Chapter 8
Probability: The Mathematics of Chance

Chapter Outline

Chapter Summary

Probability is the mathematics of random phenomena. For such phenomena, individual outcomes are uncertain but, in the long run, a regular pattern describes how frequently each outcome occurs.

A probability model for a random phenomenon consists of a sample space, which is the set of all possible outcomes, and a way of assigning probabilities to events (sets of outcomes). There are two important ways of assigning probabilities. First, assign a probability to each outcome and then determine the probability of an event by adding the probabilities of the outcomes that comprise the event. This method is particularly appropriate for finite sample spaces. Often counting methods (*combinatorics*) is used to determine how many elements are in the sample space or in a subset of the same space. Secondly, and this method is useful if the outcomes are numbers, we can assign probabilities directly to intervals of numbers as areas under a curve.

In either case, the probability of an event must be a number between 0 and 1, and the probabilities of all outcomes must add up to 1 (interpreted in the second case as: the total area under the curve is exactly 1). Moreover, if two events A and B are disjoint (meaning that they have no outcomes in common), then $P(A \text{ or } B) = P(A) + P(B)$. In the particular case of a sample space having k outcomes that are equally likely, these conditions imply that each outcome must be assigned probability $\frac{1}{k}$. A *probability histogram* gives a visual representation of a probability model. The height of each bar gives the probability of the outcome at its base, and the sum of the heights is 1.

For a random phenomenon with numerical outcomes, the average outcome to expect in the long run is called its mean, denoted μ. The mean is a weighted average of the outcomes, each outcome weighted by its probability. The law of large numbers tells us that the mean, $\bar{x}$, of actually observed outcomes will approach μ as the number of observations increases.

Probability density curves (or just density curves) are important in assigning probabilities. *Continuous probability models* such as the *uniform distribution* or the *normal distribution* assign probabilities as area under the curves. Since any normal distribution is symmetric about its mean and satisfies the 68–95–99.7 rule, this distribution is used in a variety of applications.

Sampling distributions are important in statistical inference. Random sampling ensures that each sample is equally likely to be chosen. Any number computed from a sample is called a statistic, and the term *sampling distribution* is applied to the distribution of any statistic. In particular, a statistic is a random phenomenon. An important statistic is the sample mean, $\bar{x}$. The central limit theorem tells us that the sampling distribution of this statistic is approximately normal if the sample size is large enough.

Skill Objectives

1. Explain what is meant by random phenomenon.

2. Describe the sample space for a given random phenomenon.

3. Explain what is meant by the probability of an outcome.

4. Describe a given probability model by its two parts.

5. List and apply the four rules of probability and be able to determine the validity/invalidity of a probability model by identifying which rule(s) is (are) not satisfied.

6. Compute the probability of an event when the probability model of the experiment is given.

7. Apply the addition rule to calculate the probability of a combination of several disjoint events.

8. Draw the probability histogram of a probability model, and use it to determine probabilities of events.

9. Explain the difference between a discrete and a continuous probability model.

10. Determine probabilities with equally likely outcomes.

11. Use the fundamental principal of counting to determine the number of possible outcomes involved in an event and/or the sample space.

12. List two properties of a density curve.

13. Construct basic density curves that involve geometric shapes (rectangles and triangles) and utilize them in determining probabilities.

14. State the mean and calculate the standard deviation of a sample statistic $\left(\hat{p}\right)$ taken from a normally distributed population.

15. Explain and apply the 68–95–99.7 rule to compute probabilities for the value of $\hat{p}$ from a single simple random sample (SRS).

16. Compute the mean (μ) and standard deviation (σ) of an outcome when the associated probability model is defined.

17. Explain the significance of the law of large numbers.

18. Explain the significance of the central limit theorem.

Teaching Tips

1. Probability experiments (binomial) such as tossing coins, answering questions on a true/false test, or recording the sex of each child born to a family provide an easy-to-understand approach to probability. Tree diagrams can be useful in such examples, but you may want to use the columnar-list approach.

Coin #1	Coin #2	Coin #3
H	H	H
H	H	T
H	T	H
H	T	T
T	H	H
T	H	T
T	T	H
T	T	T

Using this diagram to count the number of times a specific event occurs in the sample space helps some students set up numerical values for the probability model. It's then interesting to note that other experiments that have only two outcomes behave the same way structurally.

2. The concept of the mean of a probability model (expected value) seems to be more easily understood by some when it is placed in a monetary context. The example of betting $1 on red in a roulette game generates student interest, and the resulting mean has meaning. Initially applying the same concept to an event that is not associated with money, however, tends not to be as interesting and therefore can be confusing. Giving an explanation of mean as a kind of average and then discussing average winnings may help put it in perspective.

3. Students will need to apply the 68-95-99.7 rule throughout this chapter. Although the rule and applications were given in Chapter 5, you may choose to have them concentrate on the following diagram by inserting a specific mean and labeling the values that are one, two, and three standard deviations from the mean. Also, you may mention that in the *Student Study Guide* a page of "blank" normal distributions such as the one below appear after the Homework Help feature.
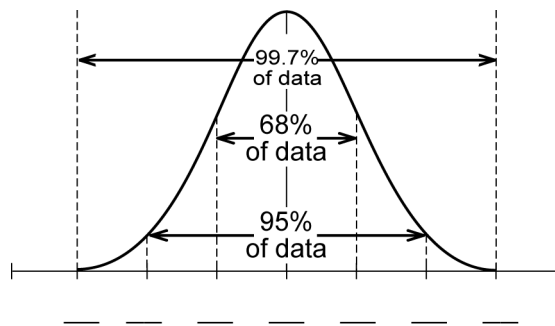
4. The first two text exercises provide nice hands-on activities that can pay dividends in terms of student understanding. Some students need this tactile approach to reinforce the concepts.

5. Another readily accessible source of a distribution of digits is a phone book. You may choose to tear pages out and ask students to collect information such as how many numbers end in an even or an odd digit. You may consider including or not the first three digits of the telephone number in other data collection activities.

