Chapter 1
Urban Services

Chapter Outline

Introduction
Section 1.1 Euler Circuits
Section 1.2 Finding Euler Circuits
Section 1.3 Beyond Euler Circuits
Section 1.4 Urban Graph Traversal Problems

Chapter Summary

Management science, or operations research, is a branch of mathematics that uses mathematical methods to find optimal solutions to management problems. Some of these problems involve finding efficient routes for services such as collecting coins from parking meters, collecting garbage, and delivering mail. The mathematical structure known as a *graph* is useful in analyzing routes.

A graph consists of a finite set of *vertices* together with edges connecting (some, all, or no) pairs of vertices. A path in a graph is a connected sequence of edges that begins and ends at a vertex. A path is called a *circuit* if it begins and ends at the same vertex. In many routing applications (e.g., mail delivery or garbage collection), the best solution would be a circuit that uses each edge (e.g., sidewalk or street) of an appropriate graph exactly once. Such circuits are called *Euler circuits*, in honor of the Swiss mathematician Leonhard Euler.

In order to have an Euler circuit, a graph must satisfy two conditions: it must be *connected* (i.e., it must have a path between any pair of its vertices); and, each of its vertices must have even *valences* (the number of edges meeting at that vertex). If the graph for a particular routing application does not have an Euler circuit, the best we can hope to do is find a circuit of minimum length. The problem of finding such a circuit is known as the *Chinese postman problem*. The solution process relies on "*eulerizing*" a graph, judiciously duplicating edges of the graph to produce a connected graph with even valences so that the total length of the duplicated edges (total number if all edges of the graph have the same length) is as small as possible. The Euler tour in the "new" graph can be traced on the original by interpreting the use of a duplicated edge as a reuse of the original edge it duplicates.

There are efficient procedures for finding good eulerizations and for solving the Chinese postman problem. The "*edge-walker*" method of the text is good for rectangular networks. More sophisticated procedures are needed in general.

Skill Objectives

1. Determine by observation if a graph is connected.

2. Identify vertices and edges of a given graph.

3. Construct the graph of a given street network.

4. Determine by observation the valence of each vertex of a graph.

5. Define an Euler circuit.

6. List the two conditions for the existence of an Euler circuit.

7. Determine whether a graph contains an Euler circuit.

8. If a graph contains an Euler circuit, list one such circuit by identifying the order in which the vertices are used by the circuit, or by identifying the order in which the edges are to be used.

9. If a graph does not contain an Euler circuit, add a minimum number of edges to“eulerize” the graph.

10. Find an Euler circuit in an eulerized graph and “squeeze” it onto the original graph. Be able to interpret, in terms of the original graph, the use of duplicated edges in the eulerization.

11. Identify management science problems whose solutions involve Euler circuits.

Teaching Tips

1. The concept of connectedness could be explored further by considering trees as opposed to circuits. This could begin to prepare the student for the work on trees in connectedness.

2. In preparation for assigning Exercise 6, you may want to explore in class discussion whether the placement of the vertices representing the cities affects the graph. In particular, consider three cities whose positions are collinear.

3. Figure 1.13 demonstrates the process of adding an edge in order to eulerize a graph. Students are sometimes confused by the fact that this added edge is curved rather than straight and attempt to attach unwarranted significance to this. A helpful explanation is that it is curved only so that it won’t be confused with the original segment. In addition, you may want to emphasize that the curve could be drawn on either side of the original edge and that it indicates a retracing of that edge.

4. Some students want to eulerize graphs by connecting odd-valent vertices in a diagonal fashion. Emphasize the practical constraints that may prevent this approach. Note that in Exercises 35 and 36, the eulerization that minimizes the total length of the edges duplicated is not the eulerization that duplicates the fewest edges.

5. Ask students to construct a graph of a several-block area of their neighborhood and then look for an Euler circuit for the letter carrier to use. If an Euler circuit does not exist, ask students to produce optimal eulerizations.

6. An underlying principle of this chapter is the fact that a mathematical model, in this case a graph, is an abstraction of reality. The solution suggested by the model need not imply that streets would be added to an existing street network simply for the purpose of creating an Euler circuit.

