Chapter 1

Rational Functions

1.1 Introduction to Rational Functions

If we add, subtract or multiply polynomial functions according to the function arithmetic rules defined in Section **??**, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. In this chapter we study **rational functions** - functions which are ratios of polynomials.

Definition 1.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions[a]

[a] According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

As we recall from Section **??**, we have domain issues anytime the denominator of a fraction is zero. In the example below, we review this concept as well as some of the arithmetic of rational expressions.

Example 1.1.1. Find the domain of the following rational functions. Write them in the form $\frac{p(x)}{q(x)}$ for polynomial functions p and q and simplify.

1. $f(x) = \dfrac{2x-1}{x+1}$
2. $g(x) = 2 - \dfrac{3}{x+1}$
3. $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$
4. $r(x) = \dfrac{2x^2-1}{x^2-1} \div \dfrac{3x-2}{x^2-1}$

Solution.

1. To find the domain of f, we proceed as we did in Section **??**: we find the zeros of the denominator and exclude them from the domain. Setting $x+1=0$ results in $x=-1$. Hence, our domain is $(-\infty,-1)\cup(-1,\infty)$. The expression $f(x)$ is already in the form requested and when we check for common factors among the numerator and denominator we find none, so we are done.

2. Proceeding as before, we determine the domain of g by solving $x+1=0$. As before, we find the domain of g is $(-\infty,-1)\cup(-1,\infty)$. To write $g(x)$ in the form requested, we need to get a common denominator

$$\begin{aligned}
g(x) &= 2 - \frac{3}{x+1} \\
&= \frac{2}{1} - \frac{3}{x+1} \\
&= \frac{(2)(x+1)}{(1)(x+1)} - \frac{3}{x+1} \\
&= \frac{(2x+2)-3}{x+1} \\
&= \frac{2x-1}{x+1}
\end{aligned}$$

This formula is also completely simplified.

3. The denominators in the formula for $h(x)$ are both x^2-1 whose zeros are $x = \pm 1$. As a result, the domain of h is $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$. We now proceed to simplify $h(x)$. Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned}
h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} \\
&= \frac{\left(2x^2-1\right)-(3x-2)}{x^2-1} \\
&= \frac{2x^2-1-3x+2}{x^2-1} \\
&= \frac{2x^2-3x+1}{x^2-1} \\
&= \frac{(2x-1)(x-1)}{(x+1)(x-1)} \\
&= \frac{(2x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\
&= \frac{2x-1}{x+1}
\end{aligned}$$

4. To find the domain of r, it may help to temporarily rewrite $r(x)$ as

$$r(x) = \frac{\dfrac{2x^2-1}{x^2-1}}{\dfrac{3x-2}{x^2-1}}$$

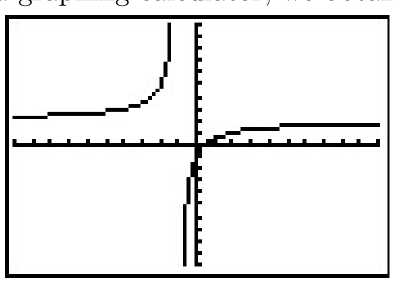
We need to set all of the denominators equal to zero which means we need to solve not only $x^2 - 1 = 0$, but also $\frac{3x-2}{x^2-1} = 0$. We find $x = \pm 1$ for the former and $x = \frac{2}{3}$ for the latter. Our domain is $(-\infty, -1) \cup \left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$. We simplify $r(x)$ by rewriting the division as multiplication by the reciprocal and then simplifying

$$\begin{aligned} r(x) &= \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1} \\ &= \frac{2x^2-1}{x^2-1} \cdot \frac{x^2-1}{3x-2} \\ &= \frac{\left(2x^2-1\right)\left(x^2-1\right)}{\left(x^2-1\right)(3x-2)} \\ &= \frac{\left(2x^2-1\right)\cancel{\left(x^2-1\right)}}{\cancel{\left(x^2-1\right)}(3x-2)} \\ &= \frac{2x^2-1}{3x-2} \end{aligned}$$

□

A few remarks about Example 1.1.1 are in order. Note that the expressions for $f(x)$, $g(x)$ and $h(x)$ work out to be the same. However, only two of these functions are actually equal. Recall that functions are ultimately sets of ordered pairs,[1] and so for two functions to be equal, they need, among other things, to have the same domain. Since $f(x) = g(x)$ and f and g have the same domain, they are equal functions. Even though the formula $h(x)$ is the same as $f(x)$, the domain of h is different than the domain of f, and thus they are different functions.

We now turn our attention to the graphs of rational functions. Consider the function $f(x) = \frac{2x-1}{x+1}$ from Example 1.1.1. Using a graphing calculator, we obtain



Two behaviors of the graph are worthy of further discussion. First, note that the graph appears to 'break' at $x = -1$. We know from our last example that $x = -1$ is not in the domain of f which means $f(-1)$ is undefined. When we make a table of values to study the behavior of f **near** $x = -1$ we see that we can get 'near' $x = -1$ from two directions. We can choose values a little less than -1, for example $x = -1.1$, $x = -1.01$, $x = -1.001$, and so on. These values are said to 'approach -1 from the **left**.' Similarly, the values $x = -0.9$, $x = -0.99$, $x = -0.999$, etc., are said

[1]You should review Sections ?? and ?? if this statement caught you off guard.

to 'approach -1 from the **right**.' If we make two tables, we find that the numerical results confirm what we see graphically.

x	$f(x)$	$(x, f(x))$
-1.1	32	$(-1.1, 32)$
-1.01	302	$(-1.01, 302)$
-1.001	3002	$(-1.001, 3002)$
-1.0001	30002	$(-1.001, 30002)$

x	$f(x)$	$(x, f(x))$
-0.9	-28	$(-0.9, -28)$
-0.99	-298	$(-0.99, -298)$
-0.999	-2998	$(-0.999, -2998)$
-0.9999	-29998	$(-0.9999, -29998)$

As the x values approach -1 from the left, the function values become larger and larger positive numbers.[2] We express this symbolically by stating as $x \to -1^-$, $f(x) \to \infty$. Similarly, using analogous notation, we conclude from the table that as $x \to -1^+$, $f(x) \to -\infty$. For this type of unbounded behavior, we say the graph of $y = f(x)$ has a **vertical asymptote** of $x = -1$. Roughly speaking, this means that near $x = -1$, the graph looks very much like the vertical line $x = -1$.

Another feature worthy of note about the graph of $y = f(x)$ is it seems to 'level off' on the left and right hand sides of the screen. This is a statement about the end behavior of the function. As we discussed in Section **??**, the end behavior of a function is its behavior as x as x attains larger[3] and larger negative values without bound, $x \to -\infty$, and as x becomes large without bound, $x \to \infty$. Making tables of values, we find

x	$f(x)$	$(x, f(x))$
-10	≈ 2.3333	$\approx (-10, 2.3333)$
-100	≈ 2.0303	$\approx (-100, 2.0303)$
-1000	≈ 2.0030	$\approx (-1000, 2.0030)$
-10000	≈ 2.0003	$\approx (-10000, 2.0003)$

x	$f(x)$	$(x, f(x))$
10	≈ 1.7273	$\approx (10, 1.7273)$
100	≈ 1.9703	$\approx (100, 1.9703)$
1000	≈ 1.9970	$\approx (1000, 1.9970)$
10000	≈ 1.9997	$\approx (10000, 1.9997)$

From the tables, we see as $x \to -\infty$, $f(x) \to 2^+$ and as $x \to \infty$, $f(x) \to 2^-$. Here the '$+$' means 'from above' and the '$-$' means 'from below'. In this case, we say the graph of $y = f(x)$ has a **horizontal asymptote** of $y = 2$. This means that the end behavior of f resembles the horizontal line $y = 2$, which explains the 'leveling off' behavior we see in the calculator's graph. We formalize the concepts of vertical and horizontal asymptotes in the following definitions.

Definition 1.2. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

[2] We would need Calculus to confirm this analytically.

[3] Here, the word 'larger' means larger in absolute value.

DEFINITION 1.3. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, either $f(x) \to c^-$ or $f(x) \to c^+$.

In our discussion following Example 1.1.1, we determined that, despite the fact that the formula for $h(x)$ reduced to the same formula as $f(x)$, the functions f and h are different, since $x = 1$ is in the domain of f, but $x = 1$ is not in the domain of h. If we graph $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$ using a graphing calculator, we are surprised to find that the graph looks identical to the graph of $y = f(x)$. There is a vertical asymptote at $x = -1$, but near $x = 1$, everything seem fine. Tables of values provide numerical evidence which supports the graphical observation.