6. For students who enjoy gambling, analyzing the game of craps with respect to the probability of winning on the first roll or losing on the first roll is a fairly simple example. Students seem to enjoy problems involving the rolling of dice.

Research Paper

A famous equation in fluid dynamics is the Bernoulli equation. It was derived by the Dutch-born mathematician, Daniel Bernoulli (1700–1782). The family name Bernoulli is also a prominent part of probability theory. Daniel Bernoulli's uncle Jakob Bernoulli (1654–1705) wrote *Ars Conjectandi* (the Art of Conjecturing). This work was groundbreaking in probability theory. In the binomial distribution, in which experiments yield a success or failure, the terms Bernoulli experiment or Bernoulli trial are used. These terms are a result Jakob's body of work which was published after his death. Students can further research the life of Bernoulli family members. To focus on probability theory, direct students to research only Jakob Bernoulli.

Note: Jakob Bernoulli's first name can also appear as Jacob, James, or Jacque.

Collaborative Learning

Estimating Probability

This exercise, involves tossing a fair coin and an unmarked (no H's or T's) version of the diagram below. An unmarked copy of the diagram below along with a table to organize the experimental results appear on the next page. Break students into groups in which they start at the top of the triangle. A student should toss the coin. If the coin lands tails, students should follow the path down and to the right. If the coin lands heads, students should follow the path down and to the left. It will take three tosses in order to land at a point (A, B, C, or D).
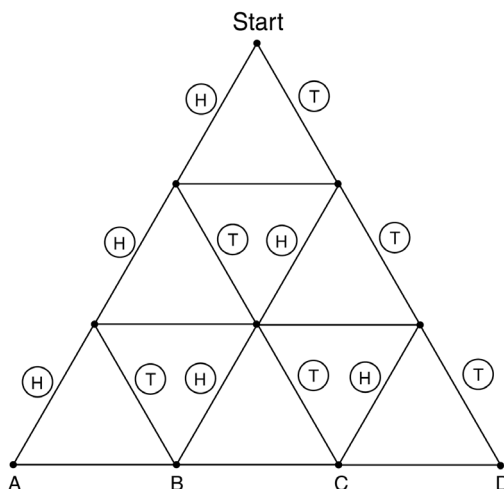

Have students in a group perform this experiment (tossing the coin three times and recording the terminal point) 40 times. Combine the results of each group on the board. Have the class find the experimental probability of terminating at one of the four points for the collective results.

Bear in mind before you perform this experiment that many students will assume the probability of landing at any of the four terminal points must be 0.25, "because there are only four possibilities, A, B, C or D."

After the results are combined, ask students to determine the possible ways of obtaining each terminal point by first examining the sample space of tossing a coin three times. After the pattern of exactly 3 heads for A, 2 heads for B, 1 head for C, 0 heads for D (or similar phrasing), ask students to construct the actual *probability model* and the *probability histogram*.

Follow this up by asking students to construct the probability model and the probability histogram based on an expanded version of the experiment. They do not need to actually perform the experiment.

Start

	A	B	C	D
Tally				

Start
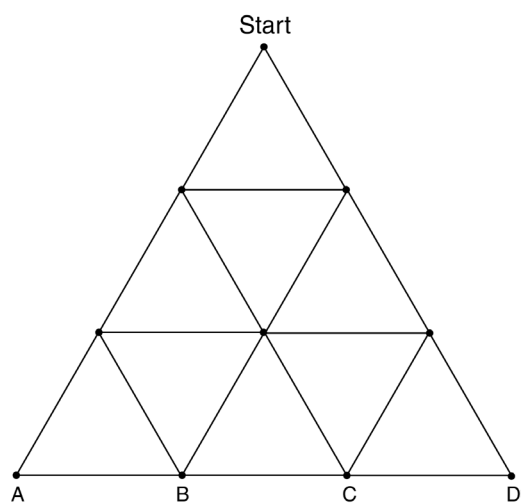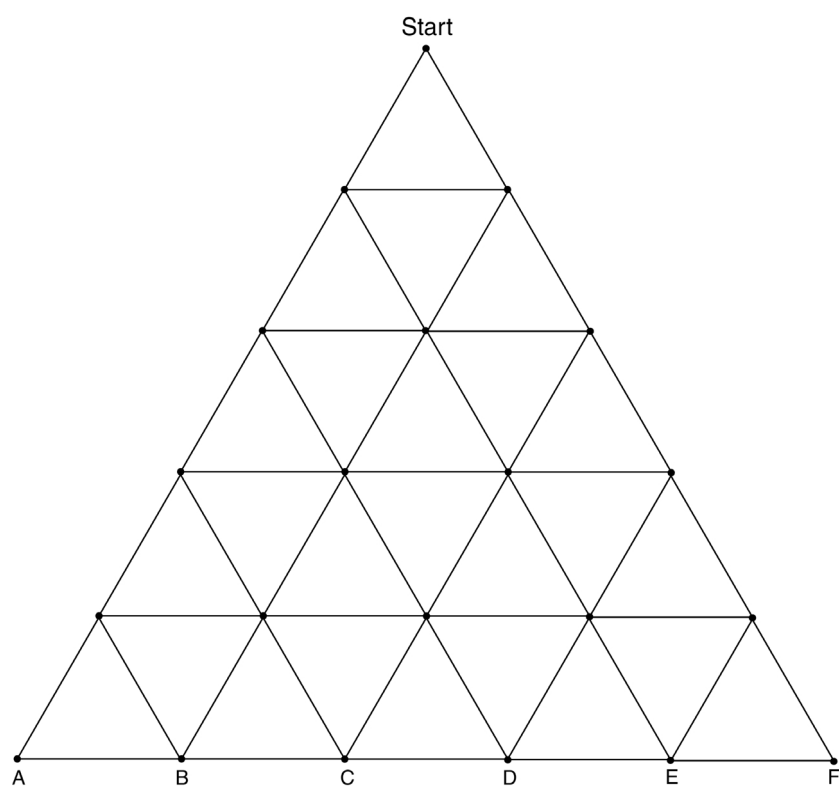
Solutions

Skills Check:

1. a   2. b   3. b   4. c   5. a   6. b   7. b   8. c   9. b   10. a

11. b   12. c   13. a   14. c   15. c   16. b   17. a   18. a   19. c   20. c

Exercises:

1. Results will vary, but the probability of a head is usually greater than 0.5 when spinning pennies. One possible explanation is the "bottle cap effect." The rim on a penny is slightly wider on the head side, so just as spinning bottle caps almost always fall with the open side up, pennies fall more often with the head side up.

