each of his/her opponents. An interesting observation regarding Condorcet's method is that group preferences need not be transitive; that is, the group can prefer *A* to *B*, *B* to *C*, and yet prefer *C* to *A*. This phenomenon is referred to as the Condorcet paradox.

Kenneth Arrow's impossibility theorem established that if the vote is between three or more candidates, then any voting method will occasionally yield paradoxical results. Furthermore, any voting method is subject to manipulation through insincere, or strategic, voting (Chapter 10). These facts paint a gloomy picture, but voting is such an integral part of democratic society that it is important to recognize the shortcomings as well as the vulnerabilities of voting methods.

Voting turns out to be a complex activity. Several things can influence the outcome of an election: the voting method, individual preference list ballots, strategic voting, and, for sequential methods, the order of the various votes.

Skill Objectives

1. Analyze and interpret preference list ballots.

2. Rearrange preference list ballots to accommodate the elimination of one or more candidates.

3. Explain the difference between majority rule and the plurality method.

4. Discuss why the majority method may not be appropriate for an election in which there are more than two candidates.

5. Apply the plurality voting method to determine the winner in an election whose preference list ballots are given.

6. Apply the Borda count method to determine the winner from preference list ballots.

7. Apply the sequential pairwise voting method to determine the winner from preference list ballots.

8. Apply the Hare system method to determine the winner from preference list ballots.

9. Apply the plurality runoff method to determine the winner from preference list ballots.

10. Apply the approval voting method to determine the winner from a vote.

11. Determine whether a Condorcet winner exists and explain the Condorcet winner criterion (CWC).

12. Explain independence of irrelevant alternatives (IIA).

13. Explain Pareto condition.

14. Explain monotonicity.

15. Explain the Condorcet paradox.

16. Explain Arrow's impossibility theorem.

Teaching Tips

1. Surprisingly, some students have difficulty reading a preference schedule and understanding the order involved. It could be helpful later to take the time in the beginning to discuss this thoroughly. You may want to poll the class on some item of common interest and then use their responses to set up preference list ballots. For example, you might ask them to rank three television programs that are being aired at the same time. This preference schedule could then serve as the basis for discussion when investigating various voting schemes.

2. The Hare system is easy for most students to understand conceptually; however, they often experience frustration when trying to reorder the preference schedule after an elimination. It helps considerably to have several copies of blank preference list ballots with elimination preference list ballot(s) following, so that the progression can be followed with ease in the classroom. Convey to students that there are such resources available to assist them with the text exercises. These resources are available in the Student Study Guide.

3. The concept of the Condorcet winner may be more understandable if the student first writes out all possible two-candidate contests. This will bring about an elimination process for which the preference list ballots will need to be altered so that only two candidates remain. Erase all other candidates from the chart first and then move the two remaining candidates vertically upward into blank spaces. It seems like a time-consuming process, but it pays dividends in terms of student learning.

4. In the Borda count method, it is useful to have the students first calculate the total number of points to be distributed. For instance, if there are 25 voters and each ballot ranks 4 candidates, with the distribution of 6 points (3-2-1-0), the total number of points is 150. The students should check their totals against this number after calculating the number of points for each candidate.

5. In the Borda count method, the text uses 0 points for the last-place candidate up to $n-1$ for the first-place candidate where n represents the number of candidates on a ballot. If we let v be the number of voters, then the total number of points is $\frac{n(n-1)}{2}\cdot v$. Ask students to determine if other formulas exist for other point distributions. Start with 25 voters and each ballot ranks 4 candidates, with the distribution of 10 points (4-3-2-1). After that, allow for other less standard scenarios like 7-3-2-0. Also, discuss whether or not the method shown in the text on page 350 (replacing candidates on ballots by boxes and counting) is valid for other point distributions discussed here.

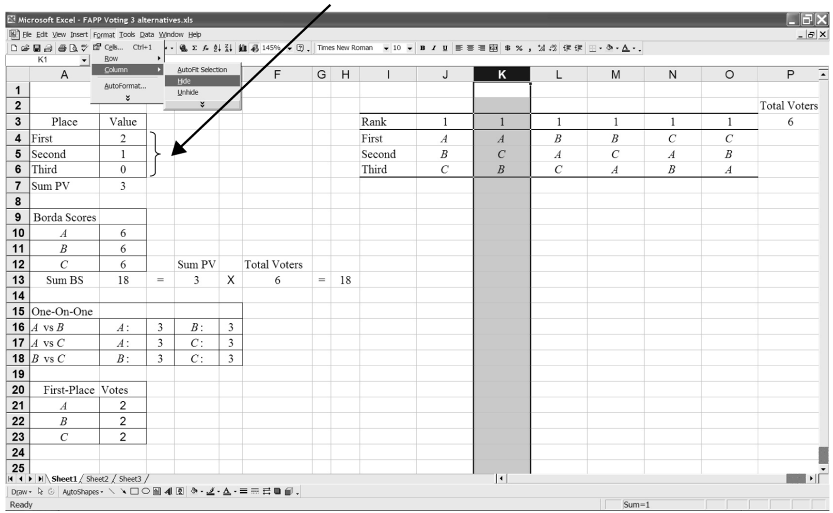
6. In Example 1 of the text, it was shown that *AG* is the winner using Condorcet's method. Explain to students that when one is determining who the winner is using Condorcet's method, one should consider all the possible one-on-one competitions. For n candidates, there will be $\frac{n!}{2\cdot(n-2)!}$ pairings to consider.

7. The plurality runoff method is a one-time runoff, even if there are three candidates in the runoff (ties for first- or second- place). If the runoff is between two candidates, you are looking for the candidate with the majority of the votes. Otherwise, as the name of the method indicates, one would look for the candidate with a plurality of votes. Ask students to determine if it would be possible to have a tie for first-place with this method. You may choose to remind students at this point that in this chapter it is assumed that there are an odd number of voters. Follow up this discussion with determining possible scenarios with an even number of voters.

8. Approval voting is an important concept because of increased public interest in this method. It may be helpful to work an example by conducting an in-class survey of campus issues of interest.

9. Spreadsheets such as Excel are very helpful in performing the tedious calculations. We have made available Excel spreadsheets for 3 as well as 4 candidates. These spreadsheets will calculate Borda scores, one-on-one competition results, and the number of first-place votes for all possible ballots. They allow the flexibility of changing the points assigned to each place in the Borda method. They are initially set up with 1 voter per ballot (even though that yields an even number of voters). If a ballot is not needed, then one can simply hide that column.

Points for Borda method



You should find these spreadsheets helpful in quickly generating examples, showing that the Borda method fails the Condorcet winner criterion, examining outcomes with an even number of voters, etc. The spreadsheet for 4 candidates contains all of the twenty-four possible ballots. You will find these spreadsheets at http://www.whfreeman.com/fapp7. Since a complete spreadsheet for 5 candidates requires 120 possible ballots, we have not included additional spreadsheets.

Research Paper

1. Have students investigate the life and contributions of Marquis of Condorcet (1743–1794), the eighteenth century French mathematician and philosopher. Other figures mentioned in this chapter to consider include Kenneth O. May (1915–1977) and Chevalier de Borda (1733–1799) aka Jean-Charles de Borda.

2. Have students investigate conditions other than those described in the text, such as the Condorcet loser criterion, and consistency criterion.

Spreadsheet Project

To do this project, go to http://www.whfreeman.com/fapp7e.

Spreadsheets are used in this project to analyze Borda counts and approval voting. Using the automatic recalculation feature of spreadsheets, these activities allow the investigation of insincere voting strategies and their impact on the voting results. This activity can provide an introduction to a topic that will be further address in Chapter 10: Manipulability.

Collaborative Learning

Group Ranking

As an ice-breaker, duplicate the following exercise and have your students discuss the problem in groups.