Research Paper

Exercise 38 indicates that the problem situation resembles the one that inspired Euler. This refers to the historical Königsberg Bridge Problem (mentioned in Spotlight 1.1). You may choose to ask students to research the history of this town (once the capital of East Prussia) and this problem. Websites can be found that contain a map of the area, including the bridges. Also, you may choose to ask students to further investigate Euler's 1736 paper on the problem, which was divided into 21 paragraphs. Another historical figure that students could research is the life and contributions of the French mathematician Louis Poinsot (1777 – 1859). Other problems posed by Euler, such as the Thirty-Six Officers Problem, may be of interest to students.

Collaborative Learning

Euler Circuits

As an introduction to this chapter, duplicate the following exercise on Euler circuits and ask your students to answer the questions after discussing them in groups. (Do not introduce technical terms such as graph, edge, vertex, valence, or Euler circuit yet.)

You will find that most of the students are successful in answering questions a and b, but perhaps not question c. If in fact they do have difficulty answering part c, call their attention to the number of edges meeting at each of the vertices. (Resist using the word "valence.") Ask them to count these numbers for each of the graphs and then try again to answer part c, by looking for a pattern.

Eulerization

Duplicate the following eulerization exercises and have the students consider the problems in groups.

a. Which of the following diagrams can be drawn without lifting your pencil from the paper?

b. Which can be drawn as in part a, but with the additional requirement that you end the drawing at the starting point?

c. What do the diagrams that could be drawn in part a have in common? What about the diagrams that could be drawn in part b? In other words, try to determine simple conditions on a diagram that enables you to predict, in advance, whether or not it can be drawn according to the requirements in parts a or b.

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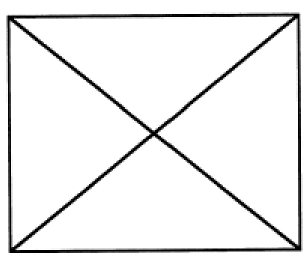

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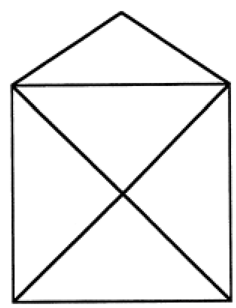

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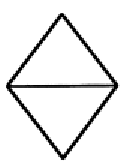

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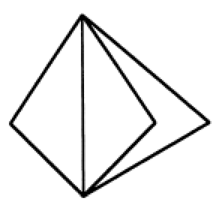

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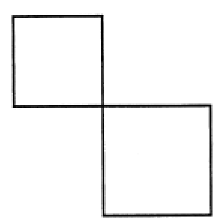

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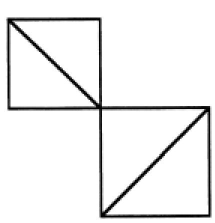

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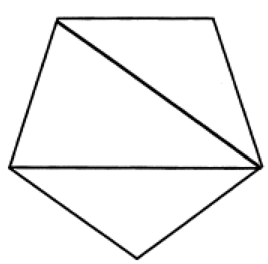

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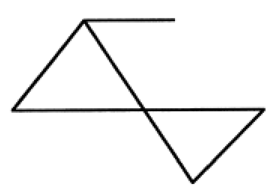
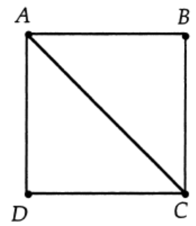
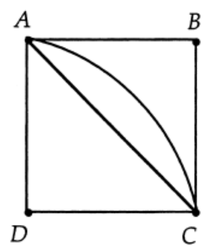


Eulerization

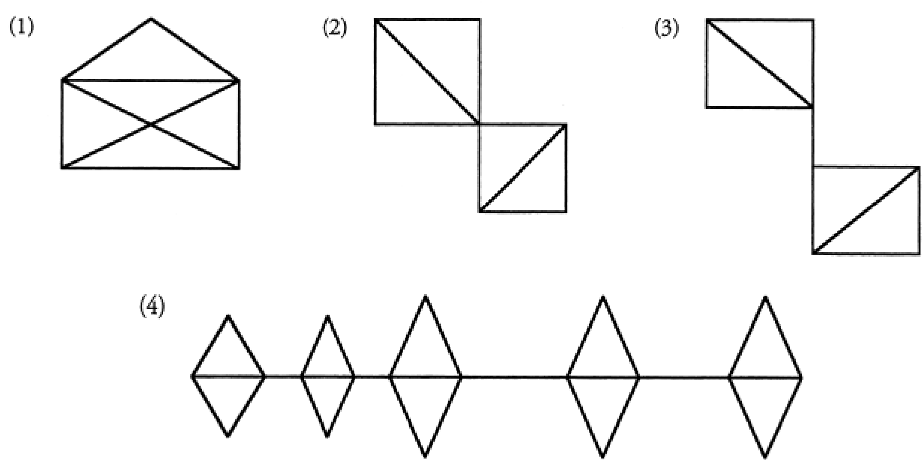
In each of the graphs below, an Euler circuit does not exist, since there are vertices with odd valences. It is possible to convert such graphs to ones having all vertices of even valence by duplicating one or more edges. For example, in the graph