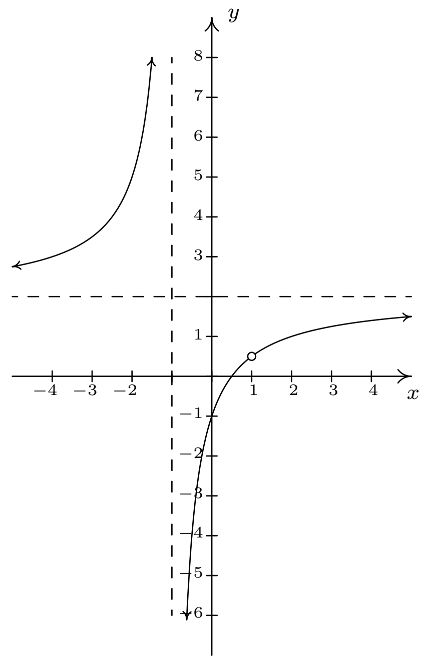
x	$h(x)$	$(x, h(x))$
0.9	≈ 0.4210	$\approx (0.9, 0.4210)$
0.99	≈ 0.4925	$\approx (0.99, 0.4925)$
0.999	≈ 0.4992	$\approx (0.999, 0.4992)$
0.9999	≈ 0.4999	$\approx (0.9999, 0.4999)$

x	$h(x)$	$(x, h(x))$
1.1	≈ 0.5714	$\approx (1.1, 0.5714)$
1.01	≈ 0.5075	$\approx (1.01, 0.5075)$
1.001	≈ 0.5007	$\approx (1.001, 0.5007)$
1.0001	≈ 0.5001	$\approx (1.0001, 0.5001)$

We see that as $x \to 1^-$, $h(x) \to 0.5^-$ and as $x \to 1^+$, $h(x) \to 0.5^+$. In other words, the points on the graph of $y = h(x)$ are approaching $(1, 0.5)$, but since $x = 1$ is not in the domain of h, it would be inaccurate to fill in a point at $(1, 0.5)$. As we've done in past sections when something like this occurs,[4] we put an open circle (also called a 'hole' in this case[5]) at $(1, 0.5)$. Below is a detailed graph of $y = h(x)$, with the vertical and horizontal asymptotes as dashed lines.

[4]For instance, graphing piecewise defined functions in Section ??.

[5]Stay tuned. In Calculus, we will see how these 'holes' can be 'plugged' when embarking on a more advanced study of contintuity.



Neither $x = -1$ nor $x = 1$ are in the domain of h, yet we see the behavior of the graph of $y = h(x)$ is drastically different near these points. The reason for this lies in the second to last step when we simplified the formula for $h(x)$ in Example 1.1.1. We had $h(x) = \frac{(2x-1)(x-1)}{(x+1)(x-1)}$. The reason $x = -1$ is not in the domain of h is because the factor $(x+1)$ appears in the denominator of $h(x)$; similarly, $x = 1$ is not in the domain of h because of the factor $(x-1)$ in the denominator of $h(x)$. The major difference between these two factors is that $(x-1)$ cancels with a factor in the numerator whereas $(x+1)$ does not. Loosely speaking, the trouble caused by $(x-1)$ in the denominator is canceled away while the factor $(x+1)$ remains to cause mischief. This is why the graph of $y = h(x)$ has a vertical asymptote at $x = -1$ but only a hole at $x = 1$. These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

THEOREM 1.1. **Location of Vertical Asymptotes and Holes:**[a] Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros.[b] Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'
[b] In other words, $r(x)$ is in lowest terms.

In English, Theorem 1.1 says if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, we have a vertical asymptote.

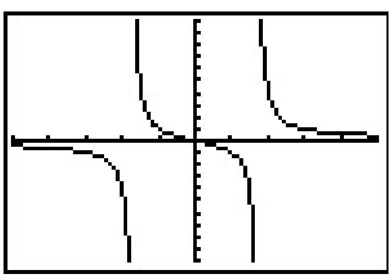
Example 1.1.2. Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions. Verify your answers using a graphing calculator.

1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$
3. $h(x) = \dfrac{x^2 - x - 6}{x^2 + 9}$
4. $r(x) = \dfrac{x^2 - x - 6}{x^2 + 4x + 4}$

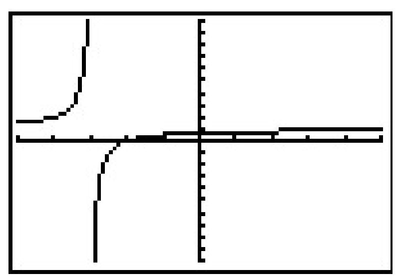
Solution.

1. To use Theorem 1.1, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms, there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. The calculator verifies this claim.

2. Solving $x^2 - 9 = 0$ gives $x = \pm 3$. In lowest terms $g(x) = \frac{x^2-x-6}{x^2-9} = \frac{(x-3)(x+2)}{(x-3)(x+3)} = \frac{x+2}{x+3}$. Since $x = -3$ continues to make trouble in the denominator, we know the line $x = -3$ is a vertical asymptote of the graph of $y = g(x)$. Since $x = 3$ no longer produces a 0 in the denominator, we have a hole at $x = 3$. To find the y-coordinate of the hole, we substitute $x = 3$ into $\frac{x+2}{x+3}$ and find the hole is at $\left(3, \frac{5}{6}\right)$. When we graph $y = g(x)$ using a calculator, we clearly see the vertical asymptote at $x = -3$, but everything seems calm near $x = 3$.



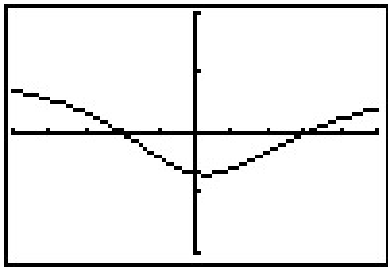
The graph of $y = f(x)$



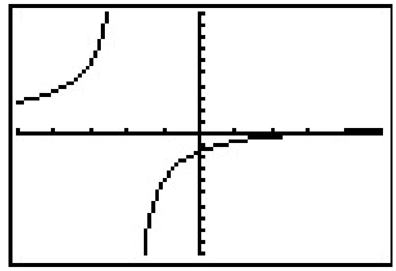
The graph of $y = g(x)$

3. The domain of h is all real numbers, since $x^2 + 9 = 0$ has no real solutions. Accordingly, the graph of $y = h(x)$ is devoid of both vertical asymptotes and holes.

4. Setting $x^2 + 4x + 4 = 0$ gives us $x = -2$ as the only real number of concern. Simplifying, we see $r(x) = \frac{x^2-x-6}{x^2+4x+4} = \frac{(x-3)(x+2)}{(x+2)^2} = \frac{x-3}{x+2}$. Since $x = -2$ continues to produce a 0 in the denominator of the reduced function, we know $x = -2$ is a vertical asymptote to the graph, which the calculator confirms.



The graph of $y = h(x)$



The graph of $y = r(x)$

$\square$

Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[6] The unfortunate name will make sense shortly.

Example 1.1.3. A mathematical model for the population P, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Determine the behavior of P as $t \to 5^-$. Interpret this result graphically and within the context of the problem.

Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$, which means 4000 bacteria are initially introduced into the environment.
2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in **thousands**. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Substituting for $P(t)$ gives the equation $\frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.
3. To determine the behavior of P as $t \to 5^-$, we can make a table

t	$P(t)$
4.9	10000
4.99	1000000
4.999	100000000
4.9999	10000000000

[6]This is a class of Calculus equations in which a population grows very rapidly.

In other words, as $t \to 5^-$, $P(t) \to \infty$. Graphically, the line $t = 5$ is a vertical asymptote of the graph of $y = P(t)$. Physically, this means the population of bacteria is increasing without bound as we near 5 days, which cannot physically happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse. $\square$

Now that we have thoroughly investigated vertical asymptotes, we now turn our attention to horizontal asymptotes. The next theorem tells us when to expect horizontal asymptotes.

THEOREM 1.2. **Location of Horizontal Asymptotes:** Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a]The use of the definite article will be justified momentarily.

Like Theorem 1.1, Theorem 1.2 is proved using Calculus. Nevertheless, we can understand the idea behind it using our example $f(x) = \frac{2x-1}{x+1}$. If we interpret $f(x)$ as a division problem, $(2x - 1) \div (x + 1)$, we find the quotient is 2 with a remainder of -3. Using what we know about polynomial division, specifically Theorem **??**, we get $2x - 1 = 2(x + 1) - 3$. Dividing both sides by $(x + 1)$ gives $\frac{2x-1}{x+1} = 2 - \frac{3}{x+1}$. (You may remember this as the formula for $g(x)$ in Example 1.1.1.) As x becomes unbounded in either direction, the quantity $\frac{3}{x+1}$ gets closer and closer to zero so that the values of $f(x)$ become closer and closer to 2. In symbols, as $x \to \pm\infty$, $f(x) \to 2$, and we have the result.[7] Notice that the graph gets close to the same y value as $x \to -\infty$ or $x \to \infty$. This means that the graph can have only one horizontal asymptote if it is going to have one at all. Thus we were justified in using 'the' in the previous theorem. (By the way, using long division to determine the asymptote will serve us well in the next section so you might want to review that topic.)

Alternatively, we can use what we know about end behavior of polynomials to help us understand this theorem. From Theorem **??**, we know the end behavior of a polynomial is determined by its leading term. Applying this to the numerator and denominator of $f(x)$, we get that as $x \to \pm\infty$,

[7]Note that as $x \to -\infty$, $f(x) \to 2^+$, whereas as $x \to \infty$, $f(x) \to 2^-$. We write $f(x) \to 2$ if we are unconcerned from which direction the function values $f(x)$ approach the number 2.

$f(x) = \frac{2x-1}{x+1} \approx \frac{2x}{x} = 2$. This last approach is useful in Calculus, and, indeed, is made rigorous there. (Keep this in mind for the remainder of this paragraph.) Applying this reasoning to the general case, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively. If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case. If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{a}{bx^{m-n}} \to 0$ as $x \to \pm\infty$. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^{n-m}}{b}$, which becomes unbounded as $x \to \pm\infty$. As we said before, if a rational function has a horizontal asymptote, then it will have only one. (This is not true for other types of functions we shall see in later chapters.)

Example 1.1.4. Determine the horizontal asymptotes, if any, of the graphs of the following functions. Verify your answers using a graphing calculator.