A small New England town is faced with the following problem:

Five capital projects have been proposed for the next year:

1. Repave the main road.

2. Plant 100 new trees.

3. Build a cabana at the town swimming pool.

4. Replace the floor of the school gym.

5. Build a bandstand in the town's central square.

The problem is that the town's income for the next year is uncertain, so there may not be enough money for all of the projects. All issues are dealt with at town meetings in which each citizen casts a vote. The citizens decide to *rank* the projects in order of importance, and to start funding from the top of the ranks. Each citizen will submit his or her rankings that will be used to determine the group ranking. Devise a method (or methods) for determining the *group* ranking from the individual rankings.

Solutions

Skills Check:

1. b	2. c	3. b	4. b	5. c	6. d	7. b	8. b	9. c	10. a
11. a	12. c	13. a	14. b	15. c	16. a	17. b	18. b	19. c	20. c

Exercises:

1. Minority Rule satisfies condition (1): An exchange of marked ballots between two voters leaves the number of votes for each candidate unchanged, so whichever candidate won on the basis of having fewer votes before the exchange still has fewer votes after the exchange. Minority rule also satisfies condition (2): Suppose candidate *X* receives *n* votes and candidate *Y* receives *m* votes, and candidate *X* wins because $n < m$. Now suppose that a new election is held, and every voter reverses his or her vote. Then candidate *X* has *m* votes and candidate *Y* has *n* votes, and so candidate *Y* is the winner. Minority rule, however, fails condition (3): Suppose, for example, that there are 3 voters and that candidate *X* wins 1 out of the 3 votes. Now suppose that one of the 2 voters who voted for candidate *Y* reverses his or her vote. Then candidate *X* would have 2 votes, and candidate *Y* would have 1 vote, thus resulting in a win for candidate *Y*.

2. Suppose that candidate *X* is the winner regardless of who votes for whom (imposed rule). Conditions (1) and (3) are satisfied: The outcome of an election is not affected by anything in particular, not by an exchange of ballots, as in condition (1), or by a change in a single ballot, as in condition (3). Condition (2), however, fails: If every voter reversed his or her vote, candidate *X* still wins, so the outcome of the election is not reversed.

3. A dictatorship satisfies condition (2): If a new election is held and every voter (in particular, the dictator) reverses his or her ballot, then certainly the outcome of the election is reversed. A dictatorship also satisfies condition (3): If a single voter changes his or her ballot from being a vote for the loser of the previous election to a vote for the winner of the previous election, then the single voter could not have been a dictator (since the dictator's ballot was not a vote for the loser of the previous election). Thus, the outcome of the new election is the same as the outcome of the previous election. A dictatorship, however, fails to satisfy condition (1): If the dictator exchanges his/her marked ballot with any voter whose marked ballot differs from that of the dictator, then the outcome of the election is certainly reversed.

4. For each of the following, assume that there are 4 voters (one of whom is named Nadia) and 2 candidates (named *A* and *B*).

(a) A voting system satisfying condition (1), but neither conditions (2) nor (3): *A* wins if and only if he or she receives 1, 2, or 4 votes.

(b) A voting system satisfying condition (2), but neither conditions (1) nor (3): *A* wins if and only Nadia votes for *B*. (Nadia is the antidictator)

(c) A voting system satisfying condition (3), but neither conditions (1) nor (2): Nadia gets 2 votes; the other three voters get 1 vote; and *A* wins if and only if he gets a total of 4 or 5 votes.

5. With an odd number of voters, each one-on-one score will have a winner because there cannot be a tie. If one of the candidates, say *A*, is the winner by Condorcet's method, then *A* would have beaten every other candidate in a one-on-one competition. It would be impossible for another candidate, say *B*, to have beaten *A*. Hence, *B* could not also be a winner.

6. Answers will vary.

7.

22%	**23%**	**15%**	**29%**	**7%**	**4%**
D	*D*	*H*	*H*	*J*	*J*
H	*J*	*D*	*J*	*H*	*D*
J	*H*	*J*	*D*	*D*	*H*

(a) We must check the one-on-one scores of *D* versus *H*, *D* versus *J*, and *H* versus *J*.

D versus *H*: *D* is over *H* on 22% + 23% + 4% = 49% of the ballots, while the reverse is true on 15% + 29% + 7% = 51%. Thus, *H* defeats *D*, 51% to 49%.

D versus *J*: *D* is over *J* on 22% + 23% + 15% = 60% of the ballots, while the reverse is true on 29% + 7% + 4% = 40%. Thus, *D* defeats *J*, 60% to 40%.

H versus *J*: *H* is over *J* on 22% + 15% + 29% = 66% of the ballots, while the reverse is true on 23% + 7% + 4% = 34%. Thus, *H* defeats *J*, 66% to 34%.

Yes, there is a Condorcet winner. Since *H* can defeat both *D* and *J* in a one-to one competition, Elizabeth Holtzman (*H*) is the winner by Condorcet's method.

(b) *D* has 22% + 23% = 45% of the first-place votes. *H* has 15% + 29% = 44% of the first-place votes. *J* has 7% + 4% = 11% of the first-place votes. Since *D* has the most first-place votes, Alfonse D'Amato (*D*) is the winner by plurality voting.

8.

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	*C*	*C*	*B*	*D*	*C*
Third	*B*	*C*	*D*	*A*	*D*	*B*	*B*
Fourth	*C*	*D*	*A*	*D*	*A*	*A*	*A*

(a) *A* has 3 + 1 = 4 first-place votes. *B* has 1 + 1 = 2 first-place votes. *C* has 1 + 1 = 2 first-place votes. *D* has 1 first-place vote. Since *A* has the most first-place votes, *A* is the winner by plurality voting.

(b)

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	4×3	0×2	1×1	4×0	13
B	2×3	2×2	5×1	0×0	15
C	2×3	3×2	1×1	3×0	13
D	1×3	4×2	2×1	2×0	13

Thus, *B* has the highest Borda score and is declared the winner.

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	□	□	*C*	*C*	*B*	*D*	*C*
Third	□	□	*D*	*A*	*D*	*B*	*B*
Fourth	□	□	*A*	□	*A*	*A*	*A*

To show that the Borda score for candidate *A* is 13, it needs to be noted that each box below *A* counts 3 times in the first column. The Borda score for *A* is $(3\times3)+(4\times1)=9+4=13$.

Continued on next page

8. (b) continued

For *B*:

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	□	□	*B*	*D*	*C*
Third	*B*	□	□	□	□	*B*	*B*
Fourth	□	□	□	□	□	□	□

To show that the Borda score for candidate *B* is 15, it needs to be noted that each box below *B* counts 3 times in the first column. The Borda score for *B* is $(3\times1)+(12\times1)=3+12=15$.

For *C*:

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	*C*	*C*	□	□	*C*
Third	*B*	*C*	□	□	□	□	□
Fourth	*C*	□	□	□	□	□	□

There are 13 boxes below *C*, each with a value of 1. The Borda score for *C* is 13.

For *D*:

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*D*
Second	*D*	*B*	*C*	*C*	*B*	*D*	□
Third	□	*C*	*D*	*A*	*D*	□	□
Fourth	□	*D*	□	*D*	□	□	□

To show that the Borda score for candidate *D* is 13, it needs to be noted that each box below *D* counts 3 times in the first column. The Borda score for *D* is $(2\times3)+(7\times1)=6+7=13$.

(c) Since *D* has the least number of first-place votes (see Part a), *D* is eliminated.

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*	*C*	*C*	*C*
Second	*B*	*B*	*C*	*C*	*B*	*B*	*B*
Third	*C*	*C*	*A*	*A*	*A*	*A*	*A*

A now has 3 + 1 = 4 first-place votes. *B* now has 1 + 1 = 2 first-place votes. *C* now has 1 + 1 + 1 = 3 first-place votes. Since *B* has the least number of first-place votes, *B* is eliminated.

	Number of voters (9)						
Rank	**3**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*A*	*C*	*C*	*C*	*C*	*C*
Second	*C*	*C*	*A*	*A*	*A*	*A*	*A*

A now has 3 + 1 = 4 first-place votes. *C* now has 1 + 1 + 1 + 1 + 1 = 5 first-place votes. Thus, *C* is the winner by the Hare system.