vertices *A* and *C* each have valence 3. Hence, if we duplicate edge *AC*, we obtain a modified graph in which each vertex has an even valence.



Do the same for each of the following graphs and then answer the questions that follow.



If a graph has 4 vertices with odd valences, what is the minimum number of duplications necessary to convert the graph to one in which all of the vertices are of even valence? Can this minimum number always be achieved?

Take a campus map (often found in the university catalog or directory), and pick out several key locations (library, dormitories, computer center, bookstore, etc.) and the streets or paths that connect them. Then ask the students whether an Euler circuit exists for this network. If not, how many edges have to be duplicated? By changing landmarks, you can obtain several new problems.

Solutions

Skills Check:

1. b	2. c	3. c	4. c	5. b	6. b	7. c	8. c	9. a	10. a
11. b	12. a	13. c	14. a	15. c	16. b	17. a	18. a	19. c	20. b

Cooperative Learning:

Euler Circuits:

a. Diagrams (2), (3), (4), (5), (7), and (8) can be drawn without lifting the pencil from the paper.

b. In (4) and (5) you can return to the starting point.

c. In part a, there are at most 2 vertices with odd valences, while in part b there are no vertices with odd valences.

Eulerization:

(1) Duplicate 1 edge.

(2) Duplicate 2 edges.

(3) Duplicate 4 edges.

(4) Duplicate 9 edges.

If there are 4 vertices with odd valences, then there will be a *minimum* of two duplications. However, this minimum can't always be achieved, as we see from numbers (3) and (4).

Exercises:

1. *E* has valence 0; *A* has valence 1; *H*, *D*, and *G* have valence 2; *B* and *F* have valence 3; *C* has valence 5. *E* is "isolated." *E* might have valence 0 because it is on an island with no road access.

2. (a) Yes.

(b) No. There is no way to get from *A* to *C*, *E*, or *H*.

3. (a) This diagram fails to be a graph because a line segment joins a single vertex to itself. The definition being used does not allow this.

(b) The edge *EC* crosses edges *AD* and *BD* at points which are not vertices; edge *AC* crosses *BD* at a point that is not a vertex.

(c) This graph has 5 vertices and 5 edges.

4. (a) 6 stores.

(b) 9 roads.

(c) *CBF*.

(d) *EDFB* or *EDCB*.

5. (a) *FDCBF*

(b) (i) *BD*; *BCD*.

(ii) *CBF*; *CDF*; *CDBF*.

(c) *CDFBC*

6. *MLB*, *MRB*, *MNLB*; Jack is right.

7. (a) 4 vertices; 4 edges.

(b) 7 vertices; 6 edges.

(c) 10 vertices; 14 edges.

8. (a) 2 + 3 + 3 + 0 = 8

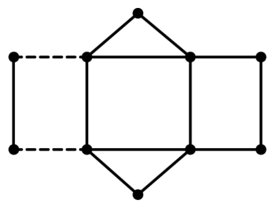
(b) 2 + 2 + 2 + 2 + 2 + 1 + 1 = 12

(c) 28

(d) The number we obtain is twice the number of edges in the graph.

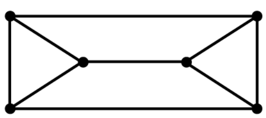
(e) The fact that the sum of the valences of the vertices of a graph is always twice the number of edges in the graph follows from noticing that each vertex of an edge contributes a total of two to the sum because the edge has two endpoints.