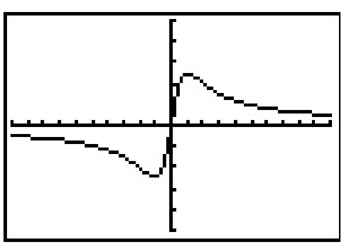
1. $f(x) = \dfrac{5x}{x^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(x) = \dfrac{6x^3-3x+1}{5-2x^3}$

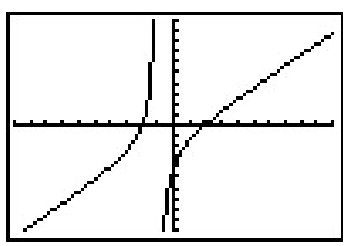
Solution.

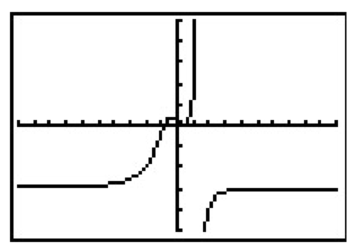
1. The numerator of $f(x)$ is $5x$, which is degree 1. The denominator of $f(x)$ is x^2+1, which is degree 2. Applying Theorem 1.2, $y = 0$ is the horizontal asymptote. Sure enough, as $x \to \pm\infty$, the graph of $y = f(x)$ gets closer and closer to the x-axis.

2. The numerator of $g(x)$, x^2-4, is degree 2, but the degree of the denominator, $x+1$, is degree 1. By Theorem 1.2, there is no horizontal asymptote. From the graph, we see the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote.[8]

3. The degrees of the numerator and denominator of $h(x)$ are both three, so Theorem 1.2 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. The calculator confirms this.


The graph of $y = f(x)$


The graph of $y = g(x)$


The graph of $y = h(x)$

□

[8]The graph does, however, seem to resemble a non-constant line as $x \to \pm\infty$. We will discuss this phenomenon in the next section.

Our last example of the section gives us a real-world application of a horizontal asymptote. Though the population below is more accurately modeled with the functions in Chapter **??**, we approximate it[9] using a rational function.

EXAMPLE 1.1.5. The number of students, N, at local college who have had the flu t months after the semester begins can be modeled by the formula $N(t) = 500 - \frac{450}{1+3t}$ for $t \geq 0$.

1. Find and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Determine the behavior of N as $t \to \infty$. Interpret this result graphically and within the context of the problem.

SOLUTION.

1. $N(0) = 500 - \frac{450}{1+3(0)} = 50$. This means that at the beginning of the semester, 50 students have had the flu.
2. We set $N(t) = 300$ to get $500 - \frac{450}{1+3t} = 300$ and solve. Isolating the fraction gives $\frac{450}{1+3t} = 200$. Clearing denominators gives $450 = 200(1 + 3t)$. Finally, we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.
3. To determine the behavior of N as $t \to \infty$, we can use a table.

t	$N(t)$
10	≈ 485.48
100	≈ 498.50
1000	≈ 499.85
10000	≈ 499.98

The table suggests that as $t \to \infty$, $N(t) \to 500$. (More specifically, 500^-.) This means as time goes by, only a total of 500 students will have ever had the flu. □

[9] Using techniques you'll see in Calculus.

1.1.1 Exercises

1. For each rational function f given below:

   - Find the domain of f.
   - Identify any vertical asymptotes of the graph of $y = f(x)$ and describe the behavior of the graph near them using proper notation.
   - Identify any holes in the graph.
   - Find the horizontal asymptote, if it exists, and describe the end behavior of f using proper notation.

   (a) $f(x) = \dfrac{x}{3x-6}$

   (b) $f(x) = \dfrac{3+7x}{5-2x}$

   (c) $f(x) = \dfrac{x}{x^2+x-12}$

   (d) $f(x) = \dfrac{x}{x^2+1}$

   (e) $f(x) = \dfrac{x+7}{(x+3)^2}$

   (f) $f(x) = \dfrac{x^3+1}{x^2-1}$

   (g) $f(x) = \dfrac{4x}{x^2+4}$

   (h) $f(x) = \dfrac{4x}{x^2-4}$

   (i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6}$

   (j) $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

   (k) $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

2. In Exercise **??** in Section **??**, the population of Sasquatch in Portage County was modeled by the function $P(t) = \frac{150t}{t+15}$, where $t = 0$ represents the year 1803. Find the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

3. In Exercise **??** in Section **??**, we fit a few polynomial models to the following electric circuit data. (The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in the table below.)[10]

Resistance: $(k\Omega)$	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

   Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power to resistance. For this circuit, it is $P(x) = \frac{25x}{(x+3.9)^2}$, where x is the resistance value, $x \geq 0$.

   (a) Graph the data along with the function $y = P(x)$ on your calculator.

   (b) Approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

   (c) Find and interpret the end behavior of $P(x)$ as $x \to \infty$.

[10] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

4. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed. (This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.) Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

1.1.2 Answers

1. (a) $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty$, $f(x) \to \frac{1}{3}^-$
As $x \to \infty$, $f(x) \to \frac{1}{3}^+$

(b) $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-$, $f(x) \to \infty$
As $x \to \frac{5}{2}^+$, $f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty$, $f(x) \to -\frac{7}{2}^+$
As $x \to \infty$, $f(x) \to -\frac{7}{2}^-$

(c) $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4, x = 3$
As $x \to -4^-$, $f(x) \to -\infty$
As $x \to -4^+$, $f(x) \to \infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

(d) $f(x) = \dfrac{x}{x^2+1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

(e) $f(x) = \dfrac{x+7}{(x+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $x = -3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[11]As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

(f) $f(x) = \dfrac{x^3+1}{x^2-1} = \dfrac{x^2-x+1}{x-1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $x = 1$
As $x \to 1^-$, $f(x) \to -\infty$
As $x \to 1^+$, $f(x) \to \infty$
Hole at $(-1, -\frac{3}{2})$
No horizontal asymptote
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$

(g) $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

[11]This is hard to see on the calculator, but trust me, the graph is below the x-axis to the left of $x = -7$.

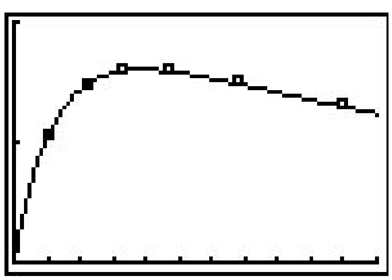
(h) $f(x) = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$
Domain: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
Vertical asymptotes: $x=-2, x=2$
As $x \to -2^-, f(x) \to -\infty$
As $x \to -2^+, f(x) \to \infty$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y=0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

(i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$
Domain: $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$
Vertical asymptote: $x=2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3,\frac{7}{5}\right)$
Horizontal asymptote: $y=1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$

(j) $f(x) = \dfrac{3x^2-5x-2}{x^2-9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
Vertical asymptotes: $x=-3, x=3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to -\infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y=3$
As $x \to -\infty, f(x) \to 3^+$
As $x \to \infty, f(x) \to 3^-$

(k) $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$
Vertical asymptote: $x=2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $(-1,0)$
No horizontal asymptote
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$

2. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y=150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

3. (a) 

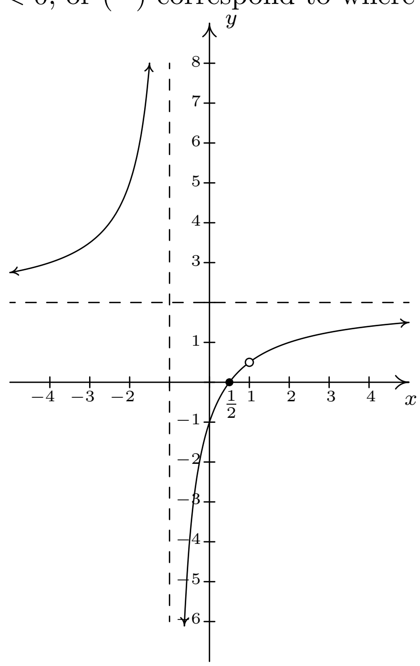
(b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

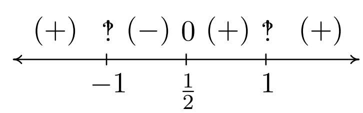
(c) As $x \to \infty,\ P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

1.2 Graphs of Rational Functions

In this section, we take a closer look at graphing rational functions. In Section 1.1, we learned that the graphs of rational functions may include vertical asymptotes, holes in the graph, and horizontal asymptotes. Theorems 1.1 and 1.2 tell us exactly when and where these behaviors will occur, and if we combine these results with what we already know about graphing functions, we will quickly be able to generate reasonable graphs of rational functions.

One of the standard tools we will use is the sign diagram which was first introduced in Section **??**, and then revisited in Section **??**. In those sections, we operated under the belief that a function couldn't change its sign without its graph crossing through the x-axis. The major theorem we used to justify this belief was the Intermediate Value Theorem, Theorem **??**. It turns out the Intermediate Value Theorem applies to all **continuous** functions,[1] not just polynomials. Although rational functions are continuous on their domains,[2] Theorem 1.1 tells us vertical asymptotes and holes occur at the values excluded from their domains. In other words, rational functions aren't continuous at these excluded values which leaves open the possibility that the function could change sign **without** crossing through the x-axis. Consider the graph of $y = h(x)$ from Example 1.1.1, recorded below for convenience. We have added its x-intercept at $\left(\frac{1}{2}, 0\right)$ for the discussion that follows. Suppose we wish to construct a sign diagram for $h(x)$. Recall that the intervals where $h(x) > 0$, or $(+)$, correspond to the x-values where the graph of $y = h(x)$ is **above** the x-axis; the intervals on which $h(x) < 0$, or $(-)$ correspond to where the graph is **below** the x-axis.