(d) In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, we first pit *A* against *B*. There are 4 voters that prefer *A* to *B* and 5 prefer *B* to *A*. Thus, *B* wins by a score of 5 to 4. *A* is therefore eliminated, and *B* moves on to confront *C*.

There are 6 voters who prefer *B* to *C* and 3 prefer *C* to *B*. Thus, *B* wins by a score of 6 to 3. *C* is therefore eliminated, and *B* moves on to confront *D*.

There are 4 voters who prefer *B* to *D* and 5 prefer *D* to *B*. Thus, *D* wins by a score of 5 to 4.

Thus, *D* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*.

9.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*A*
Second	*A*	*A*	*D*	*D*	*D*
Third	*B*	*C*	*A*	*A*	*B*
Fourth	*D*	*B*	*B*	*C*	*C*

(a) *A* has 1 first-place vote. *B* has 1 first-place vote. *C* has 2 + 1 = 3 first-place votes. *D* has 2 first-place votes. Since *C* has the most first-place votes, *C* is the winner by plurality voting.

(b)

Preference	1st place votes × 3	2nd place votes × 2	3rd place votes × 1	4th place votes × 0	Borda score
A	1×3	4×2	2×1	0×0	13
B	1×3	0×2	3×1	3×0	6
C	3×3	0×2	2×1	2×0	11
D	2×3	3×2	0×1	2×0	12

Thus, *A* has the highest Borda score and is declared the winner.

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*A*
Second	*A*	*A*	*D*	*D*	$\square$
Third	$\square$	$\square$	*A*	*A*	$\square$
Fourth	$\square$	$\square$	$\square$	$\square$	$\square$

To show that the Borda score for candidate *A* is 13, it needs to be noted that each box below *A* counts 2 times in the first and second columns. The Borda score for *A* is therefore $(4\times2)+(5\times1)=8+5=13$.

For *B*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*A*
Second	*A*	*A*	*D*	$\square$	*D*
Third	*B*	*C*	*A*	$\square$	*B*
Fourth	$\square$	*B*	*B*	$\square$	$\square$

To show that the Borda score for candidate *B* is 6, it needs to be noted that each box below *B* counts 2 times in the first column. The Borda score for *B* is $(1\times2)+(4\times1)=2+4=6$.

For *C*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*A*
Second	$\square$	*A*	$\square$	*D*	*D*
Third	$\square$	*C*	$\square$	*A*	*B*
Fourth	$\square$	$\square$	$\square$	*C*	*C*

To show that the Borda score for candidate *C* is 11, it needs to be noted that each box below *C* counts 2 times in the first and second columns. The Borda score for *C* is therefore $(4\times2)+(3\times1)=8+3=11$.

Continued on next page

9. (b) continued

For *D*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*A*
Second	*A*	□	*D*	*D*	*D*
Third	*B*	□	□	□	□
Fourth	*D*	□	□	□	□

To show that the Borda score for candidate *D* is 12, it needs to be noted that each box below *D* counts 2 times in the second column. The Borda score for *D* is $(3\times2)+(6\times1)=6+6=12$.

(c) Since *A* and *B* have the least number of first-place votes (see Part a), both are eliminated.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*D*	*C*	*D*	*D*
Second	*D*	*C*	*D*	*C*	*C*

C now has 2 + 1 = 3 first-place votes. *D* now has 2 + 1 + 1 = 4 first-place votes. Thus, *D* is the winner by the Hare system.

(d) In sequential pairwise voting with the agenda *B*, *D*, *C*, *A*, we first pit *B* against *D*. There are 3 voters who prefer *B* to *D* and 4 prefer *D* to *B*. Thus, *D* wins by a score of 4 to 3. *B* is therefore eliminated, and *D* moves on to confront *C*.

There are 4 voters who prefer *D* to *C* and 3 prefer *C* to *D*. Thus, *D* wins by a score of 4 to 3. *C* is therefore eliminated, and *D* moves on to confront *A*.

There are 4 voters who prefer *D* to *A* and 3 prefer *A* to *D*. Thus, *D* wins by a score of 4 to 3.

Thus, *D* is the winner by sequential pairwise voting with the agenda *B*, *D*, *C*, *A*.

10.

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	*B*	*B*	*D*	*E*	*E*	*E*
Third	*C*	*D*	*C*	*C*	*D*	*A*
Fourth	*D*	*C*	*B*	*D*	*A*	*B*
Fifth	*E*	*A*	*E*	*A*	*B*	*C*

(a) *A* has 2 + 1 = 3 first-place votes. *B* has 1 first-place vote. *C* has 1 first-place vote. *D* has 1 first-place vote. *E* has 2 first-place votes. Since *A* has the most first-place votes, *A* is the winner by plurality voting.

(b)

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	3×4	0×3	1×2	1×1	3×0	15
B	1×4	4×3	0×2	2×1	1×0	18
C	1×4	0×3	4×2	2×1	1×0	14
D	1×4	1×3	3×2	3×1	0×0	16
E	2×4	3×3	0×2	0×1	3×0	17

Thus, *B* has the highest Borda score and is declared the winner.

Continued on next page

10. (b) continued

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	□	*B*	□	*E*	*E*	*E*
Third	□	*D*	□	*C*	*D*	*A*
Fourth	□	*C*	□	*D*	*A*	□
Fifth	□	*A*	□	*A*	□	□

To show that the Borda score for candidate *A* is 15, it needs to be noted that each box below *A* counts 2 times in the first column. The Borda score for *A* is $(4\times 2)+(7\times 1)=8+7=15$.

For *B*:

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	*B*	*B*	*D*	□	*E*	*E*
Third	□	□	*C*	□	*D*	*A*
Fourth	□	□	*B*	□	*A*	*B*
Fifth	□	□	□	□	*B*	□

To show that the Borda score for candidate *B* is 18, it needs to be noted that each box below *B* counts 2 times in the first and second columns. The Borda score for *B* is therefore $(6\times 2)+(6\times 1)=12+6=18$.

For *C*:

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	*B*	*B*	*D*	*E*	□	*E*
Third	*C*	*D*	*C*	*C*	□	*A*
Fourth	□	*C*	□	□	□	*B*
Fifth	□	□	□	□	□	*C*

To show that the Borda score for candidate *C* is 14, it needs to be noted that each box below *C* counts 2 times in the first and second columns. The Borda score for *C* is therefore $(3\times 2)+(8\times 1)=6+8=14$.

For *D*:

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	*B*	*B*	*D*	*E*	*E*	□
Third	*C*	*D*	□	*C*	*D*	□
Fourth	*D*	□	□	*D*	□	□
Fifth	□	□	□	□	□	□

To show that the Borda score for candidate *D* is 16, it needs to be noted that each box below *D* counts 2 times in the first and second columns. The Borda score for *D* is $(3\times 2)+(10\times 1)=6+10=16$.

Continued on next page

10. (b) continued

For E:

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*B*	*C*	*D*
Second	*B*	$\square$	*D*	*E*	*E*	*E*
Third	*C*	$\square$	*C*	$\square$	$\square$	$\square$
Fourth	*D*	$\square$	*B*	$\square$	$\square$	$\square$
Fifth	*E*	$\square$	*E*	$\square$	$\square$	$\square$

To show that the Borda score for candidate E is 17, it needs to be noted that each box below E counts 2 times in the second column. The Borda score for E is $(4\times 2)+(9\times 1)=8+9=17$.

(c) Since B, C, and D have the least number of first-place votes (see Part a), they are all eliminated.

	Number of voters (8)					
Rank	**2**	**2**	**1**	**1**	**1**	**1**
First	*A*	*E*	*A*	*E*	*E*	*E*
Second	*E*	*A*	*E*	*A*	*A*	*A*

A now has 2 + 1 = 3 first-place votes. E now has 2 + 1 + 1 + 1 = 5 first-place votes. Thus, E is the winner by the Hare system.