9. Remove the edges dotted in the figure below and the remaining graph will be disconnected.

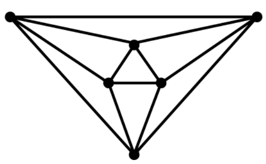


10. In any of these graphs, two edges can be removed and the graph will become disconnected. One of the disconnected pieces will be a single vertex.

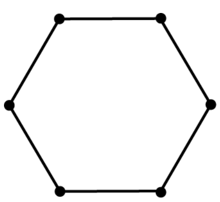
11. (a)



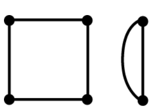
(b)



12. (a)

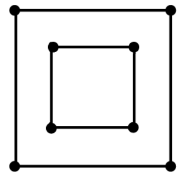


(b)

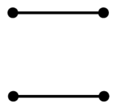


(c) Yes. The sum of the valences of a graph with 6 vertices each of valence 2 is 12. Thus, all such graphs have 6 edges.

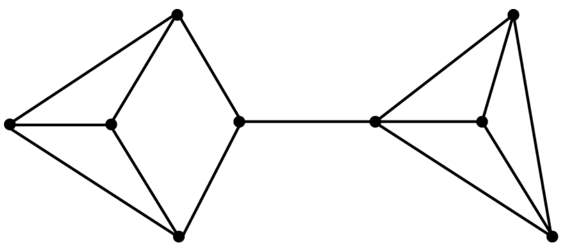
13. Yes, a disconnected graph can arise. One, possible example is shown below:



which gives rise to the disconnected graph:

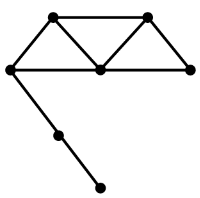


14. (a)

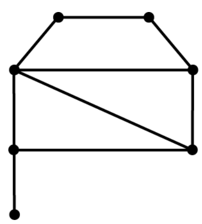


(b) The edge might represent a bridge or tunnel. Recently, when a bridge collapsed because it was hit by a barge, there was a major disruption to the communities near the bridge on opposite sides of the river.

15. (a)



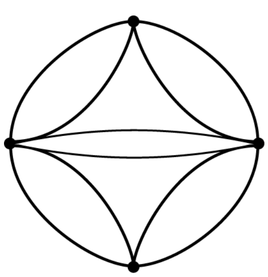
(b)



16. The street direction will matter for a problem involving how long it will take to get between two street intersections and for routing a street sweeper that follows traffic rules. The street direction may not matter for an inspector checking "manholes" located in the middle of streets or a service that involves walking along either side of the street such as inspecting sidewalks.

17. The supervisor is not satisfied because all of the edges are not traveled upon by the postal worker. The worker is unhappy because the end of the worker's route wasn't the same point as where the worker began. The original job description is unrealistic because there is no Euler circuit in the graph.

18.

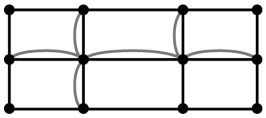


19. There is such an efficient route. The appropriate graph model has an additional edge joining the same pair of vertices for each of the edges shown in the graph of Exercise 17. Since this graph is connected and even-valent, it has an Euler circuit, any one of which will provide a route for the snowplow. Routes without 180-degree turns are better choices.

20. (a) Pothole inspection or inspecting the centerline for possible repainting because it had faded.

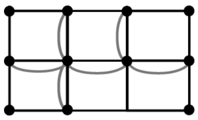
(b) Street sweeping, snow removal, and curb inspections in urban areas.

21.



22. (a) The graph is a rectangular network with two rows and three columns No extra edges need to be added.

(b)



23. (a) The largest number of such paths is 3. One set of such paths is *AF*, *ABEF*, and *ACF*.

(b) This task is simplified by noticing there are many symmetries in this graph.

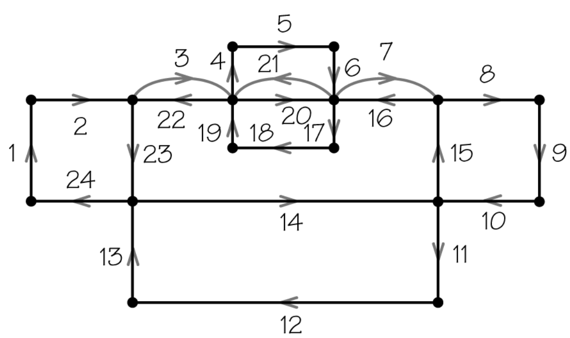
(c) In a communication system such a graph offers redundant ways to get messages between pairs of points even when the failure of some of the communication links (edges removed) occurs.