2. Results will vary.

3. The first five lines contain 200 digits, of which 21 are zeros. The proportion of zeros is $\frac{21}{200} = 0.105.$

TABLE 7.1	Random	Digits						
101	19223	95034	05756	28713	96409	12531	42544	82853
102	73676	47150	99400	01927	27754	42648	82425	36290
103	45467	71709	77558	00095	32863	29485	82226	90056
104	52711	38889	93074	60227	40011	85848	48767	52573
105	95592	94007	69971	91481	60779	53791	17297	59335
106	68417	35013	15529	72765	85089	57067	50211	47487

4. (a) Probability 0.
    (b) Probability 1.
    (c) Probability 0.01, once per 100 trials on the average in the long run.
    (d) Probability 0.6.

5. (a) $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
    (b) $S = \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$.
    (c) $S = \{\text{Yes, No}\}$.

6. (a) $S = \{\text{Female, Male}\}$.
    (b) $S = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.
    (c) $S = $ whole numbers from 50 to 180 (use judgment for lower and upper limits).

7. (a) $S = \{$HHHH, HHHM, HHMH, HMHH, MHHH, HHMM, HMMH, MMHH, HMHM, MHHM, MHMH, HMMM, MMMH, MHMM, MMHM, MMMM$\}$.
    (b) $S = \{0, 1, 2, 3, 4\}$.

8. (a) $S = \{\text{Right; Left}\}$.
    (b) $S = $ whole numbers from 48 to 84 (use judgment for lower and upper limits).
    (c) $S = $ whole numbers from 0 to 360 (use judgment for lower and upper limits).

9. (a) The given probabilities have sum 0.81, so the probability of any other topic is $1 - 0.81 = 0.19.$
    (b) The probability of adult or scam is $0.145 + 0.142 = 0.287.$

10. (a) $1-0.71=0.28,$ because the probabilities of all education levels must sum to 1.

(b) $1-0.12=0.88,$ or $0.31+0.28+0.29=0.88.$

11. Answers will vary. Any two events that can occur together will do.

A = a student is female and B = a student is taking a mathematics course.
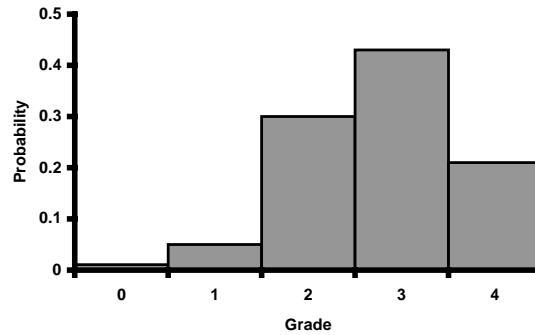
12. (a) The probability of choosing one of the most popular colors is as follows.

$$0.201+0.184+0.116+0.115+0.088+0.085=0.789$$

Thus, the probability of choosing any other color than the six listed is $1-0.789=0.211.$

(b) $0.201+0.184=0.385.$

13. (a) Here is the probability histogram:

(b) $0.43+0.21=0.64.$

14. The probability histograms show that owner-occupied housing units tend to have more rooms than rented units. The center is around 6 rooms, as opposed to around 4 rooms for rented housing. Presumably more of the owner-occupied units are houses, while more rented units are apartments. The distribution for rented units is also more strongly peaked.
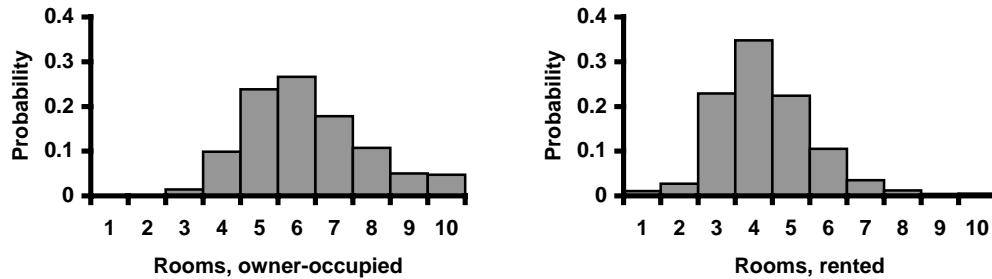
15. (a) Yes: the probabilities are between 0 and 1, inclusively, and have sum 1.

$$0+\tfrac{1}{6}+\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{6}+0=\tfrac{1}{6}+\tfrac{2}{6}+\tfrac{2}{6}+\tfrac{1}{6}=\tfrac{6}{6}=1$$

(Think of a die with no 1 or 6 face and two 3 and 4 faces.)

(b) No: the probabilities are between 0 and 1, but the sum is greater than 1.

$$0.56+0.24+0.44+0.17=1.41$$

(c) Yes: the probabilities are between 0 and 1, inclusively, and have sum 1.

$$\tfrac{12}{52}+\tfrac{12}{52}+\tfrac{12}{52}+\tfrac{16}{52}=\tfrac{52}{52}=1$$

16. For owner-occupied units, we have the following probability.

$$P(5,6,7,8,9,10) = 0.238 + 0.266 + 0.178 + 0.107 + 0.050 + 0.047 = 0.866$$

For rented units, we have the following probability.

$$P(5,6,7,8,9,10) = 0.224 + 0.105 + 0.035 + 0.012 + 0.004 + 0.005 = 0.385$$

17. Each count between 1 and 12 occurs 3 times in the 36 possible outcomes. For example, 1 and 7 can only occur when the first die shows a 1.