(d) In sequential pairwise voting with the agenda B, D, C, A, E, we first pit B against D. There are 5 voters who prefer B to D and 3 prefer D to B. Thus, B wins by a score of 5 to 3. D is therefore eliminated, and B moves on to confront C.

There are 6 voters who prefer B to C and 2 prefer C to B. Thus, B wins by a score of 6 to 2. C is therefore eliminated, and B moves on to confront A.

There are 3 voters who prefer B to A and 5 prefer A to B. Thus, A wins by a score of 5 to 3. B is therefore eliminated, and A moves on to confront E.

There are 3 voters that prefer A to E and 5 prefer E to A. Thus, E wins by a score of 5 to 3.

Thus, E is the winner by sequential pairwise voting with the agenda B, D, C, A, E.

11.

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*C*	*B*	*C*	*D*
Third	*E*	*A*	*E*	*A*	*C*
Fourth	*D*	*D*	*D*	*E*	*A*
Fifth	*C*	*E*	*A*	*B*	*B*

(a) A has 1 first-place vote. B has 1 first-place vote. C has 1 first-place vote. D has 1 first-place vote. E has 1 first-place vote. Since all candidates have the same number of first-place votes, they tie.

(b)

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	1×4	0×3	2×2	1×1	1×0	9
B	1×4	2×3	0×2	0×1	2×0	10
C	1×4	2×3	1×2	0×1	1×0	12
D	1×4	1×3	0×2	3×1	0×0	10
E	1×4	0×3	2×2	1×1	1×0	9

Thus, C has the highest Borda score and is declared the winner.

Continued on next page

11. (b) continued

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	□	*C*	*B*	*C*	*D*
Third	□	*A*	*E*	*A*	*C*
Fourth	□	□	*D*	□	*A*
Fifth	□	□	*A*	□	□

The Borda score for candidate *A* is 9 since there are 9 boxes.

For *B*:

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	□	*B*	*C*	*D*
Third	□	□	□	*A*	*C*
Fourth	□	□	□	*E*	*A*
Fifth	□	□	□	*B*	*B*

The Borda score for candidate *B* is 10 since there are 10 boxes.

For *C*:

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*C*	□	*C*	*D*
Third	*E*	□	□	□	*C*
Fourth	*D*	□	□	□	□
Fifth	*C*	□	□	□	□

The Borda score for candidate *C* is 12.

For *D*:

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*C*	*B*	□	*D*
Third	*E*	*A*	*E*	□	□
Fourth	*D*	*D*	*D*	□	□
Fifth	□	□	□	□	□

The Borda score for candidate *D* is 10 since there are 10 boxes.

For *E*:

	Number of voters (5)				
Rank	**1**	**1**	**1**	**1**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*C*	*B*	*C*	□
Third	*E*	*A*	*E*	*A*	□
Fourth	□	*D*	□	*E*	□
Fifth	□	*E*	□	□	□

The Borda score for candidate *E* is 9 since there are 9 boxes.

Continued on next page

11. continued

(c) Since all candidates have the same least number of first-place votes (see Part a), they all tie.

(d) In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*, we first pit *A* against *B*. There are 3 voters who prefer *A* to *B* and 2 prefer *B* to *A*. Thus, *A* wins by a score of 3 to 2. *B* is therefore eliminated, and *A* moves on to confront *C*.

There is 1 voter who prefers *A* to *C* and 4 prefer *C* to *A*. Thus, *C* wins by a score of 4 to 1. *A* is therefore eliminated, and *C* moves on to confront *D*.

There are 2 voters who prefer *C* to *D* and 3 prefer *D* to *C*. Thus, *D* wins by a score of 3 to 2. *C* is therefore eliminated, and *D* moves on to confront *E*.

There are 2 voters who prefer *D* to *E* and 3 prefer *E* to *D*. Thus, *E* wins by a score of 3 to 2.

Thus, *E* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*.

12.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*C*	*D*
Second	*D*	*D*	*B*	*B*	*B*
Third	*C*	*A*	*D*	*D*	*A*
Fourth	*B*	*C*	*C*	*A*	*C*

(a) *A* has 2 + 1 = 3 first-place votes. *B* has 2 first-place votes. *C* has 1 first-place vote. *D* has 1 first-place vote. Since *A* has the most number of the first-place votes, *A* is the winner by plurality voting.

(b)

Preference	1^{st} place votes × 3	2^{nd} place votes × 2	3^{rd} place votes × 1	4^{th} place votes × 0	Borda score
A	3×3	0×2	3×1	1×0	12
B	2×3	3×2	0×1	2×0	12
C	1×3	0×2	2×1	4×0	5
D	1×3	4×2	2×1	0×0	13

Thus, *D* has the highest Borda score and is declared the winner.

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*C*	*D*
Second	□	*D*	□	*B*	*B*
Third	□	*A*	□	*D*	*A*
Fourth	□	□	□	*A*	□

To show that the Borda score for candidate *A* is 12, it needs to be noted that each box below *A* counts 2 times in the first and second columns. The Borda score for *A* is therefore $(4\times 2)+(4\times 1)=8+4=12$.

continued on next page

12. (b) continued

For *B*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*C*	*D*
Second	*D*	☐	*B*	*B*	*B*
Third	*C*	☐	☐	☐	☐
Fourth	*B*	☐	☐	☐	☐

To show that the Borda score for candidate *B* is 12, it needs to be noted that each box below *B* counts 2 times in the second column. The Borda score for *B* is $(3\times 2)+(6\times 1)=6+6=12$.

For *C*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*C*	*D*
Second	*D*	*D*	*B*	☐	*B*
Third	*C*	*A*	*D*	☐	*A*
Fourth	☐	*C*	*C*	☐	*C*

To show that the Borda score for candidate *C* is 5, it needs to be noted that each box below *C* counts 2 times in the first column. The Borda score for *C* is $(1\times 2)+(3\times 1)=2+3=5$.

For *D*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*C*	*D*
Second	*D*	*D*	*B*	*B*	☐
Third	☐	☐	*D*	*D*	☐
Fourth	☐	☐	☐	☐	☐

To show that the Borda score for candidate *D* is 13, it needs to be noted that each box below *D* counts 2 times in first and second columns. The Borda score for *D* is $(4\times 2)+(5\times 1)=8+5=13$.

(c) Since *C* and *D* have the least number of first-place votes (see Part a), both are eliminated.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*A*	*B*	*A*	*B*	*B*
Second	*B*	*A*	*B*	*A*	*A*

A now has 2 + 1 = 3 first-place votes. *B* now has 2 + 1 + 1 = 4 first-place votes. Thus, *B* is the winner by the Hare system.

(d) In sequential pairwise voting with the agenda *B*, *D*, *C*, *A*, we first pit *B* against *D*. There are 4 voters who *B* to *D* and 3 prefer *D* to *B*. Thus, *B* wins by a score of 4 to 3. *D* is therefore eliminated, and *B* moves on to confront *C*.

There are 4 voters who prefer *B* to *C* and 3 prefer *C* to *B*. Thus, *B* wins by a score of 4 to 3. *C* is therefore eliminated, and *B* moves on to confront *A*.

There are 4 voters who prefer *B* to *A* and 3 prefer *A* to *B*. Thus, *B* wins by a score of 4 to 3.

Thus, *B* is the winner by sequential pairwise voting with the agenda *B*, *D*, *C*, *A*.

13.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	*B*	*A*	*E*	*E*
Third	*D*	*D*	*D*	*A*	*C*
Fourth	*A*	*C*	*E*	*C*	*D*
Fifth	*B*	*A*	*B*	*B*	*B*

(a) *A* has 1 first-place vote. *B* has 0 first-place votes. *C* has 2 + 1 = 3 first-place votes. *D* has 1 first-place vote. *E* has 2 first-place votes. Since *C* has the most first-place votes, *C* is the winner by plurality voting.