24. Both are circuits; however, only graph (b) is an Euler circuit.

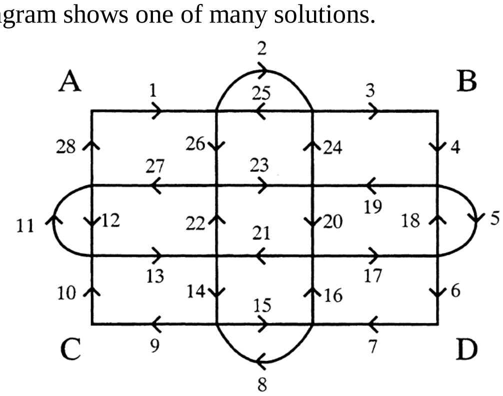
25. Do not choose edge 2, but edges 1 or 10 could be chosen.

26. Do not choose edge 3, but edges 9 or 10 could be chosen.

27.



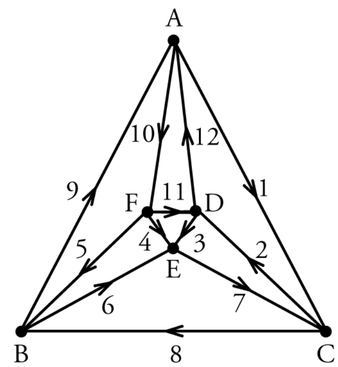
28. (a) The following diagram shows one of many solutions.



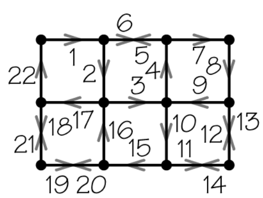
(b) Answers will vary; there are many Euler circuits in the graph.

29. Two edges need to be dropped to produce a graph with an Euler circuit. Persons who parked along these stretches of sidewalk without putting coins in the meters would not need to fear that they would get tickets.

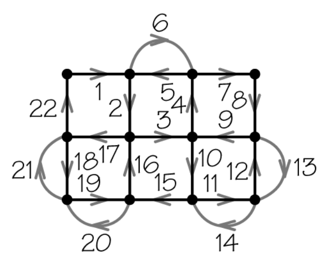
30. If one was outlining garden plots with a sprinkler hose, this tour would allow having the hoses as flat as possible because one hose would not have to cross another hose.



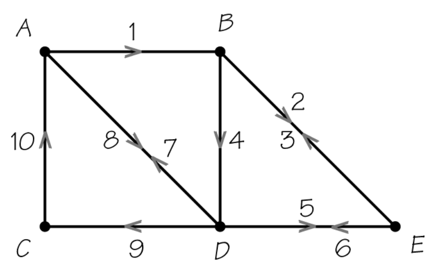
31. (a)



(b)



32. The curved edges on the first graph become double-traversals on the straight edges of the second graph.



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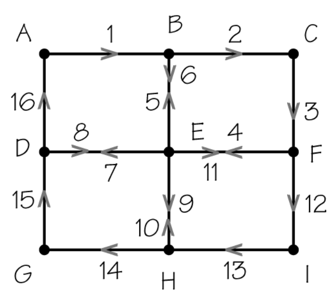


ABEBDEDADCA

33. (a) There are four 3-valent vertices. By properly removing two edges adjacent to these four vertices, (edge between left two 3-valent vertices and edge between right two 3-valent vertices) one can make the graph even-valent.

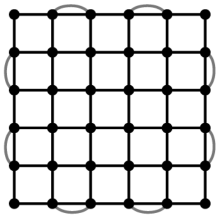
(b) Yes, because the resulting graph is connected and even-valent.

(c) It is possible to remove two edges and have the resulting graph be even-valent.

(d) No, because the resulting graph is not connected, even though it is even-valent.

34. A minimum of three edges must be added: one edge along the horizontal segment in the first parallelogram and a segment along two opposite edges of the second parallelogram.