Outcome	1	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

18. There are 16 possible outcomes for the two dice, all equally likely $\left(\text{probability } \frac{1}{16}\right)$. Counting outcomes and adding 1 to the sum gives the model

Intelligence	3	4	5	6	7	8	9
Probability	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

The probability of intelligence 7 or higher is $\frac{3}{16} + \frac{2}{16} + \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$.

19. All 90 guests are equally likely to get the prize, so $P(\text{woman}) = \dfrac{42}{90} = \dfrac{7}{15}$.

20. (a) Using first letters to stand for names, the possible choices are: AD, AJ, AS, AR, DJ, DS, DR, JS, JR, SR.

(b) There are 10 choices, so each has probability $\dfrac{1}{10} = 0.1$.

(c) Four choices include Julie, so the probability is $\dfrac{4}{10} = 0.4$.

(d) Three choices qualify, so the probability is $\dfrac{3}{10} = 0.3$.

21. (a) $2\times2\times2\times2\times2\times2\times2\times2\times2\times2 = 2^{10} = 1024.$

(b) $\dfrac{2}{1024} = \dfrac{1}{512}.$

22. (a) $24\times24\times24 = 24^3 = 13,824.$

(b) $\dfrac{14\times14\times14}{13,824} = \dfrac{2744}{13,824} = \dfrac{343}{1728} \approx 0.1985.$

23. There are $36\times36\times36 = 36^3 = 46,656$ different codes. The probability of no x is as follows.

$$\frac{35\times35\times35}{46,656} = \frac{42,875}{46,656} \approx 0.919$$

The probability of no digits is $\dfrac{26\times26\times26}{46,656} = \dfrac{17,576}{46,656} = \dfrac{2197}{5832} \approx 0.377.$

24. $\dfrac{14\times10\times14}{24\times24\times24} = \dfrac{1960}{13,824} = \dfrac{245}{1728} \approx 0.1418.$

25. The possibilities are *ags*, *asg*, *gas*, *gsa*, *sag*, *sga*, of which "*gas*" and "*sag*" are English words. The probability is $\frac{2}{6} = \frac{1}{3} \approx 0.333.$

26. The number of IDs is the sum of the numbers of 3-, 4-, and 5-character IDs, or the following.

$$26^3 + 26^4 + 26^5 = 17,576 + 456,976 + 11,881,376 = 12,355,928$$

The number of IDs with no repeats, again adding over the three ID lengths, is as follows.

$$(26)(25)(24)+(26)(25)(24)(23)+(26)(25)(24)(23)(22)=8,268,000$$

The probability of no repeats is $\dfrac{8,268,000}{12,355,928} \approx 0.669.$

27. There are $26\times36^2 + 26\times36^3 + 26\times36^4 = 44,916,768$ possible IDs. The number of IDs with no numbers is the sum of the numbers of 3-, 4-, and 5-character IDs, or the following.

$$26^3 + 26^4 + 26^5 = 15,576 + 456,976 + 11,881,376 = 12,355,928.$$

The probability is therefore $\dfrac{12,355,928}{44,916,768} \approx 0.275.$

28. (a) The probability for each square face is $\frac{0.72}{6} = 0.12$ because the 6 square faces are equally likely. The probability of a triangle is $1 - 0.72 = 0.28$, so the probability for each triangle face is $\frac{0.28}{8} = 0.035.$

(b) Answers will vary.
Start with a different probability for squares. If each square face has probability 0.1, the 6 square faces have combined probability 0.6, the 8 triangle faces have combined probability 0.4, and each triangle face has probability 0.05.
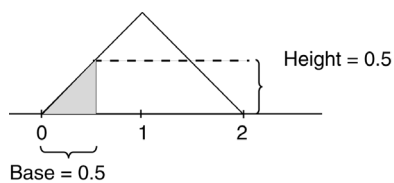
29. (a) The area is $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2)(1) = 1$.
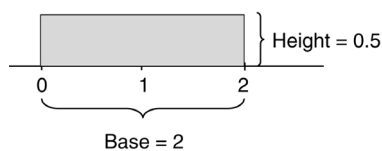

Base = 2
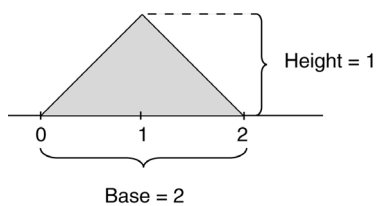
(b) Probability $\frac{1}{2}$ by symmetry or finding the area, $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(1)(1) = \frac{1}{2}$..

Base = 1
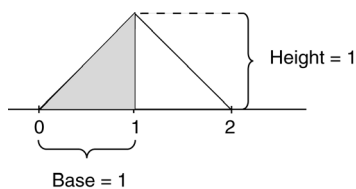
(c) The area representing this event is $\left(\frac{1}{2}\right)(0.5)(0.5) = 0.125$.
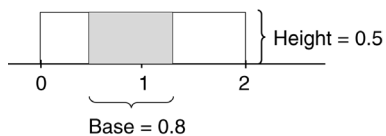

Base = 0.5

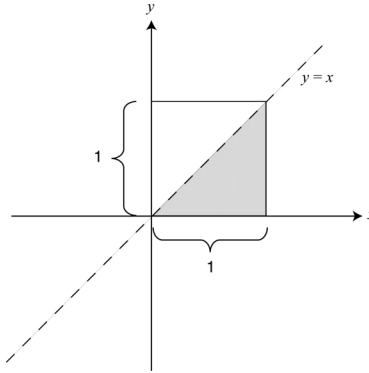**30.** (a) Height 0.5 between 0 and 2, height 0 elsewhere.

Base = 2

(b) Probability $\frac{1}{2}$ by symmetry or finding the area, $\text{base} \times \text{height} = (1)(0.5) = 0.5$.

Base = 1
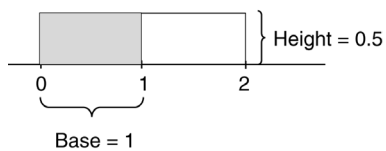
(c) Probability = Area = $\text{base} \times \text{height} = (0.8)(0.5) = 0.4$.