As we examine the graph of $y = h(x)$, reading from left to right, we note that from $(-\infty, -1)$,

[1] Recall that, for our purposes, this means the graphs are devoid of any breaks, jumps or holes

[2] Another result from Calculus.

the graph is above the x-axis, so $h(x)$ is $(+)$ there. At $x = -1$, we have a vertical asymptote, at which point the graph 'jumps' across the x-axis. On the interval $\left(-1, \frac{1}{2}\right)$, the graph is below the x-axis, so $h(x)$ is $(-)$ there. The graph crosses through the x-axis at $\left(\frac{1}{2}, 0\right)$ and remains above the x-axis until $x = 1$, where we have a 'hole' in the graph. Since $h(1)$ is undefined, there is no sign here. So we have $h(x)$ as $(+)$ on the interval $\left(\frac{1}{2}, 1\right)$. Continuing, we see that on $(1, \infty)$, the graph of $y = h(x)$ is above the x-axis, and so we mark $(+)$ there. To construct a sign diagram from this information, we not only need to denote the zero of h, but also the places not in the domain of h. As is our custom, we write '0' above $\frac{1}{2}$ on the sign diagram to remind us that it is a zero of h. We need a different notation for -1 and 1, and we have chosen to use '‽' - a nonstandard symbol called the interrobang. We use this symbol to convey a sense of surprise, caution, and wonderment - an appropriate attitude to take when approaching these points. The moral of the story is that when constructing sign diagrams for rational functions, we include the zeros as well as the values excluded from the domain.

Steps for Constructing a Sign Diagram for a Rational Function

Suppose r is a rational function.

1. Place any values excluded from the domain of r on the number line with an '‽' above them.
2. Find the zeros of r and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $r(x)$ for each test value in step 3, and write that sign above the corresponding interval.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Find the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable.
3. Find the x- and y-intercepts of the graph of $y = r(x)$, if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of r on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of r. Use long division, as needed.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = r(x)$.

Example 1.2.1. Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2 - 4}$.

Solution. We follow the six step procedure outlined above.

1. As usual, we set the denominator equal to zero to get $x^2 - 4 = 0$. We find $x = \pm 2$, so our domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
2. To reduce $f(x)$ to lowest terms, we factor the numerator and denominator which yields $f(x) = \frac{3x}{(x-2)(x+2)}$. There are no common factors which means $f(x)$ is already in lowest terms.
3. To find the x-intercepts of the graph of $y = f(x)$, we set $y = f(x) = 0$. Solving $\frac{3x}{(x-2)(x+2)} = 0$ results in $x = 0$. Since $x = 0$ is in our domain, $(0, 0)$ is the x-intercept. To find the y-intercept, we set $x = 0$ and find $y = f(0) = 0$, so that $(0, 0)$ is our y-intercept as well.[3]
4. The two numbers excluded from the domain of f are $x = -2$ and $x = 2$. Since $f(x)$ didn't reduce at all, both of these values of x still cause trouble in the denominator, and so, by Theorem 1.1, $x = -2$ and $x = 2$ are vertical asymptotes of the graph. We can actually go a step farther at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,[4] it is good practice for those heading off to Calculus. For the discussion that follows, it is best to use the factored form of $f(x) = \frac{3x}{(x-2)(x+2)}$.

[3] As we mentioned at least once earlier, since functions can have at most one y-intercept, once we find $(0, 0)$ is on the graph, we know it is the y-intercept.

[4] The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

- *The behavior of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use x-values a little less than -2, say -2.1, -2.01 and -2.001. While there is no harm in actually building a table like we did in Section 1.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to -6, the quantity $(x-2)$ would be very close to -4, and the factor $(x+2)$ would be very close to 0. More specifically, $(x+2)$ would be a little less than 0, in this case, -0.000001. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value. So, mentally, as $x \to -2^-$, we estimate

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)\,(\text{very small }(-))} = \frac{3}{2\,(\text{very small }(-))}$$

Now, the closer x gets to -2, the smaller $(x+2)$ will become, and so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

The term 'very big $(-)$' means a number with a large absolute value which is negative.[5] What all of this means is that as $x \to -2^-$, $f(x) \to -\infty$. Now suppose we wanted to determine the behavior of $f(x)$ as $x \to -2^+$. If we imagine substituting something a little larger than -2 in for x, say -1.999999, we mentally estimate

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small }(+))} = \frac{3}{2\,(\text{very small }(+))} \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$$

We conclude that as $x \to -2^+$, $f(x) \to \infty$.

- *The behavior of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

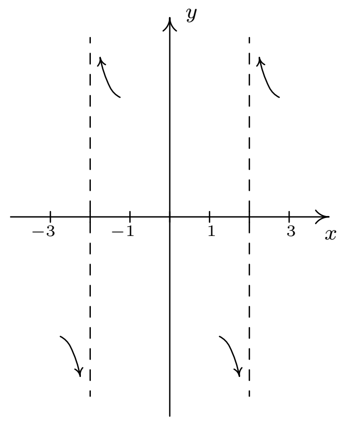
$$f(x) \approx \frac{6}{(\text{very small }(-))\,(4)} = \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

We conclude that as $x \to 2^-$, $f(x) \to -\infty$. Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$, we get $f(x) \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$. So as $x \to 2^+$, $f(x) \to \infty$.

Graphically, we have that near $x = -2$ and $x = 2$ the graph of $y = f(x)$ looks like[6]

[5] The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.

[6] We have deliberately left off the labels on the y-axis because we know only the behavior near $x = \pm 2$, not the actual function values.



5. Next, we determine the end behavior of the graph of $y = f(x)$. Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 1.2 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \frac{3x}{x^2-4}$.

   - *The behavior of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behavior of f as $x \to -\infty$, we would substitute very 'large' negative numbers in for x, say, for example, $x = -1$ billion. The numerator $3x$ would then be -3 billion, whereas the denominator $x^2 - 4$ would be $(-1 \text{ billion})^2 - 4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

   $$f\left(-1 \text{ billion}\right) \approx \frac{-3\,\text{billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small } (-)$$

   Notice that if we substituted in $x = -1$ trillion, essentially the same kind of cancellation would happen, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit **below** the x-axis as we move to the far left.

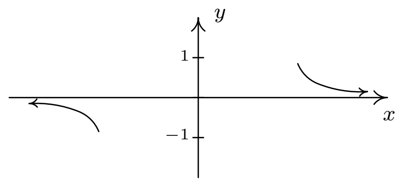
   - *The behavior of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for x and looking at the behavior of $f(x)$. For example, let $x = 1$ billion. Proceeding as before, we get

   $$f\left(1 \text{ billion}\right) \approx \frac{3\,\text{billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small } (+)$$

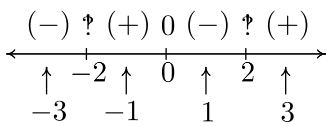
   The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit **above** the x-axis as we look toward the far right.

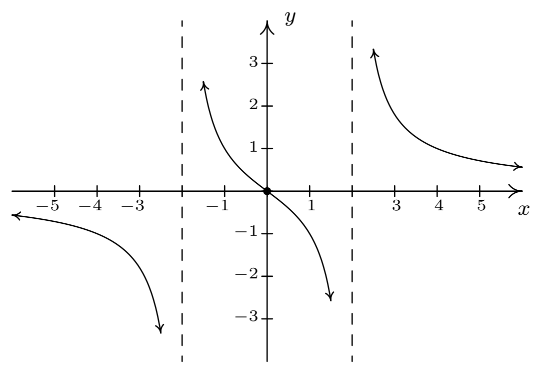
   Graphically, we have[7]

[7] As with the vertical asymptotes in the previous step, we know only the behavior of the graph as $x \to \pm\infty$. For that reason, we provide no x-axis labels.



6. Lastly, we construct a sign diagram for $f(x)$. The x-values excluded from the domain of f are $x = \pm 2$, and the only zero of f is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and we choose the test values[8] we $x = -3$, $x = -1$, $x = 1$, and $x = 3$. We find $f(-3)$ is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$, and $f(3)$ is $(+)$. Combining this with our previous work, we get the graph of $y = f(x)$ below.





□

A couple of notes are in order. First, the graph of $y = f(x)$ certainly seems to possess symmetry with respect to the origin. In fact, we can check $f(-x) = -f(x)$ to see that f is an odd function. In some textbooks, checking for symmetry is part of the standard procedure for graphing rational functions; but since it happens comparatively rarely[9] we'll just point it out when we see it. Also note that while $y = 0$ is the horizontal asymptote, the graph of f nevertheless crosses the x-axis at $(0, 0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false, as we shall see again in our next example.

Example 1.2.2. Sketch a detailed graph of $g(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$.

Solution.

1. Setting $x^2 - x - 6 = 0$ gives $x = -2$ and $x = 3$. Our domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

2. Factoring $g(x)$ gives $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$. There is no cancellation, so $g(x)$ is in lowest terms.

[8] In this particular case, we can eschew test values, since our analysis of the behavior of f near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.

[9] And Jeff doesn't think much of it to begin with...