(b)

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	1×4	1×3	1×2	2×1	2×0	11
B	0×4	2×3	0×2	0×1	5×0	6
C	3×4	0×3	1×2	3×1	0×0	17
D	1×4	0×3	5×2	1×1	0×0	15
E	2×4	4×3	0×2	1×1	0×0	21

Thus, *E* has the highest Borda score and is declared the winner.

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	*B*	*A*	*E*	$\square$
Third	*D*	*D*	$\square$	*A*	$\square$
Fourth	*A*	*C*	$\square$	$\square$	$\square$
Fifth	$\square$	*A*	$\square$	$\square$	$\square$

To show that the Borda score for candidate *A* is 11, it needs to be noted that each box below *A* counts 2 times in the first column. The Borda score for *A* is $(1\times2)+(9\times1)=2+9=11$.

For *B*:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	*B*	*A*	*E*	*E*
Third	*D*	$\square$	*D*	*A*	*C*
Fourth	*A*	$\square$	*E*	*C*	*D*
Fifth	*B*	$\square$	*B*	*B*	*B*

To show that the Borda score for candidate *B* is 6, it needs to be noted that each box below *B* counts 2 times in the second column. The Borda score for *B* is $3\times2=6$.

Continued on next page

13. (b) continued

For C:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	□	*B*	□	*E*	*E*
Third	□	*D*	□	*A*	*C*
Fourth	□	*C*	□	*C*	□
Fifth	□	□	□	□	□

To show that the Borda score for candidate C is 17, it needs to be noted that each box below C counts 2 times in the first and second columns. The Borda score for C is therefore $(5\times2)+(7\times1)=10+7=17$.

For D:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	*B*	*A*	□	*E*
Third	*D*	*D*	*D*	□	*C*
Fourth	□	□	□	□	*D*
Fifth	□	□	□	□	□

To show that the Borda score for candidate D is 15, it needs to be noted that each box below D counts 2 times in the first and second columns. The Borda score for D is therefore $(4\times2)+(7\times1)=8+7=15$.

For E:

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	□	*A*	*E*	*E*
Third	□	□	*D*	□	□
Fourth	□	□	*E*	□	□
Fifth	□	□	□	□	□

To show that the Borda score for candidate E is 21, it needs to be noted that each box below E counts 2 times in the first and second columns. The Borda score for E is $(7\times2)+(7\times1)=14+7=21$.

(c) Since B has the least number of first-place votes (see Part a), B is eliminated.

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*D*	*A*
Second	*E*	*D*	*A*	*E*	*E*
Third	*D*	*C*	*D*	*A*	*C*
Fourth	*A*	*A*	*E*	*C*	*D*

A now has 1 first-place vote. C now has 2 + 1 = 3 first-place votes. D now has 1 first-place vote. E has 2 first-place votes. Since A and D have the least number of first-place votes, they are both eliminated.

Continued on next page

13. (c) continued

	Number of voters (7)				
Rank	**2**	**2**	**1**	**1**	**1**
First	*C*	*E*	*C*	*E*	*E*
Second	*E*	*C*	*E*	*C*	*C*

C now has 2 + 1 = 3 first-place votes. *E* now has 2 + 1 + 1 = 4 first-place votes. Thus, *E* is the winner by the Hare system.

(d) In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*, we first pit *A* against *B*. There are 5 voters who prefer *A* to *B* and 2 prefer *B* to *A*. Thus, *A* wins by a score of 5 to 2. *B* is therefore eliminated, and *A* moves on to confront *C*.

There are 2 voters who prefer *A* to *C* and 5 prefer *C* to *A*. Thus, *C* wins by a score of 5 to 2. *A* is therefore eliminated, and *C* moves on to confront *D*.

There are 4 voters who prefer *C* to *D* and 3 prefer *D* to *C*. Thus, *C* wins by a score of 4 to 3. *D* is therefore eliminated, and *C* moves on to confront *E*.

There are 3 voters who prefer *C* to *E* and 4 prefer *E* to *C*. Thus, *E* wins by a score of 4 to 3.

Thus, *E* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*.

14.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	*A*	*E*	*D*	*D*	*E*	*A*
Third	*E*	*E*	*D*	*A*	*A*	*A*	*E*
Fourth	*B*	*C*	*A*	*E*	*C*	*B*	*B*
Fifth	*D*	*B*	*B*	*C*	*B*	*C*	*D*

(a) *A* has 0 first-place votes. *B* has 1 first-place vote. *C* has 3 first-place votes. *D* has 2 first-place vote. *E* has 1 first-place votes. Since *C* has the most first-place votes, *C* is the winner by plurality voting.

(b)

Preference	1st place votes × 4	2nd place votes × 3	3rd place votes × 2	4th place votes × 1	5th place votes × 0	Borda score
A	0×4	3×3	3×2	1×1	0×0	16
B	1×4	0×3	0×2	3×1	3×0	7
C	3×4	0×3	0×2	2×1	2×0	14
D	2×4	2×3	1×2	0×1	2×0	16
E	1×4	2×3	3×2	1×1	0×0	17

Thus, *E* has the highest Borda score and is declared the winner.

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	*A*	*E*	*D*	*D*	*E*	*A*
Third	□	□	*D*	*A*	*A*	*A*	□
Fourth	□	□	*A*	□	□	□	□
Fifth	□	□	□	□	□	□	□

The Borda score for candidate *A* is 16 since there are 16 boxes.

Continued on next page

14. (b) continued

For *B*:

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	*A*	*E*	□	*D*	*E*	*A*
Third	*E*	*E*	*D*	□	*A*	*A*	*E*
Fourth	*B*	*C*	*A*	□	*C*	*B*	*B*
Fifth	□	*B*	*B*	□	*B*	□	□

The Borda score for candidate *B* is 7 since there are 7 boxes.

For *C*:

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	□	*A*	□	*D*	*D*	*E*	□
Third	□	*E*	□	*A*	*A*	*A*	□
Fourth	□	*C*	□	*E*	*C*	*B*	□
Fifth	□	□	□	*C*	□	*C*	□

The Borda score for candidate *C* is 14 since there are 14 boxes.

For *D*:

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	□	*E*	*D*	*D*	□	*A*
Third	*E*	□	*D*	□	□	□	*E*
Fourth	*B*	□	□	□	□	□	*B*
Fifth	*D*	□	□	□	□	□	*D*

The Borda score for candidate *D* is 16 since there are 16 boxes.

For *E*:

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	*A*	*E*	*D*	□	*E*	*A*
Third	*E*	*E*	□	*A*	□	□	*E*
Fourth	□	□	□	*E*	□	□	□
Fifth	□	□	□	□	□	□	□

The Borda score for candidate *E* is 17 since there are 17 boxes.

(c) In sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*, we first pit *A* against *B*. There are 6 voters who prefer *A* to *B* and 1 prefers *B* to *A*. Thus, *A* wins by a score of 6 to 1. *B* is therefore eliminated, and *A* moves on to confront *C*.

There are 4 voters who prefer *A* to *C* and 3 prefer *C* to *A*. Thus, *A* wins by a score of 4 to 3. *C* is therefore eliminated, and *A* moves on to confront *D*.

There are 2 voters who prefer *A* to *D* and 5 prefer *D* to *A*. Thus, *D* wins by a score of 5 to 2. *A* is therefore eliminated, and *D* moves on to confront *E*.

There are 3 voters who prefer *D* to *E* and 4 prefer *E* to *D*. Thus, *E* wins by a score of 4 to 3.

Thus, *E* is the winner by sequential pairwise voting with the agenda *A*, *B*, *C*, *D*, *E*.

Continued on next page

14. continued

(d) Since *A* has the least number of first-place votes (see Part a), *A* is eliminated.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*E*	*E*	*E*	*D*	*D*	*E*	*E*
Third	*B*	*C*	*D*	*E*	*C*	*B*	*B*
Fourth	*D*	*B*	*B*	*C*	*B*	*C*	*D*

B now has 1 first-place vote. *C* now has 3 first-place votes. *D* now has 2 first-place votes. *E* has 1 first-place vote. Since *B* and *E* have the least number of first-place votes, they are both eliminated.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*D*	*D*	*D*	*C*
Second	*D*	*C*	*D*	*C*	*C*	*C*	*D*

C now has 3 first-place votes. *D* now has 4 first-place votes. Thus, *D* is the winner by the Hare system.