Base = 0.8

31. The area is the half of the square below the $y = x$ line.   The probability is the area, $\left(\frac{1}{2}\right)(1)(1) = \frac{1}{2}$.

The area is half the square.

32. Because earnings are \$400 times sales, the probability model is as follows.

Earnings	\$0	\$400	\$800	\$1200
Probability	0.3	0.4	0.2	0.1

The mean for this model is as follows.

$$(0)(0.3)+(400)(0.4)+(800)(0.2)+(1200)(0.1) = \ 0+160+160+120 = \$440$$

33. The mean is as follows.

$$\mu = (0)(0.01)+(1)(0.05)+(2)(0.30)+(3)(0.43)+(4)(0.21)$$
$$= \ 0+0.05+0.60+1.29+0.84 = 2.78$$

The variance is as follows.

$$\sigma^2 = (0-2.78)^2(0.01)+(1-2.78)^2(0.05)+(2-2.78)^2(0.30)+(3-2.78)^2(0.43)+(4-2.78)^2(0.21)$$
$$= (-2.78)^2(0.01)+(-1.78)^2(0.05)+(-0.78)^2(0.30)+(0.22)^2(0.43)+(1.22)^2(0.21)$$
$$= (7.7284)(0.01)+(3.1684)(0.05)+(0.6084)(0.30)+(0.0484)(0.43)+(1.4884)(0.21)$$
$$= 0.077284+0.15842+0.18252+0.020812+0.312564$$
$$= 0.7516$$

Thus, the standard deviation is $\sigma = \sqrt{0.7516} \approx 0.8669$.

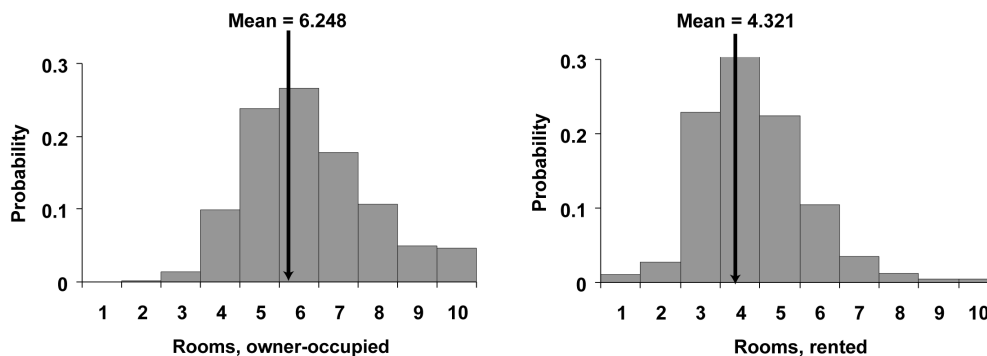**34.** The mean intelligence is

$$\mu = (3)\left(\frac{1}{16}\right)+(4)\left(\frac{2}{16}\right)+(5)\left(\frac{3}{16}\right)+(6)\left(\frac{4}{16}\right)+(7)\left(\frac{3}{16}\right)+(8)\left(\frac{2}{16}\right)+(9)\left(\frac{1}{16}\right)$$
$$= \frac{3}{16}+\frac{8}{16}+\frac{15}{16}+\frac{24}{16}+\frac{21}{16}+\frac{16}{16}+\frac{9}{16} = \frac{96}{16} = 6,$$

as the symmetry of the model demands.

35. The mean for owner-occupied units is $\mu = (1)(0.000) + (2)(0.001) + ... + (10)(0.047) = 6.248$.

For rented units, $\mu = (1)(0.011) + (2)(0.027) + ... + (10)(0.005) = 4.321$.

36. For nonword errors, we have the following.

$$\mu = (0)(0.1) + (1)(0.2) + (2)(0.3) + (3)(0.3) + (4)(0.1)$$
$$= 0 + 0.2 + 0.6 + 0.9 + 0.4 = 2.1$$

For word errors, we have the following.

$$\mu = (0)(0.4) + (1)(0.3) + (2)(0.2) + (3)(0.1) = 0 + 0.3 + 0.4 + 0.3 = 1$$

The models show that there are likely to be fewer word errors than nonword errors, and the smaller mean number of word errors describes this fact.

37. Both models have mean 1, because both density curves are symmetric about 1.

38. Answers will vary.

Selling 12 policies collects just $3000 plus costs and profit. One loss, though unlikely, would be catastrophic. If the company sells thousands of policies, the law of large numbers says that its mean payout per policy will be very close to the average loss of $250. It gets to keep its costs and profit.

39. (a) $\mu = (1)\left(\frac{1}{6}\right) + (2)\left(\frac{1}{6}\right) + (3)\left(\frac{1}{6}\right) + (4)\left(\frac{1}{6}\right) + (5)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{6}\right) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = 3.5$.

(b)

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{12}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The mean is as follows.

$$\mu = (2)\left(\frac{1}{36}\right) + (3)\left(\frac{2}{36}\right) + (4)\left(\frac{3}{36}\right) + (5)\left(\frac{4}{36}\right) + (6)\left(\frac{5}{36}\right) + (7)\left(\frac{6}{36}\right)$$
$$+ (8)\left(\frac{5}{36}\right) + (9)\left(\frac{4}{36}\right) + (10)\left(\frac{3}{36}\right) + (11)\left(\frac{2}{36}\right) + (12)\left(\frac{1}{36}\right)$$
$$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} = \frac{252}{36} = 7$$

(c) Answers will vary.

We could roll two dice separately and add the spots later. We expect the average outcome for two dice to be twice the average for one die. Remember that expected values are averages, so they behave like averages.

40. (a) Twelve of the 38 slots win, so the probability of winning is $\frac{12}{38}$. The probability model is as follows.