3. To find the x-intercept we set $y = g(x) = 0$. Using the factored form of $g(x)$ above, we find the zeros to be the solutions of $(2x-5)(x+1) = 0$. We obtain $x = \frac{5}{2}$ and $x = -1$. Since both of these numbers are in the domain of g, we have two x-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we set $x = 0$ and find $y = g(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

4. Since $g(x)$ was given to us in lowest terms, we have, once again by Theorem 1.1 vertical asymptotes $x = -2$ and $x = 3$. Keeping in mind $g(x) = \frac{(2x-5)(x+1)}{(x-3)(x+2)}$, we proceed to our analysis near each of these values.

   - *The behavior of $y = g(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have

     $$g(x) \approx \frac{(-9)(-1)}{(-5)(\text{very small }(-))} \approx \frac{9}{\text{very small }(+)} \approx \text{very big }(+)$$

     so as $x \to -2^-$, $g(x) \to \infty$. On the flip side, as $x \to -2^+$, we get

     $$g(x) \approx \frac{9}{\text{very small }(-)} \approx \text{very big }(-)$$

     so $g(x) \to -\infty$.
   - *The behavior of $y = g(x)$ as $x \to 3$:* As $x \to 3^-$, we imagine plugging in a number just shy of 3. We have

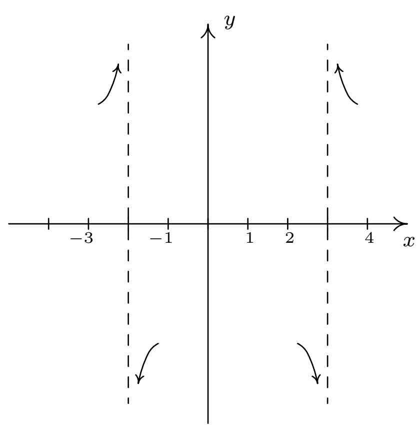
     $$g(x) \approx \frac{(1)(4)}{(\text{ very small }(-))(5)} \approx \frac{4}{\text{very small }(-)} \approx \text{very big }(-)$$

     Hence, as $x \to 3^-$, $g(x) \to -\infty$. As $x \to 3^+$, we get

     $$g(x) \approx \frac{4}{\text{very small }(+)} \approx \text{very big }(+)$$

     so $g(x) \to \infty$.

   Graphically, we have (again, without labels on the y-axis)



5. Since the degrees of the numerator and denominator of $g(x)$ are the same, we know from Theorem 1.2 that we can find the horizontal asymptote of the graph of g by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behavior which would be lost otherwise.[10] As in the discussion following Theorem 1.2, we use the result of the long division $\left(2x^2-3x-5\right) \div \left(x^2-x-6\right)$ to rewrite $g(x) = \frac{2x^2-3x-5}{x^2-x-6}$ as $g(x) = 2 - \frac{x-7}{x^2-x-6}$. We focus our attention on the term $\frac{x-7}{x^2-x-6}$.

   - *The behavior of $y = g(x)$ as $x \to -\infty$:* If imagine substituting $x = -1$ billion into $\frac{x-7}{x^2-x-6}$, we estimate $\frac{x-7}{x^2-x-6} \approx \frac{-1 \text{ billion}}{1\text{billion}^2} \approx$ very small $(-)$.[11] Hence,

   $$g(x) = 2 - \frac{x-7}{x^2-x-6} \approx 2 - \text{very small } (-) = 2 + \text{very small } (+)$$

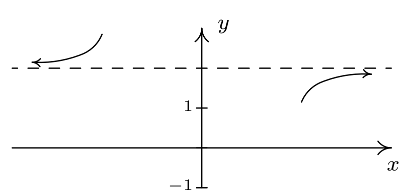
   In other words, as $x \to -\infty$, the graph of $y = g(x)$ is a little bit **above** the line $y = 2$.

   - *The behavior of $y = g(x)$ as $x \to \infty$.* To consider $\frac{x-7}{x^2-x-6}$ as $x \to \infty$, we imagine substituting $x = 1$ billion and, going through the usual mental routine, find

   $$\frac{x-7}{x^2-x-6} \approx \text{very small } (+)$$

   Hence, $g(x) \approx 2 -$ very small $(+)$, in other words, the graph of $y = g(x)$ is just **below** the line $y = 2$ as $x \to \infty$.

   On $y = g(x)$, we have (again, without labels on the x-axis)

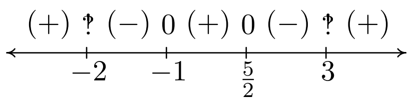


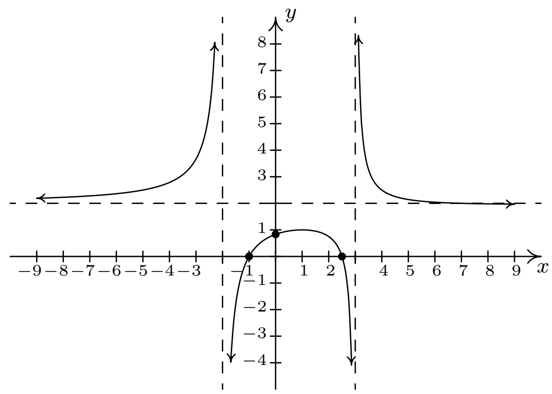
6. Finally we construct our sign diagram. We place an '‽' above $x = -2$ and $x = 3$, and a '0' above $x = \frac{5}{2}$ and $x = -1$. Choosing test values in the test intervals gives us $f(x)$ is $(+)$ on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$, and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$. As we piece together all of the information, we note that the graph must cross the horizontal asymptote at some point after $x = 3$ in order for it to approach $y = 2$ from underneath. This is the subtlety that we would have missed had we skipped the long division and subsequent end behavior analysis. We can, in fact, find exactly when the graph crosses $y = 2$. As a result

[10] That is, if you use a calculator to graph. Once again, Calculus is the ultimate graphing power tool.

[11] In the denominator, we would have $(1\text{billion})^2 - 1\text{billion} - 6$. It's easy to see why the 6 is insignificant, but to ignore the 1 billion seems criminal. However, compared to $(1 \text{ billion})^2$, it's on the insignificant side; it's 10^{18} versus 10^9. We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the x^2 term wins out over the x term.

of the long division, we have $g(x) = 2 - \frac{x-7}{x^2-x-6}$. For $g(x) = 2$, we would need $\frac{x-7}{x^2-x-6} = 0$. This gives $x - 7 = 0$, or $x = 7$. Note that $x - 7$ is the remainder when $2x^2 - 3x - 5$ is divided by $x^2 - x - 6$, and so it makes sense that for $g(x)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7) = 2$. Moreover, it stands to reason that g must attain a relative minimum at some point past $x = 7$. Calculus verifies that at $x = 13$, we have such a minimum at exactly $(13, 1.96)$. The reader is challenged to find calculator windows which show the graph crossing its horizontal asymptote on one window, and the relative minimum in the other.





□

Our next example gives us not only a hole in the graph, but also some slightly different end behavior.

Example 1.2.3. Sketch a detailed graph of $h(x) = \dfrac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$.

Solution.

1. For domain, you know the drill. Solving $x^2 + 3x + 2 = 0$ gives $x = -2$ and $x = -1$. Our answer is $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

2. To reduce $h(x)$, we need to factor the numerator and denominator. We get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} = \frac{(2x+1)(x+1)^{\cancelto{1}{2}}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

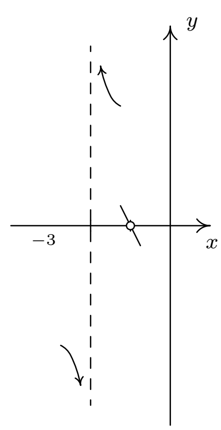
   We will use this reduced formula for $h(x)$ as long as we're not substituting $x = -1$. To make this exclusion specific, we write $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

3. To find the x-intercepts, as usual, we set $h(x) = 0$ and solve. Solving $\frac{(2x+1)(x+1)}{x+2} = 0$ yields $x = -\frac{1}{2}$ and $x = -1$. The latter isn't in the domain of h, so we exclude it. Our only x-intercept is $\left(-\frac{1}{2}, 0\right)$. To find the y-intercept, we set $x = 0$. Since $0 \neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0) = \frac{1}{2}$ for a y-intercept of $\left(0, \frac{1}{2}\right)$.

4. From Theorem 1.1, we know that since $x = -2$ still poses a threat in the denominator of the reduced function, we have a vertical asymptote there. As for $x = -1$, we note the factor $(x+1)$ was canceled from the denominator when we reduced $h(x)$, and so it no longer causes trouble there. This means we get a hole when $x = -1$. To find the y-coordinate of the hole, we substitute $x = -1$ into $\frac{(2x+1)(x+1)}{x+2}$, per Theorem 1.1 and get 0. Hence, we have a hole on the x-axis at $(-1, 0)$. It should make you uncomfortable plugging $x = -1$ into the reduced formula for $h(x)$, especially since we've made such a big deal concerning the stipulation about not letting $x = -1$ for that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near $x = -1$ which we will show below. Speaking of which, for the discussion that follows, we will use the formula $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

   - *The behavior of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small }(-))} \approx \frac{3}{(\text{very small }(-))} \approx$ very big $(-)$ and so as $x \to -2^-$, $h(x) \to -\infty$. On the other side of -2, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big $(+)$, so $h(x) \to \infty$.
   - *The behavior of $y = h(x)$ as $x \to -1$.* As $x \to -1^-$, we imagine plugging in a number a bit less than $x = -1$. We have $h(x) \approx \frac{(-1)(\text{very small }(-))}{1} =$ very small $(+)$ Hence, as $x \to -1^-$, $h(x) \to 0^+$. This means, as $x \to -1^-$, the graph is a bit above the point $(-1, 0)$. As $x \to -1^+$, we get $h(x) \approx \frac{(-1)(\text{very small }(+))}{1} =$ very small $(-)$. This gives us that as $x \to -1^+$, $h(x) \to 0^-$, so the graph is a little bit lower than $(-1, 0)$ here.