15.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*B*	*E*	*D*	*C*
Second	*A*	*A*	*E*	*D*	*D*	*E*	*A*
Third	*E*	*E*	*D*	*A*	*A*	*A*	*E*
Fourth	*B*	*C*	*A*	*E*	*C*	*B*	*B*
Fifth	*D*	*B*	*B*	*C*	*B*	*C*	*D*

(a) *A* has 0 last-place votes. *B* has 3 last-place votes. *C* has 2 last-place votes. *D* has 2 last-place votes. *E* has 0 last-place votes. Since *B* has the most last-place votes, *B* is eliminated.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*C*	*D*	*C*	*D*	*E*	*D*	*C*
Second	*A*	*A*	*E*	*A*	*D*	*E*	*A*
Third	*E*	*E*	*D*	*E*	*A*	*A*	*E*
Fourth	*D*	*C*	*A*	*C*	*C*	*C*	*D*

A has 1 last-place vote. *C* has 4 last-place votes. *D* has 2 last-place votes. *E* has 0 last-place votes. Since *C* has the most last-place votes, *C* is eliminated.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*A*	*D*	*E*	*D*	*E*	*D*	*A*
Second	*E*	*A*	*D*	*A*	*D*	*E*	*E*
Third	*D*	*E*	*A*	*E*	*A*	*A*	*D*

A has 3 last-place votes. *D* has 2 last-place votes. *E* has 2 last-place votes. Since *A* has the most number of the last-place votes, *A* is eliminated.

	Number of voters (7)						
Rank	**1**	**1**	**1**	**1**	**1**	**1**	**1**
First	*E*	*D*	*E*	*D*	*E*	*D*	*E*
Second	*D*	*E*	*D*	*E*	*D*	*E*	*D*

D now has 4 last-place votes. *E* now has 3 last-place votes. Thus, *E* is the winner by the procedure of Clyde Coombs.

Continued on next page

15. continued

(b) To show that it is possible for two voters and three candidates to result in different outcomes using different methods (Coombs procedure and the Hare method), we need to find an example that illustrates such an occurrence.

One possible scenario is having candidates *A*, *B*, and *C* with the following preference lists.

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*B*	*B*
Third	*C*	*A*

Using the Coombs procedure, both *C* and *A* have the same number of last-place votes. They are both therefore eliminated leaving only *B*. *B* therefore is the winner using the Coombs procedure.

Using the Hare method, *B* has the least number of first-place votes. *B* is therefore eliminated resulting in the following preference list.

	Number of voters (2)	
Rank	**1**	**1**
First	*A*	*C*
Second	*C*	*A*

Since *A* and *C* both have the same number of first-place votes, a tie is declared.

16. (a) Condorcet's method satisfies the Pareto condition because if a candidate, say *A*, is the winner by Condorcet's method, *A* would have beaten every other candidate in a one-on-one contest. Thus, none of the other candidates would be a winner.

(b) Condorcet's method satisfies the monotonicity because if a candidate, say *A*, is the winner by Condorcet's method, *A* would have beaten every other candidate in a one-on-one contest. To move *A* higher on the preference list would not alter this outcome. *A* would still beat every other candidate in a one-on-one contest and thus would still be the winner.

17. (a) Plurality voting satisfies the Pareto condition because if everyone prefers *B* to *D*, for example, then *D* has no-first place votes at all. Thus, *D* cannot be among the winners in plurality voting.

(b) Plurality voting satisfies the monotonicity because if a candidate wins on the basis of having the most first-place votes, then moving that candidate up one spot on some list (and making no other changes) neither decreases the number of first-place votes for the winning candidate nor increases the number of first-place votes for any other candidate. Hence, the original winner remains a winner in plurality voting.

18. (a) Borda count satisfies the Pareto condition because if everyone prefers *B* to *D*, for example, then *B* receives more points from each list than *D*. Thus, *B* receives a higher total than *D* and so *D* is certainly not among the winners.

(b) Borda count satisfies the monotonicity because suppose *X*'s position with the candidate above *X* on some list adds one point to the score of *X*, subtracts one point from the score of the candidate that had been above *X* on that list, and leaves the score of all other candidates the same.

19. (a) Sequential pairwise voting satisfies the Condorcet winner criterion because a Condorcet winner always wins the kind of one-on-one contest that is used to produce the winner in sequential pairwise voting.

(b) Sequential pairwise voting satisfies the monotonicity because moving a candidate up on some list only improves that candidate's chances in one-on-one contests.

20. The Hare system satisfies the Pareto condition because if everyone prefers *B* to *D*, for example, then *D* is not on top of any list. Thus, either we have immediate winners and *D* is not among them, or the procedure moves on and *D* is eliminated at the very next stage. Hence, *D* is not among the winners.

21. In the plurality runoff method, in order to have one candidate's ranking be consistently higher than another candidate's would imply that only one candidate would be considered. This candidate would have received all first-place votes and is therefore the winner. Thus, none of the other candidates are being considered and cannot be the winner. Pareto condition is therefore satisfied.

22.

	Number of voters (5)		
Rank	**2**	**2**	**1**
First	*A*	*B*	*C*
Second	*C*	*C*	*B*
Third	*B*	*A*	*A*

Since *A* and *B* have the most number of first-place votes, *C* is eliminated.

	Number of voters (5)		
Rank	**2**	**2**	**1**
First	*A*	*B*	*B*
Second	*B*	*A*	*A*

A now has 2 first-place votes. *B* has 2 + 1 = 3 first-place votes. Since *B* has the majority of the first-place votes, *B* is the winner by the plurality runoff method.

Now using Condorcet's method, we must check the one-on-one scores of *A* versus *B*, *A* versus *C*, and *B* versus *C*.

A versus *B*: *A* is over *B* on 2 ballots, while the reverse is true on 2 + 1 = 3 ballots. Thus, *B* defeats *A*, 3 to 2.

A versus *C*: *A* is over *C* on 2 ballots, while the reverse is true on 2 + 1 = 3 ballots. Thus, *C* defeats *A*, 3 to 2.

B versus *C*: *B* is over *C* on 2 ballots, while the reverse is true on 2 + 1 = 3 ballots. Thus, *C* defeats *B*, 3 to 2.

Thus, the winner by Condorcet's method is *C*. Thus, using this example we see that the plurality runoff method does not satisfy the CWC since there is a winner by Condorcet's method (*C*) and it is not the same as the winner by the plurality runoff method (*B*).

23.

	Number of voters (13)				
Rank	**4**	**3**	**3**	**2**	**1**
First	*A*	*B*	*C*	*D*	*E*
Second	*B*	*A*	*A*	*B*	*D*
Third	*C*	*C*	*B*	*C*	*C*
Fourth	*D*	*D*	*D*	*A*	*B*
Fifth	*E*	*E*	*E*	*E*	*A*

Since *A* has the highest number of first-place votes and *B* and *C* have the same number of second-place votes, *D* and *E* are eliminated.

	Number of voters (13)				
Rank	**4**	**3**	**3**	**2**	**1**
First	*A*	*B*	*C*	*B*	*C*
Second	*B*	*A*	*A*	*C*	*B*
Third	*C*	*C*	*B*	*A*	*A*

A now has 4 first-place votes. *B* has 3 + 2 = 5 first-place votes. *C* has 3 + 1 =4 first-place votes. Since *B* has the most first-place votes, *B* is the winner by the plurality runoff method.

Now if the single voter on the far right in our original preference lists moved *B* to the top, we would have the following.