Outcome	Win $2	Lose $1
Probability	$\frac{12}{38}$	$\frac{26}{38}$

(b) Joe gains $2 if he wins and otherwise loses $1. So, the mean is the following.

$$\mu = (2)\left(\tfrac{12}{38}\right) + (-1)\left(\tfrac{26}{38}\right) = \tfrac{24}{38} + \left(-\tfrac{26}{38}\right) = -\tfrac{2}{38} \approx -\$0.053 \text{ (a loss of 5.3 cents)}$$

This is the same as the mean for bets on red or black in Example 13. The variance is as follows.

$$\left[2-(-0.053)\right]^2 \left(\tfrac{12}{38}\right) + \left[-1-(-0.053)\right]^2 \left(\tfrac{26}{38}\right) = (2.053)^2 \left(\tfrac{12}{38}\right) + (-0.947)^2 \left(\tfrac{26}{38}\right)$$

$$= (4.214809)\left(\tfrac{12}{38}\right) + (0.896809)\left(\tfrac{26}{38}\right)$$

$$= \tfrac{50.577708}{38} + \tfrac{23.317034}{38} = \tfrac{73.894742}{38} \approx 1.9446$$

The standard deviation is $\sqrt{1.9446} \approx 1.394$.

(c) The law of large numbers says that in the very long run Joe will lose an average of close to 5.3 cents per bet.

41. (a) $\mu = (1)\left(\tfrac{1}{6}\right) + (3)\left(\tfrac{1}{6}\right) + (4)\left(\tfrac{1}{6}\right) + (5)\left(\tfrac{1}{6}\right) + (6)\left(\tfrac{1}{6}\right) + (8)\left(\tfrac{1}{6}\right) = \tfrac{1}{6} + \tfrac{3}{6} + \tfrac{4}{6} + \tfrac{5}{6} + \tfrac{6}{6} + \tfrac{8}{6} = \tfrac{27}{6} = 4.5.$

(b) $\mu = (1)\left(\tfrac{1}{6}\right) + (2)\left(\tfrac{2}{6}\right) + (3)\left(\tfrac{2}{6}\right) + (4)\left(\tfrac{1}{6}\right) = \tfrac{1}{6} + \tfrac{4}{6} + \tfrac{6}{6} + \tfrac{4}{6} = \tfrac{15}{6} = 2.5.$

(c) The mean count for the two dice is 7. This is the same as for rolling two standard dice, with mean 3.5 for each. See the answer to Exercise 39.

42. Your digits can appear in six orders, so six of 1000 three-digit numbers win. So the mean is $\mu = (81.33)(0.006) + (-1)(0.994) = 0.48798 + (-0.994) = -0.50602$, or essentially an average loss per ticket of 51 cents.

43. (a) Since $0.00039 + 0.00044 + 0.00051 + 0.00057 + 0.00060 = 0.00251$,   the probability is therefore $1 - 0.00251 = 0.99749$.

(b) The probability model for the company's cash intake is as follows.

Probability	Outcome
0.00039	$175 - 100,000 = -99,825$
0.00044	$2(175) - 100,000 = -99,650$
0.00051	$3(175) - 100,000 = -99,475$
0.00057	$4(175) - 100,000 = -99,300$
0.00060	$5(175) - 100,000 = -99,125$
0.99749	875

From this table, the mean is as follows.

$$(-99,825)(0.00039) + (-99,650)(0.00044) + (-99,475)(0.00051)$$

$$+ (-99,300)(0.00057) + (-99,125)(0.00060) + (875)(0.99749)$$

$$= (-38.93175) + (-43.846) + (-50.73225) + (-56.601) + (-59.475) + 872.80375$$

$$= 623.21775 \approx 623.218$$

44. The mean is $(\mu - \sigma)(0.5) + (\mu + \sigma)(0.5) = 0.5\mu - 0.5\sigma + 0.5\mu + 0.5\sigma = \mu$

The variance is as follows.

$$[(\mu - \sigma) - \mu]^2 (0.5) + [(\mu + \sigma) - \mu]^2 (0.5) = (\mu - \sigma - \mu)^2 (0.5) + (\mu + \sigma - \mu)^2 (0.5)$$
$$= \sigma^2 (0.5) + \sigma^2 (0.5) = \sigma^2$$

45. Sample means $\bar{x}$ have a sampling distribution close to normal with mean $\mu = 0.15$ and standard deviation $\dfrac{\sigma}{\sqrt{n}} = \dfrac{0.4}{\sqrt{400}} = \dfrac{0.4}{20} = 0.02.$ Therefore, 95% of all samples have an $\bar{x}$ between $0.15 - 2(0.02) = 0.15 - 0.04 = 0.11$ and $0.15 + 2(0.02) = 0.15 + 0.04 = 0.19$.

46. (a) The mean is 300, so the probability of a higher score is about 0.5. A score of 335 is one standard deviation above the mean, so by the 68 part of the 68-95-99.7 rule the probability of a higher score is half of 0.32, or 0.16.

(b) The average score of $n = 4$ students has mean 300 and standard deviation. $\dfrac{\sigma}{\sqrt{n}} = \dfrac{35}{\sqrt{4}} = \dfrac{35}{2} = 17.5.$ The probability of an average score higher than 300 is still 0.5 Because 335 is now two standard deviations above the mean, the 95 part of the 68-95-99.7 rule says that the probability of a higher average score is 0.025.

47. (a) The standard deviation of the average measurement is $\dfrac{\sigma}{\sqrt{n}} = \dfrac{10}{\sqrt{3}} \approx 5.77$ mg.

(b) To cut the standard deviation in half (from 10 mg to 5 mg), we need $n = 4$ measurements because $\dfrac{\sigma}{\sqrt{n}}$ is then $\dfrac{\sigma}{\sqrt{4}} = \dfrac{\sigma}{2}.$ Averages of several measurements are less variable than individual measurements, so an average is more likely to give about the same result each time.