   Graphically, we have



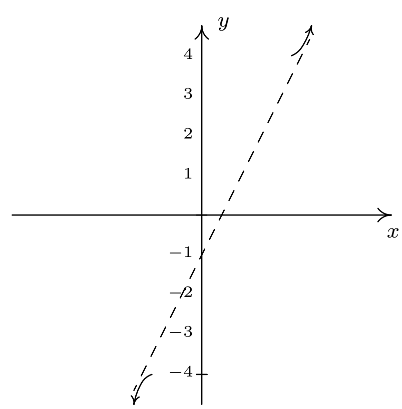
5. For end behavior, we note that the degree of the numerator of $h(x)$, $2x^3 + 5x^2 + 4x + 1$ is 3, and the degree of the denominator, $x^2 + 3x + 2$, is 2. Theorem 1.2 is of no help here, since the degree of the numerator is greater than the degree of the denominator. That won't stop us, however, in our analysis. Since for end behavior we are considering values of x as $x \to \pm\infty$, we are far enough away from $x = -1$ to use the reduced formula, $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \frac{2x^2+3x+1}{x+2}$, $x \neq -1$,

and, as a result, we rewrite $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$. As in the previous example, we focus our attention on the term generated from the remainder, $\frac{3}{x+2}$.

- *The behavior of $y = h(x)$ as $x \to -\infty$:* Substituting $x = -1$ billion into $\frac{3}{x+2}$, we get the estimate $\frac{3}{-1 \text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x-1+\frac{3}{x+2} \approx 2x-1+$very small $(-)$. This means the graph of $y = h(x)$ is a little bit **below** the line $y = 2x - 1$ as $x \to -\infty$.

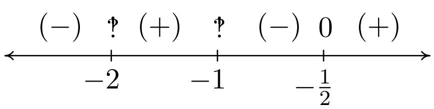
- *The behavior of $y = h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\frac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x - 1 +$ very small $(+)$, or that the graph of $y = h(x)$ is a little bit **above** the line $y = 2x - 1$ as $x \to \infty$.

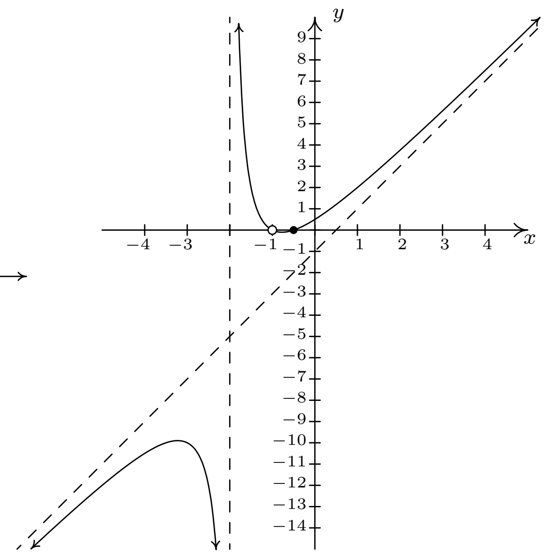
This is end behavior unlike any we've ever seen. Instead of approaching a horizontal line, the graph is approaching a slanted line. For this reason, $y = 2x - 1$ is called a **slant asymptote**[12] of the graph of $y = h(x)$. A slant asymptote will always arise when the degree of the numerator is exactly one more than the degree of the denominator, and there's no way to determine exactly what it is without going through the long division. Graphically we have



6. To make our sign diagram, we place an '‽' above $x = -2$ and $x = -1$ and a '0' above $x = -\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2, -1)$ and $\left(-\frac{1}{2}, \infty\right)$ and $h(x)$ is $(-)$ on $(-\infty, -2)$ and $\left(-1, -\frac{1}{2}\right)$. Putting all of our work together yields the graph below.

[12] Also called an 'oblique' asymptote in some texts.





We could ask whether the graph of $y = h(x)$ crosses its slant asymptote. From the formula $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$, we see that if $h(x) = 2x - 1$, we would have $\frac{3}{x+2} = 0$. Since this will never happen, we conclude the graph never crosses its slant asymptote.[13] □

We end this section with an example that shows it's not all pathological weirdness when it comes to rational functions and technology still has a role to play in studying their graphs at this level.

Example 1.2.4. Sketch the graph of $r(x) = \dfrac{x^4+1}{x^2+1}$.

Solution.

1. The denominator $x^2 + 1$ is never zero so the domain is $(-\infty, \infty)$.

2. With no real zeros in the denominator, $x^2 + 1$ is an irreducible quadratic. Our only hope of reducing $r(x)$ is if $x^2 + 1$ is a factor of $x^4 + 1$. Performing long division gives us

$$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$$

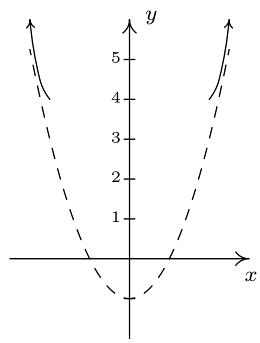
The remainder is not zero so $r(x)$ is already reduced.

3. To find the x-intercept, we'd set $r(x) = 0$. Since there are no real solutions to $\frac{x^4+1}{x^2+1} = 0$, we have no x-intercepts. Since $r(0) = 1$, so we get $(0, 1)$ for the y-intercept.

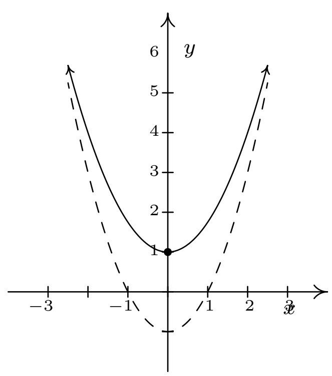
4. This step doesn't apply to r, since its domain is all real numbers.

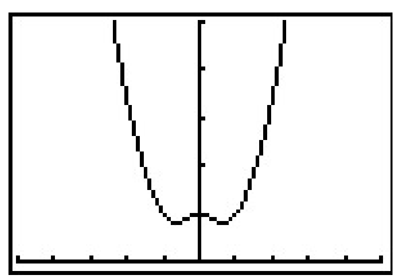
[13]But rest assured, some graphs do!

5. For end behavior, once again, since the degree of the numerator is greater than that of the denominator, Theorem 1.2 doesn't apply. We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \frac{2}{x^2+1}$, and so we focus our attention on the term corresponding to the remainder, $\frac{2}{x^2+1}$ It should be clear that as $x \to \pm\infty$, $\frac{2}{x^2+1} \approx$ very small $(+)$, which means $r(x) \approx x^2 - 1 +$ very small $(+)$. So the graph $y = r(x)$ is a little bit **above** the graph of the parabola $y = x^2 - 1$ as $x \to \pm\infty$. Graphically,



6. There isn't much work to do for a sign diagram for $r(x)$, since its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty, \infty)$, and since we know $r(0) = 1$, we conclude $r(x)$ is $(+)$ for all real numbers. At this point, we don't have much to go on for a graph. Below is a comparison of what we have determined analytically versus what the calculator shows us. We have no way to detect the relative extrema analytically[14] apart from brute force plotting of points, which is done more efficiently by the calculator.





□

[14] Without appealing to Calculus, of course.

1.2.1 Exercises

1. Find the slant asymptote of the graph of the rational function.

   (a) $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$

   (b) $f(x) = \dfrac{2x^2 + 5x - 3}{3x + 2}$

   (c) $f(x) = \dfrac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$

   (d) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

2. Use the six-step procedure to graph each rational function given. Be sure to draw any asymptotes as dashed lines.

   (a) $f(x) = \dfrac{4}{x + 2}$

   (b) $f(x) = \dfrac{5x}{6 - 2x}$

   (c) $f(x) = \dfrac{1}{x^2}$

   (d) $f(x) = \dfrac{1}{x^2 + x - 12}$

   (e) $f(x) = \dfrac{2x - 1}{-2x^2 - 5x + 3}$

   (f) $f(x) = \dfrac{x}{x^2 + x - 12}$

   (g) $f(x) = \dfrac{4x}{x^2 + 4}$

   (h) $f(x) = \dfrac{4x}{x^2 - 4}$

   (i) $f(x) = \dfrac{x^2 - x - 12}{x^2 + x - 6}$

   (j) $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9}$

   (k) $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$

   (l) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$

   (m) [15] $f(x) = \dfrac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$

3. Example 1.2.4 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function. Without Calculus, we need to use our graphing calculators to reveal the hidden mysteries of rational function behavior. Working with your classmates, use a graphing calculator to examine the graphs of the following rational functions. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

   (a) $f(x) = \dfrac{1}{x^2 + 1}$

   (b) $f(x) = \dfrac{x}{x^2 + 1}$

   (c) $f(x) = \dfrac{x^2}{x^2 + 1}$

   (d) $f(x) = \dfrac{x^3}{x^2 + 1}$

[15] Once you've done the six-step procedure, use your calculator to graph this function on the viewing window $[0, 12] \times [0, 0.25]$. What do you see?