35. (a)



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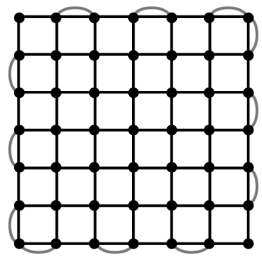

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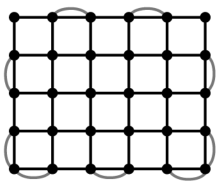

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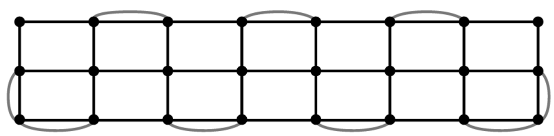
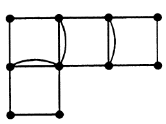


Nine is the best one can do.

36. (a)



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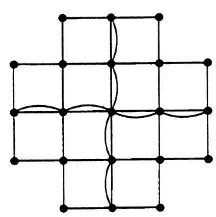

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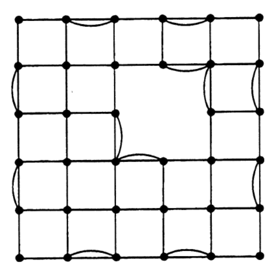
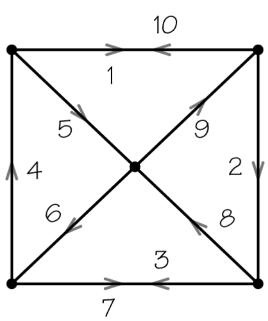
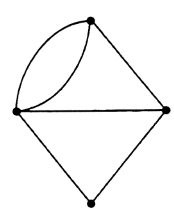
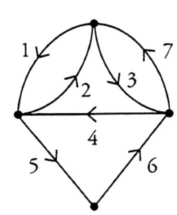


37.



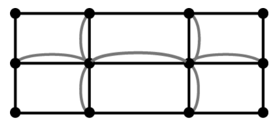
38. Represent each riverbank by a vertex and each island by a vertex. Represent each bridge by an edge. This produces the graph on the left. After eulerization we produce the graph on the right. An Euler circuit is shown on this graph. After squeezing this circuit into the original graph, we have a circuit with one repeated edge.





39. There are many different circuits which will involve three reuses of edges. These are the edges which join up the six 3-valent vertices in pairs.

40. The minimum length (36,000 feet) is obtained for any Euler circuit in the graph with edges duplicated as shown below. For minimizing total length it is better to repeat many shorter edges rather than a few long ones.



41. There are many circuits that achieve a length of 44,000 feet. The number of edges reused is eight because a shorter length tour can be found by repeating more shorter edges than fewer longer edges.

42. (a) The cheapest route has cost 49 and repeats edges *BC*, *CD*, and *DF*.

(b) Three edges.

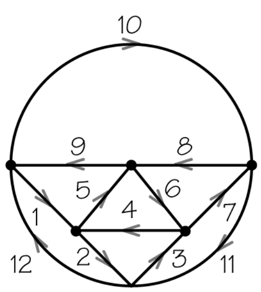
(c) When there are different weights on the edges of a graph, the discussion about good eulerizations must be modified to take the size of the weights into account. It turns out there is an efficient, though complex, algorithm for finding minimum cost solutions to such problems.

(d) The weight might represent time. Two blocks of the same physical length can take different times to traverse due to construction or other factors.

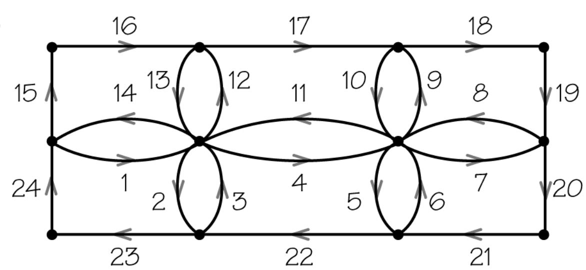
(e) The weight might represent traversal time, traffic volume, number of potholes, number of stop signs, etc.

43. Both graphs (b) and (c) have Euler circuits. The valences of all of the vertices in (a) are odd, which makes it impossible to have an Euler circuit there.