	Number of voters (13)				
Rank	**4**	**3**	**3**	**2**	**1**
First	*A*	*B*	*C*	*D*	*B*
Second	*B*	*A*	*A*	*B*	*E*
Third	*C*	*C*	*B*	*C*	*D*
Fourth	*D*	*D*	*D*	*A*	*C*
Fifth	*E*	*E*	*E*	*E*	*A*

Since *A* and *B* have the same number of first-place votes, *C*, *D* and *E* are eliminated.

	Number of voters (13)				
Rank	**4**	**3**	**3**	**2**	**1**
First	*A*	*B*	*A*	*B*	*B*
Second	*B*	*A*	*B*	*A*	*A*

A now has 4 + 3 = 7 first-place votes. *B* now has 3 + 2 + 1 = 6 first-place votes. Since *A* has the majority of the first-place votes, *A* is the winner by the plurality runoff method. This does not satisfy monotonicity since a ballot change favorable to *B* resulted in *B* losing.

24. (a) In the first election, if plurality voting is used; *A* wins with two first-place votes.

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*C*
Second	*B*	*B*	*C*	*B*
Third	*C*	*C*	*A*	*A*

If the second is calculated the same way, then *A* and *B* tie with two first-place votes.

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*
Second	*B*	*B*	*C*	*C*
Third	*C*	*C*	*A*	*A*

Thus, *B* has gone from non-winner status to winner status even though no voter reversed the order in which he or she had ranked *B* and the winning candidate from the previous election (i.e. *A*).

(b) Using the Hare system in the first election, we eliminate both *B* and *C* since they lave the least number of first-place votes. *A* is thus the winner of the first election.

If the second is calculated the same way, *C* is eliminated since *C* has the least number of first-place votes.

	Number of voters (4)			
Rank	**1**	**1**	**1**	**1**
First	*A*	*A*	*B*	*B*
Second	*B*	*B*	*A*	*A*

A and *B* now tie with two first-place votes.

Thus, *B* has gone from non-winner status to winner status even though no voter reversed the order in which he or she had ranked *B* and the winning candidate from the previous election (i.e. *A*).

25. One possible scenario is having candidates *A*, *B*, and *C* with the following preference lists.

	Number of voters (5)	
Rank	**3**	**2**
First	*A*	*B*
Second	*B*	*C*
Third	*C*	*A*

Preference	1^{st} place votes $\times$ 2	2^{nd} place votes $\times$ 1	3^{rd} place votes $\times$ 0	Borda score
A	3×2	0×1	2×0	6
B	2×2	3×1	0×0	7
C	0×2	2×1	3×0	2

Thus, *B* has the highest Borda score and is declared the winner.

Continued on next page

25. continued

Note, the Borda score for each candidate could also have been determined firstly by individually replacing the candidates below the one you are determining the score for by a box.

For *A*:

	Number of voters (5)	
Rank	**3**	**2**
First	*A*	*B*
Second	$\square$	*C*
Third	$\square$	*A*

To show that the Borda score for candidate *A* is 6, it needs to be noted that each box below *A* counts 3 times in the first column. The Borda score for *A* is $3 \times 2 = 6$.

For *B*:

	Number of voters (5)	
Rank	**3**	**2**
First	*A*	*B*
Second	*B*	$\square$
Third	$\square$	$\square$

To show that the Borda score for candidate *B* is 7, it needs to be noted that each box below *B* counts 3 times in the first column and 2 times in the second column. The Borda score for *B* is $(1 \times 3) + (2 \times 2) = 3 + 4 = 7$.

For *C*:

	Number of voters (5)	
Rank	**3**	**2**
First	*A*	*B*
Second	*B*	*C*
Third	*C*	$\square$

To show that the Borda score for candidate *C* is 2, it needs to be noted that each box below *C* counts 2 times in the second columns. The Borda score for *C* is $1 \times 2 = 2$.

Now using Condorcet's method, we must check the one-on-one scores of *A* versus *B*, *A* versus *C*, and *B* versus *C*.

A versus *B*: *A* is over *B* on 3 ballots, while the reverse is true on 2 ballots. Thus, *A* defeats *B*, 3 to 2.

A versus *C*: *A* is over *C* on 3 ballots, while the reverse is true on 2 ballots. Thus, *A* defeats *C*, 3 to 2.

B versus *C*: *B* is over *C* on all 5 ballots. Thus, *B* defeats *C*, 5 to 0.

The winner by Condorcet's method is *A*.

In this case the Borda count produces *B* as the winner while *A* is the Condorcet winner. Thus, this example shows that the Borda count does not satisfy the Condorcet winner criterion.

26.

	Number of voters (17)			
Rank	**7**	**5**	**4**	**1**
First	*A*	*C*	*B*	*D*
Second	*D*	*A*	*C*	*B*
Third	*B*	*B*	*D*	*A*
Fourth	*C*	*D*	*A*	*C*

Since *D* has the least number of first-place votes, *D* is eliminated.

	Number of voters (17)			
Rank	**7**	**5**	**4**	**1**
First	*A*	*C*	*B*	*B*
Second	*B*	*A*	*C*	*A*
Third	*C*	*B*	*A*	*C*

A now has 7 first-place votes. *B* now has 4 + 1 = 5 first-place votes. *C* now has 5 first-place votes. Since *B* and *C* have the least number of first-place votes, they are both eliminated. Thus, *A* is the winner by the Hare system.

Now suppose the voter on the far right moves *A* up. Our new preference list would be as follows.

	Number of voters (17)			
Rank	**7**	**5**	**4**	**1**
First	*A*	*C*	*B*	*D*
Second	*D*	*A*	*C*	*A*
Third	*B*	*B*	*D*	*B*
Fourth	*C*	*D*	*A*	*C*

Using the Hare system again, *D* is eliminated since *D* has the least number of first-place votes.

	Number of voters (17)			
Rank	**7**	**5**	**4**	**1**
First	*A*	*C*	*B*	*A*
Second	*B*	*A*	*C*	*B*
Third	*C*	*B*	*A*	*C*

A now has 7 + 1 = 8 first-place votes. *B* now has 4 first-place votes. *C* now has 5 first-place votes. Since *B* has the least number of first-place votes, *B* is eliminated.

	Number of voters (17)			
Rank	**7**	**5**	**4**	**1**
First	*A*	*C*	*C*	*A*
Second	*C*	*A*	*A*	*C*

A now has 7 + 1 = 8 first-place votes. *C* now has 5 + 4 = 9 first-place votes. Thus, *C* is now the winner by the Hare system.

This demonstrates nonmonotonicity since there was a change that was favorable to candidate *A*, but *A* was not the winner in the second calculation.

27.

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*B*	*C*	*D*
Second	*B*	*A*	*B*	*C*
Third	*C*	*C*	*A*	*B*
Fourth	*D*	*D*	*D*	*A*

(a) Since *D* has the least number of first-place votes, *D* is eliminated.

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*B*	*C*	*C*
Second	*B*	*A*	*B*	*B*
Third	*C*	*C*	*A*	*A*

A now has 7 first-place votes. *B* now has 6 first-place votes. *C* now has 5 + 3 = 8 first-place votes. Since *B* has the least number of first-place votes, *B* is eliminated.

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*A*	*C*	*C*
Second	*C*	*C*	*A*	*A*

A now has 7 + 6 = 13 first-place votes. *C* now has 5 + 3 = 8 first-place votes. Thus, *A* is the unique winner by the Hare system.

(b)

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*B*	*C*	*A*
Second	*B*	*A*	*B*	*D*
Third	*C*	*C*	*A*	*C*
Fourth	*D*	*D*	*D*	*B*

Since *D* has the least number of first-place votes, *D* is eliminated.

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*B*	*C*	*A*
Second	*B*	*A*	*B*	*C*
Third	*C*	*C*	*A*	*B*

A now has 7 + 3 =10 first-place votes. *B* now has 6 first-place votes. *C* now has 5 first-place votes. Since *C* has the least number of first-place votes, *C* is eliminated.