4. Graph the following rational functions by applying transformations to the graph of $y = \dfrac{1}{x}$.

   (a) $f(x) = \dfrac{1}{x-2}$

   (b) $g(x) = 1 - \dfrac{3}{x}$

   (c) $h(x) = \dfrac{-2x+1}{x}$ (Hint: Divide)

   (d) $j(x) = \dfrac{3x-7}{x-2}$ (Hint: Long division)

   Discuss with your classmates how you would graph $f(x) = \dfrac{ax+b}{cx+d}$. What restrictions must be placed on a, b, c and d so that the graph is indeed a transformation of $y = \dfrac{1}{x}$?

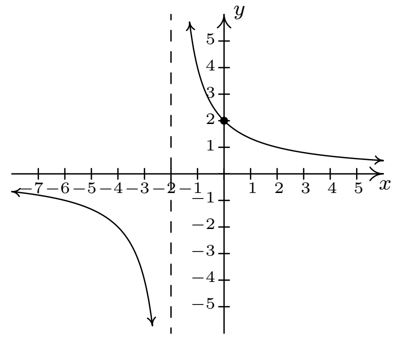
5. In Example **??** in Section **??** we showed that $p(x) = \frac{4x+x^3}{x}$ is not a polynomial even though its formula reduced to $4 + x^2$ for $x \neq 0$. However, it is a rational function similar to those studied in the section. With the help of your classmates, graph $p(x)$.

6. Let $g(x) = \dfrac{x^4 - 8x^3 + 24x^2 - 72x + 135}{x^3 - 9x^2 + 15x - 7}$. With the help of your classmates, find the x- and y- intercepts of the graph of g. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Find all of the asymptotes of the graph of g and any holes in the graph, if they exist. Be sure to show all of your work including any polynomial or synthetic division. Sketch the graph of g, using more than one picture if necessary to show all of the important features of the graph.

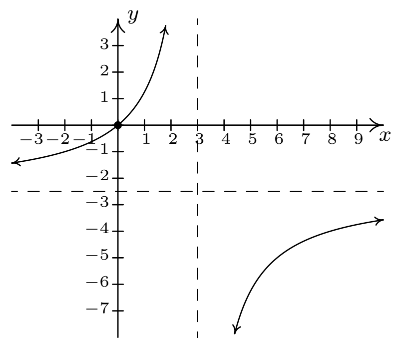
1.2.2 Answers

1. (a) $y = x$
   (b) $y = \frac{2}{3}x + \frac{11}{9}$
   (c) $y = -5x - 18$
   (d) $y = -x$

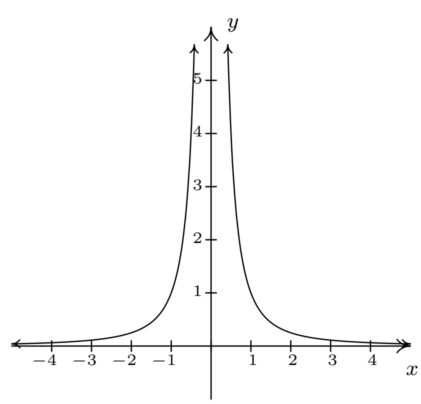
2. (a) $f(x) = \dfrac{4}{x+2}$
   Domain: $(-\infty, -2) \cup (-2, \infty)$
   No x-intercepts
   y-intercept: $(0, 2)$
   Vertical asymptote: $x = -2$
   As $x \to -2^-$, $f(x) \to -\infty$
   As $x \to -2^+$, $f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty$, $f(x) \to 0^-$
   As $x \to \infty$, $f(x) \to 0^+$



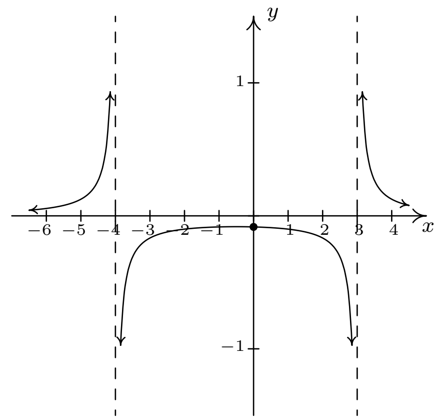
   (b) $f(x) = \dfrac{5x}{6-2x}$
   Domain: $(-\infty, 3) \cup (3, \infty)$
   x-intercept: $(0, 0)$
   y-intercept: $(0, 0)$
   Vertical asymptote: $x = 3$
   As $x \to 3^-$, $f(x) \to \infty$
   As $x \to 3^+$, $f(x) \to -\infty$
   Horizontal asymptote: $y = -\frac{5}{2}$
   As $x \to -\infty$, $f(x) \to -\frac{5}{2}^+$
   As $x \to \infty$, $f(x) \to -\frac{5}{2}^-$



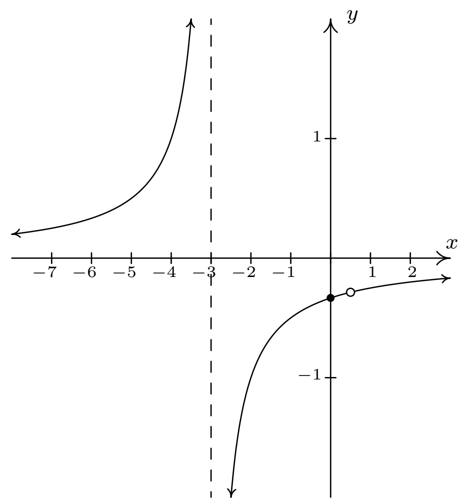
   (c) $f(x) = \dfrac{1}{x^2}$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   No x-intercepts
   No y-intercepts
   Vertical asymptote: $x = 0$
   As $x \to 0^-$, $f(x) \to \infty$
   As $x \to 0^+$, $f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty$, $f(x) \to 0^+$
   As $x \to \infty$, $f(x) \to 0^+$



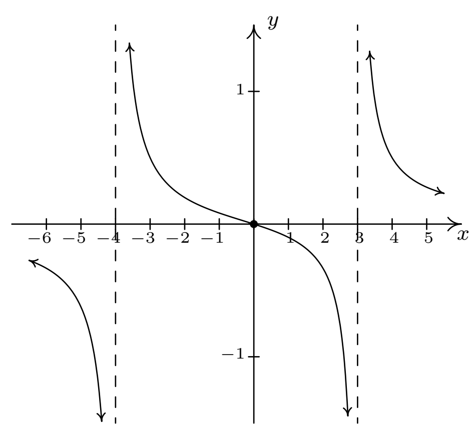
(d) $f(x) = \dfrac{1}{x^2 + x - 12}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
No x-intercepts
y-intercept: $(0, -\frac{1}{12})$
Vertical asymptotes: $x = -4$ and $x = 3$
As $x \to -4^-$, $f(x) \to \infty$
As $x \to -4^+$, $f(x) \to -\infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^+$



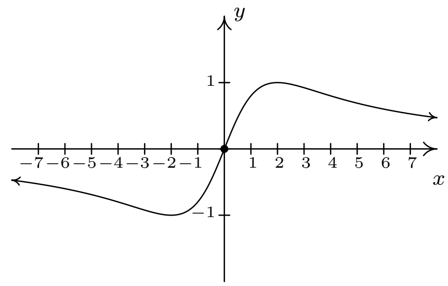
(e) $f(x) = \dfrac{2x - 1}{-2x^2 - 5x + 3}$
Domain: $(-\infty, -3) \cup (-3, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$
No x-intercepts
y-intercept: $(0, -\frac{1}{3})$
$f(x) = \dfrac{-1}{x+3}$, $x \neq \frac{1}{2}$
Hole in the graph at $(\frac{1}{2}, -\frac{2}{7})$
Vertical asymptote: $x = -3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^+$



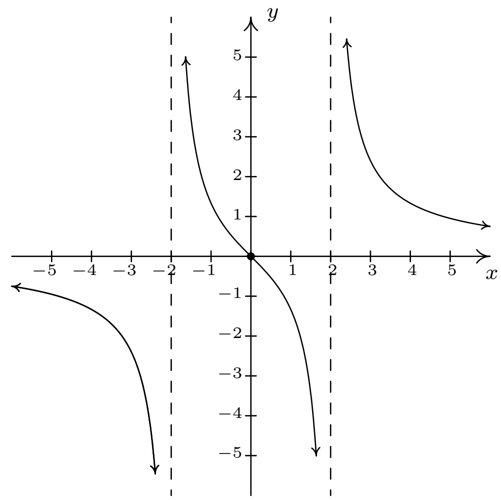
(f) $f(x) = \dfrac{x}{x^2 + x - 12}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -4$ and $x = 3$
As $x \to -4^-$, $f(x) \to -\infty$
As $x \to -4^+$, $f(x) \to \infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$



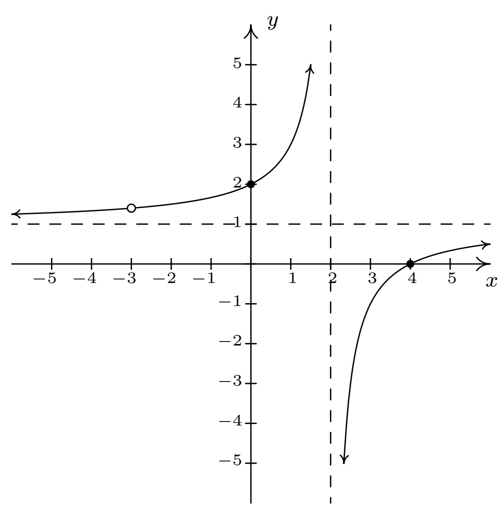
(g) $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
x-intercept: $(0,0)$
y-intercept: $(0,0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y=0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