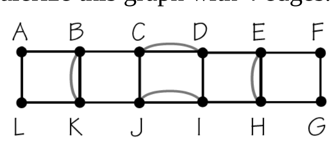
(b)



(c)



44. There are 5 different ways to eulerize this graph with 4 edges. One of them is shown below:

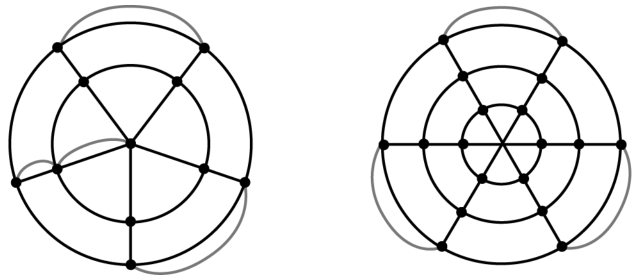


An Euler circuit in the original graph that repeats 4 edges is: *ABCDEHGFEHIJCDIJKBKLA.*

45. If the graph *G* is connected, the newly constructed graph will be even-valent and, thus, will have an Euler circuit. If *G* is not connected, the new graph will not have an Euler circuit because it, too, will not be connected.

46. A good eulerization duplicates the 5 “spokes” that go from the inner pentagon to the outer one. There are many Euler circuits in the eulerized graph.

47. (a)



(b) The best eulerization for the four-circle, four-ray case adds two edges.

(c) Hint: Consider the cases where r is even and odd separately.

48. Pick any vertex and try to start an Euler circuit for the graph there. At some point the circuit traverses this special edge, crossing from the starting part of the graph to the other part. This special edge is the only connection between the parts, so we cannot return to the starting part and thus cannot have an Euler circuit. Since there is no Euler circuit, somewhere there must be a vertex with an odd valence.

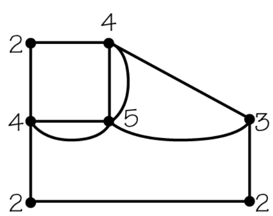
49. A graph with six vertices where each vertex is joined to every other vertex will have valence 5 for each vertex.

50. Both graphs (a) and (c) have Euler circuits. In graph (b), there is no Euler circuit because some vertices have odd valences.

51. When you attach a new edge to an existing graph, it gets attached at two ends. At each of its ends, it makes the valence of the existing vertex go up by one. Thus the increase in the sum of the valences is two. Therefore, if the graph had an even sum of the valences before, it still does, and if its valence sum was odd before, it still is.

52. Dots without edges all have valence zero, and so the number of odd-valent vertices is zero, which is an even number. As edges are added, the number of odd-valent vertices will always increase by either 0 or 2. Thus, any graph has an even number of odd-valent vertices.

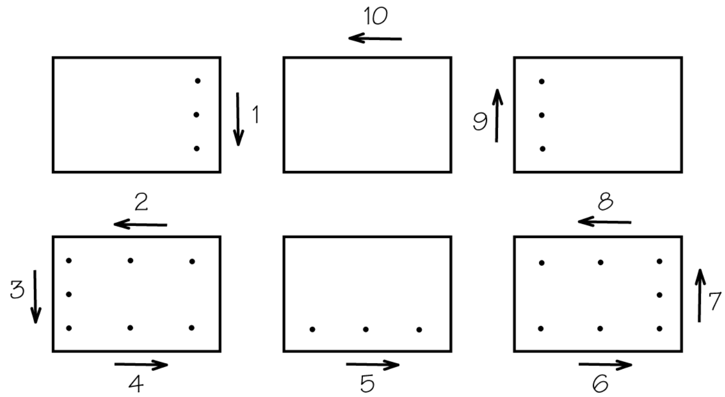
53.



54. When $r = 1$, a formula for the number of repeated edges is $(s-1)$. If r and s are odd, where $r = 2a+1$ and $s = 2b+1$ (a and b positive integers which are at least 1) then a formula for the number of repeated edges is $2(a+b)$. Similar formulas hold for the cases where both r and s are even or one of them is even and the other odd. The exact form of the formula depends on the way one expresses these situations.

55. In chemistry when we say, for example, that hydrogen has valence 1, we mean that it forms one chemical bond with other elements. This usage has similarities with the graph theory concept of valence.

56. The thinking used to solve the Chinese postman problem does not apply directly for the situation described because the modeling assumptions for that problem are not met here.



57. A tour that begins and ends at vertex *A* and which respects the traffic directions would be: *ABDEFBEBFEDBACDCA.* The cutting machine has to make "sharp turns" at some intersections.

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