	Number of voters (21)			
Rank	**7**	**6**	**5**	**3**
First	*A*	*B*	*B*	*A*
Second	*B*	*A*	*A*	*B*

A now has 7 + 3 = 10 first-place votes. *B* now has 6 + 5 = 11 first-place votes. Thus, *B* is the winner by the Hare system.

28. (a) In sequential pairwise voting, if there is an odd number of voters then each one-on-one competition will yield a unique candidate that has preference over the other because a majority of voters will have a preference. Since each of these one-on-one competitions yields a unique winner, ultimately there will be a unique winner at the end of the sequential one-on-one competitions.

(b) In the Hare system, if the final comparison is between two candidates they cannot both have the same number of votes if the number of voters is odd. Thus, we cannot have exactly two winners.

29. If a candidate, say *B*, is ranked last on a majority of votes (over 50% of the votes are for last-place) then this candidate may or may not be considered in the runoff. Obviously, if this candidate is not considered in the runoff, then this candidate cannot be among the winners.

Now suppose this candidate is considered in the runoff. Since we assume there are no ties for second-place, then candidate *B* is having a runoff against another candidate, say candidate *A*. Since candidate *B* has the majority of the last-place votes, candidate *A* must have the majority of the first place votes and is hence the winner.

30.

	Number of voters (7)		
Rank	**3**	**2**	**2**
First	*C*	*A*	*B*
Second	*A*	*B*	*A*
Third	*B*	*C*	*C*

C has the majority of the last-place votes (over 50%), but is the winner by the plurality voting since *C* has the highest number of first-place votes.

31. Consider the following set of preference lists.

Rank	**Number of voters (3)**		
First	*A*	*B*	*A*
Second	*B*	*C*	*C*
Third	*C*	*A*	*B*

Checking the one-on-one scores of *A* versus *B* and *A* versus *C*, we see that candidate *A* is a Condorcet winner.

A versus *B*: *A* is over *B* on 2 of the ballots, while the reverse is true on 1. Thus, *A* defeats *B*, 2 to 1.

A versus *C*: *A* is over *C* on 2 of the ballots, while the reverse is true on 1. Thus, *A* defeats *C*, 2 to 1.

Since candidate *A* is the Condorcet winner, it must be the unique winner under our hypothetical voting rule. Therefore, *A* is a winner and *B* is a nonwinner for these preference lists.

Because our hypothetical voting rule satisfies IIA, we know that candidate *B* will remain a non-winner as long as no voter reverses his or her ordering of *B* and *A*. But to arrive at the preference lists from the voting paradox, we can move *C* (the candidate that is irrelevant to *A* and *B*) up one slot on the third voter's list.

Rank	**Number of voters (3)**		
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Thus, because of IIA, we know that candidate *B* is a nonwinner when our voting rule is confronted by the preference lists from the voting paradox of Condorcet.

Continued on next page

31. continued

Now consider the following set of preference lists.

Rank	Number of voters (3)		
First	*B*	*B*	*C*
Second	*A*	*C*	*A*
Third	*C*	*A*	*B*

Checking the one-on-one scores of *B* versus *A* and *B* versus *C*, we see that candidate *B* is a Condorcet winner.

B versus *A*: *B* is over *A* on 2 of the ballots, while the reverse is true on 1. Thus, *B* defeats *A*, 2 to 1.

B versus *C*: *B* is over *C* on 2 of the ballots, while the reverse is true on 1. Thus, *B* defeats *C*, 2 to 1.

Since candidate *B* is the Condorcet winner, it must be the unique winner under our hypothetical voting rule. Therefore, *B* is a winner and *C* is a nonwinner for these preference lists.

Because our hypothetical voting rule satisfies IIA, we know that alternative *C* will remain a non-winner as long as no voter reverses his or her ordering of *C* and *B*. But to arrive at the preference lists from the voting paradox, we can move *A* (the candidate that is irrelevant to *B* and *C*) up one slot on the first voter's list.

Rank	Number of voters (3)		
First	*A*	*B*	*C*
Second	*B*	*C*	*A*
Third	*C*	*A*	*B*

Thus, because of IIA, we know that candidate *C* is a nonwinner when our voting rule is confronted by the preference lists from the voting paradox of Condorcet.

32.

	Voters									
Candidates	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
A	X	X	X			X	X	X		X
B		X	X	X	X	X	X	X	X	
C			X					X		
D	X	X	X	X	X		X	X	X	X
E	X		X		X		X		X	
F	X		X	X	X	X	X	X		X
G	X	X	X	X	X			X		
H		X		X		X		X		X

A has 7 approval votes, *B* has 8, *C* has 2, *D* has 9, *E* has 5, *F* has 8, *G* has 6, and *H* has 5. Ranking the candidates we have, *D* (9), *B* and *F* (8), *A* (7), *G* (6), *E* and *H* (5), and *C* (2).

(a) Since *D* has the most votes, *D* is chosen for the board.

(b) The top four are *A*, *B*, *D*, and *F*.

(c) Candidates *B*, *D* and *F* have at least 80% (8 out of 10) approval.

(d) Candidates *A*, *B*, *D*, *F* and *G* have at least 60% (6 out of 10) approval. But since at most four candidates can be elected, only *A*, *B*, *D*, and *F* are considered.

33.

Nominee	Number of voters (45)							
	7	**8**	**9**	**9**	**6**	**3**	**1**	**2**
A	X			X	X		X	
B		X		X		X	X	
C			X		X	X	X	

A has $7+9+6+1=23$ approval votes. *B* has $8+9+3+1=21$ approval votes.

C has $9+6+3+1=19$ approval votes.

(a) Since *A* has the most approval votes, *A* is selected for the award.

(b) Since *B* has the second highest number of approval votes, *B* is announced as the runner-up.

(c) Suppose *A* had *a* approval votes, *B* had *b*, and *C* had *c*, where $a \geq b \geq c$. If an additional voter abstains, then clearly there is no change in the arrangement since $a+0=a$, $b+0=b$, and $c+0=c$. If an additional voter approves all nominees, then *A* would have $a+1$ votes, *B* would have $b+1$ votes, and *C* would have $c+1$ votes. Since we assumed $a \geq b \geq c$, clearly $a+1 \geq b+1 \geq c+1$. Thus, there would be no difference in the arrangement of the nominees.

Word Search Solution

```
P L U R A L I T Y R U N O F F A I K X A S T F S S N
P L B E K P F I E G U C R P S E Y N I G X I U C Z C
R R I N M B L S G J M J J O G D C S A E H I D S Q X
A M B N I C S Y S B P H H N O P C M R N X A S J N K
P L A I V Z E P V T F L H X M A H E T D E M I E K A
S N A W Y K A F E X I S S D M N H E O A A N N N T M
G N I T O V L A V O R P P A T N U O C A D R O B A S
O B Q E C N G S N E E F J K H I G S S R M T F P G Y
A N Z C S W R M M O N O T O N I C I T Y F J O T A T
I Q F R E W C E P I R G N I T O V Y T I L A R U L P
M E R O E H T Y T I L I B I S S O P M I S W O R R A
C O N D O R C E T S V O T I N G P A R A D O X Y X R
P L A N R M A Y S T H E O R E M O C I O G E G W N E
T E P O B I R S E T V V S V N D S T L S G I O P R T
Z E G C R U M F B R S B T D R X R G H M M L D H D O
R S H F L A V I T R H D L P M J T M T Z N U E A F C
A L I E D A R C F E E P S T A A B P O S E S U R Q O
T O L L A B T S I L E C N E R E F E R P E K C E N N
G N I T O V E S I W R I A P L A I T N E U Q E S R D
J C Q C X T L N C A Y P O Y C E X F S N V E N Y L I
R X S N S T O K T N I H I O A C N M S N U A I S A T
F M M R C I N E T N P Z E N E C H W F S V P E T F I
F R R A P E X E R B F P O G T D U I I Q H A E E T O
N B S N A O H M E I N T E E O X L F U Q C E R M D N
C K U C N Z P T E P N E N E W H X N Y T P T N X A N
X W I Z N E H L Q A L H F D O O A T S J O Y J A J M
```