48. The average winnings per bet has mean $\mu = -0.053$ for any number of bets. The standard deviation of the average winnings is $\dfrac{1.394}{\sqrt{n}}.$

(a) After 100 bets, $\dfrac{1.394}{\sqrt{100}} = \dfrac{1.394}{10} = 0.1394.$ Thus, the spread of average winnings is as follows.

$$-0.053 - 3(0.1394) = -0.053 - 0.4182 = -0.4712$$
$$\text{to}$$
$$-0.053 + 3(0.1394) = -0.053 + 0.4182 = 0.3652$$

(b) After 1000 bets, $\dfrac{1.394}{\sqrt{1000}} \approx 0.0441.$ Thus, the spread of average winnings is as follows.

$$-0.053 - 3(0.0441) = -0.053 - 0.1323 = -0.1853$$
$$\text{to}$$
$$-0.053 + 3(0.0441) = -0.053 + 0.1323 = 0.0793$$

49. (a) Sketch a normal curve and mark the center at 4600 and the change-of-curvature points at 4590 and 4610. The curve will extend from about 4570 to 4630. This is the curve for one measurement. The mean of $n = 3$ measurements has mean $\mu = 4600$ mg and standard deviation 5.77 mg. Mark points about 5.77 above and below 4600 and sketch a second curve.
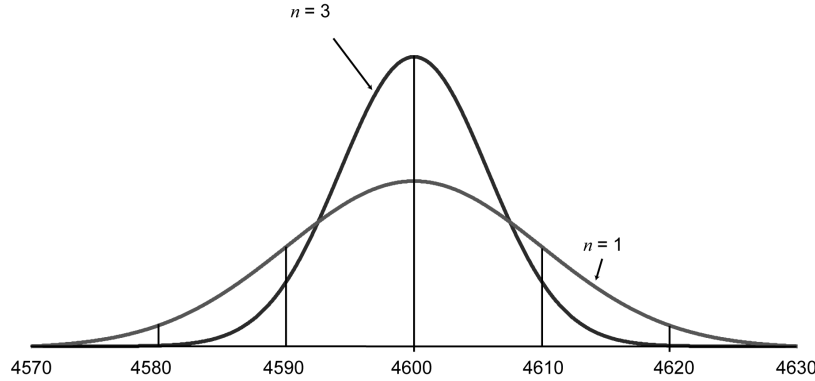
(b) Use the 95 part of the 68-95-99.7 rule with $\sigma = 10$.

$$4600 - 2(10) = 4600 - 20 = 4580 \text{ to } 4600 + 2(5.77) = 4600 + 20 = 4620$$

(c) Now the standard deviation is 5.77, so we have the following.

$$4600 - 2(5.77) = 4600 - 11.54 = 4588.46 \text{ to } 4600 + 2(10) = 4600 + 11.54 = 4611.54$$

50. The mean intelligence (from Exercise 34) is $\mu = 6$. The variance is as follows.

$$\sigma^2 = (3-6)^2 \left(\tfrac{1}{16}\right) + (4-6)^2 \left(\tfrac{2}{16}\right) + (5-6)^2 \left(\tfrac{3}{16}\right) + (6-6)^2 \left(\tfrac{4}{16}\right) + (7-6)^2 \left(\tfrac{3}{16}\right) + (8-6)^2 \left(\tfrac{2}{16}\right) + (9-6)^2 \left(\tfrac{1}{16}\right)$$

$$= (-3)^2 \left(\tfrac{1}{16}\right) + (-2)^2 \left(\tfrac{2}{16}\right) + (-1)^2 \left(\tfrac{3}{16}\right) + (0)^2 \left(\tfrac{4}{16}\right) + (1)^2 \left(\tfrac{3}{16}\right) + (2)^2 \left(\tfrac{2}{16}\right) + (3)^2 \left(\tfrac{1}{16}\right)$$

$$= (9)\left(\tfrac{1}{16}\right) + (4)\left(\tfrac{2}{16}\right) + (1)\left(\tfrac{3}{16}\right) + (0)\left(\tfrac{4}{16}\right) + (1)\left(\tfrac{3}{16}\right) + (4)\left(\tfrac{2}{16}\right) + (9)\left(\tfrac{1}{16}\right)$$

$$= \tfrac{9}{16} + \tfrac{8}{16} + \tfrac{3}{16} + \tfrac{0}{16} + \tfrac{3}{16} + \tfrac{8}{16} + \tfrac{9}{16} = \tfrac{40}{16} = 2.5$$

Thus, $\sigma = \sqrt{2.5} \approx 1.58$. By the central limit theorem, the average score in 100 games is approximately normal with mean 6 and standard deviation $\dfrac{1.58}{\sqrt{100}} = \dfrac{1.58}{10} = 0.158$. Therefore, the middle 68% of average scores lie within one standard deviation of the mean as follows.

$$6 + 0.158 = 5.842 \text{ to } 6 + 0.158 = 6.158$$

51.

ACT exam scores
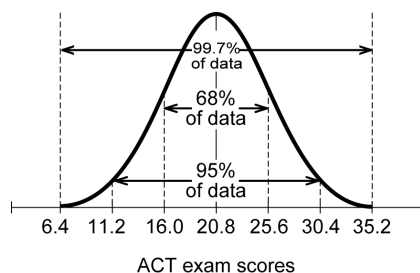
(a) Because 25.6 is one standard deviation above the mean, the probability is about 0.16.

ACT exam scores
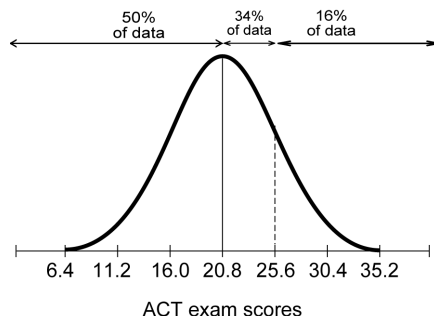
(b) The mean remains $\mu = 20.8$. The standard deviation is $\dfrac{\sigma}{\sqrt{9}} = \dfrac{4.8}{3} = 1.6$.

(c) Because $25.6 = 20.8 + 4.8 = 20.8 + 3(1.6)$ is three standard deviations above the mean, the probability is about 0.0015. (This is half of the 0.003 probability for outcomes more than three standard deviations from the mean, using the 99.7 part of the 68-95-99.7 rule.)

52. (a) The population proportion of single-occupant vehicles is $p = 0.7$. The sample proportion, $\hat{p}$, of single-occupant vehicles in a random sample of $n = 84$ has mean $p = 0.7$ and standard deviation $\sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.7(1-0.7)}{84}}\sqrt{\dfrac{0.7(0.3)}{84}} = \sqrt{\dfrac{0.21}{84}} = \sqrt{0.0025} = 0.05$.

(b) Because $0.6 = 0.7 - 0.1 = 0.7 - 2(0.05)$ is two standard deviations below the mean, the probability is 0.975.
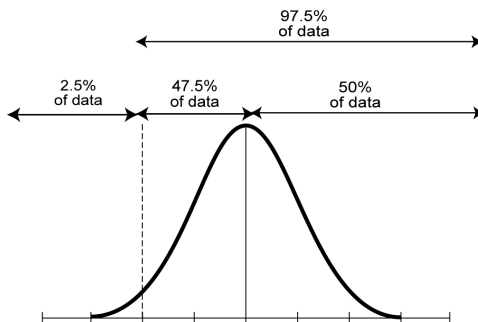
53. (a) There are $26 \times 10 \times 10 \times 26 \times 26 \times 26 = 45,697,600$ different license plates of this form.

(b) There are $26 \times 10 \times 10 = 2600$ plates ending in AAA, because that leaves only the first three characters free.

(c) The probability is $\dfrac{2600}{45,697,600} \approx 0.0000569$.

54. (a) There are only 100 plates like this, because Jerry has specified all four letters exactly. The probability is $\dfrac{100}{45,697,600} \approx 0.0000022$.

(b) The number of possible plates that meet Jerry's new specification is as follows.

$$4 \times 10 \times 10 \times 4 \times 4 \times 4 = 25,600$$

The probability that he will get such a plate is $\dfrac{25,600}{45,697,600} \approx 0.00056$.

55. (a) The probability is $0.07 + 0.08 = 0.15$.

(b) The complement to the event of working out at least one day is working out no days. Thus, using the complement rule, the desired probability is $1 - 0.68 = 0.32$.

56. The mean is as follows.

$$\mu = (0)(0.68) + (1)(0.05) + (2)(0.07) + (3)(0.08)$$
$$+ (4)(0.05) + (5)(0.04) + (6)(0.01) + (7)(0.02)$$
$$= 0 + 0.05 + 0.14 + 0.24 + 0.20 + 0.20 + 0.06 + 0.14 = \ 1.03 \text{ days}$$

As you interview more and more people, the average number of days, $\bar{x}$, that these people work out will always get closer and closer to 1.03.

57. (a) The variance is as follows.

$$\sigma^2 = (0 - 1.03)^2(0.68) + (1 - 1.03)^2(0.05) + (2 - 1.03)^2(0.07) + (3 - 1.03)^2(0.08)$$
$$+ (4 - 1.03)^2(0.05) + (5 - 1.03)^2(0.04) + (6 - 1.03)^2(0.01) + (7 - 1.03)^2(0.02)$$
$$= (-1.03)^2(0.68) + (-0.03)^2(0.05) + (0.97)^2(0.07) + (1.97)^2(0.08)$$
$$+ (2.97)^2(0.05) + (3.97)^2(0.04) + (4.97)^2(0.01) + (5.97)^2(0.02)$$
$$= (1.0609)(0.68) + (0.0009)(0.05) + (0.9409)(0.07) + (3.8809)(0.08)$$
$$+ (8.8209)(0.05) + (15.7609)(0.04) + (24.7009)(0.01) + (35.6409)(0.02)$$
$$= 0.721412 + 0.000045 + 0.065863 + 0.310472 + 0.441045 + 0.630436 + 0.247009 + 0.712818$$
$$= 3.1291$$

Thus, the standard deviation is $\sigma = \sqrt{3.1291} \approx 1.7689$ days.

(b) The mean, $\bar{x}$, of $n = 100$ observations has mean $\mu = 1.03$ and standard deviation

$$\frac{\sigma}{\sqrt{n}} = \frac{1.7689}{\sqrt{100}} = \frac{1.7689}{10} \approx 0.177.$$

The central limit theorem says that $\bar{x}$ is approximately normal with this mean and standard deviation. The 95 part of the 68-95-99.7 rule says that with probability 0.95, values of $\bar{x}$ lie between $1.03 - 2(0.177) = 1.03 - 0.354 = 0.676$ days and $1.03 + 2(0.177) = 1.03 + 0.354 = 1.384$ days.

Word Search Solution

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A D R T I M R O E D N D E M R V E H M Z E A E G U
J O C M X T G G N J X C I E L A X F A R S A E R O
C O M B I N A T O R I C S A E E T L G B H T L P V
B S M N C I H T E S G O L N Z O A M X Y T O W F C
N C I D V P O I W R A N D O M P H E N O M E N O N
P O E D L E Y Z S N P P S F Z W E R E U O S T A D
R M Q H H A H W A Z P R M A W R H O J P J O Y R O
F P F P X W J C M N R O H P N T R E R W R V A V G
A L R R S F S O P E O B J R P E E H O L E F N A N
Z E K O J Y T I L I B A B O R P E T E R C S I D G
E M R B I M N Z I E A B S B E R I T A I A L E D S
F E P B N R S N E B I E A H A Y I A D P R E I J
P N I B G E V R G I I L M B E F M O E S V D T Q
X T I I D S E S D J L I X I G I P I E E L K I O
P R G L E G T K I T I T F L N D A L R N L E P O D
F U S I E H N M S R T Y E I Z T A L T I P N I N J
E L J T D E I I T Q Y H E T C O A A S W M E J R J
O E F Z I S O A R S M I F Y N C J R E M A I T U B
I D O N S I J X I I O S M M M T G T E A S Q A L O
L P V X U N S P B G D T T O E Q N N K A A F A E N
I O H S E E I M U T E O M D I O T E H G J L U R S
S B F T Z E D Q T Z L G F E V R U C Y T I S N E D
P E Y M A G E P I E H R X L T G E T S Q E I J M R
D E A G N L A W O F L A R G E N U M B E R S D T M
P M I E V E U E N E O M O H X N H R R R T H N E F
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