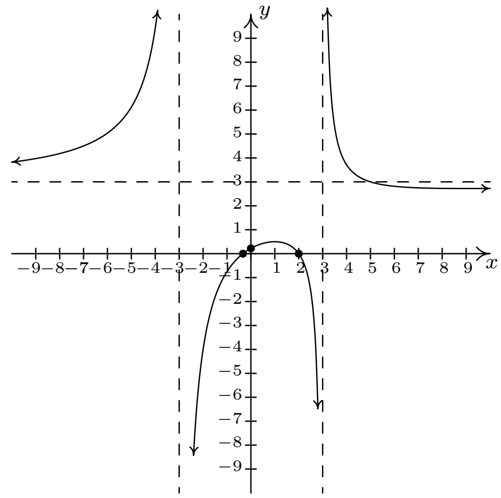
(h) $f(x) = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
x-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $x=-2, x=2$
As $x \to -2^-, f(x) \to -\infty$
As $x \to -2^+, f(x) \to \infty$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y=0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



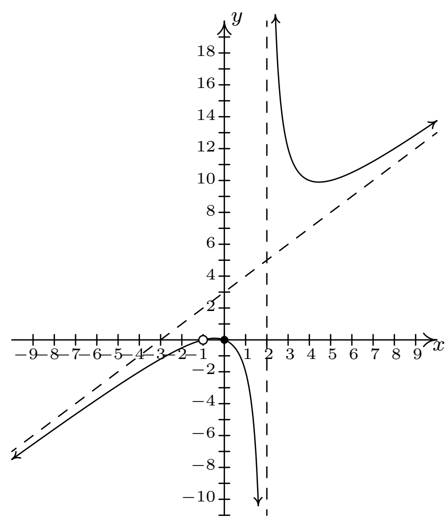
(i) $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
x-intercept: $(4,0)$
y-intercept: $(0,2)$
Vertical asymptote: $x=2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y=1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$



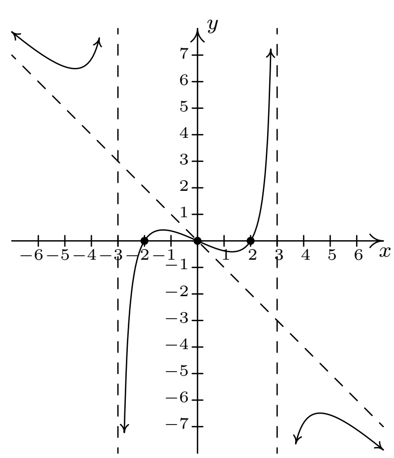
(j) $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
x-intercepts: $\left(-\frac{1}{3}, 0\right)$, $(2, 0)$
y-intercept: $\left(0, \frac{2}{9}\right)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to -\infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty, f(x) \to 3^+$
As $x \to \infty, f(x) \to 3^-$



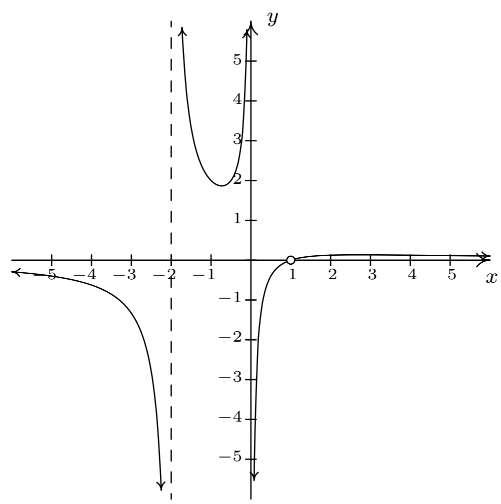
(k) $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty, f(x) \to -\infty$
As $x \to \infty, f(x) \to \infty$



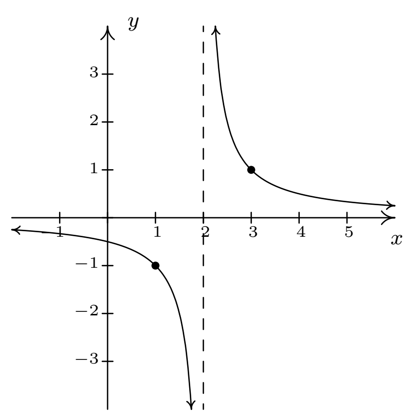
(l) $f(x) = \dfrac{-x^3 + 4x}{x^2 - 9}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
x-intercepts: $(-2, 0), (0, 0), (2, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-,\ f(x) \to \infty$
As $x \to -3^+,\ f(x) \to -\infty$
As $x \to 3^-,\ f(x) \to \infty$
As $x \to 3^+,\ f(x) \to -\infty$
Slant asymptote: $y = -x$
As $x \to -\infty,\ f(x) \to \infty$
As $x \to \infty,\ f(x) \to -\infty$



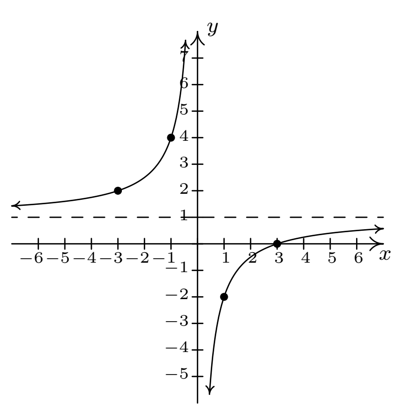
(m) $f(x) = \dfrac{x^2 - 2x + 1}{x^3 + x^2 - 2x}$
Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)$
$f(x) = \dfrac{x-1}{x(x+2)},\ x \neq 1$
No x-intercepts
No y-intercepts
Vertical asymptotes: $x = -2$ and $x = 0$
As $x \to -2^-,\ f(x) \to -\infty$
As $x \to -2^+,\ f(x) \to \infty$
As $x \to 0^-,\ f(x) \to \infty$
As $x \to 0^+,\ f(x) \to -\infty$
Hole in the graph at $(1, 0)$
Horizontal asymptote: $y = 0$
As $x \to -\infty,\ f(x) \to 0^-$
As $x \to \infty,\ f(x) \to 0^+$



4. (a) $f(x) = \dfrac{1}{x-2}$
Shift the graph of $y = \dfrac{1}{x}$ to the right 2 units.

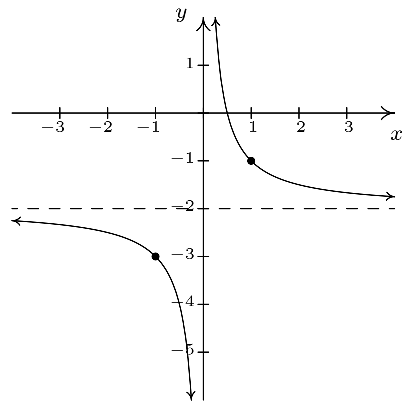


(b) $g(x) = 1 - \dfrac{3}{x}$
Vertically stretch the graph of $y = \dfrac{1}{x}$ by a factor of 3.
Reflect the graph of $y = \dfrac{3}{x}$ about the x-axis.
Shift the graph of $y = -\dfrac{3}{x}$ up 1 unit.



(c) $h(x) = \dfrac{-2x+1}{x} = -2 + \dfrac{1}{x}$

Shift the graph of $y = \dfrac{1}{x}$ down 2 units.

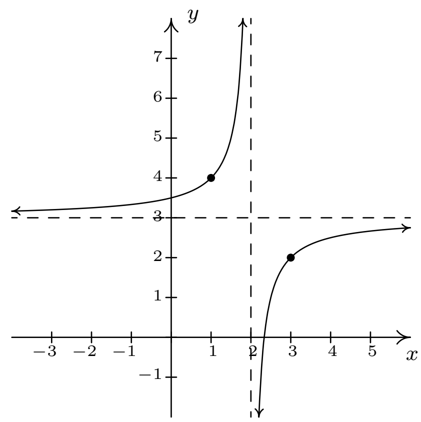


(d) $j(x) = \dfrac{3x-7}{x-2} = 3 - \dfrac{1}{x-2}$

Shift the graph of $y = \dfrac{1}{x}$ to the right 2 units.

Reflect the graph of $y = \dfrac{1}{x-2}$ about the x-axis.

Shift the graph of $y = -\dfrac{1}{x-2}$ up 3 units.



1.3 Rational Inequalities and Applications

In this section, we use sign diagrams to solve rational inequalities including some that arise from real-world applications. Our first example showcases the critical difference in procedure between solving a rational equation and a rational inequality.

Example 1.3.1.

1. Solve $\dfrac{x^3-2x+1}{x-1} = \dfrac{1}{2}x - 1$.

2. Solve $\dfrac{x^3-2x+1}{x-1} \geq \dfrac{1}{2}x - 1$.

3. Use your calculator to graphically check your answers to 1 and 2.

Solution.

1. To solve the equation, we clear denominators

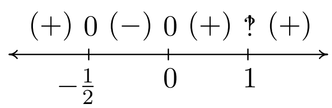
$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & = & \dfrac{1}{2}x-1 & \\
\left(\dfrac{x^3-2x+1}{x-1}\right)\cdot 2(x-1) & = & \left(\dfrac{1}{2}x-1\right)\cdot 2(x-1) & \\
2x^3-4x+2 & = & x^2-3x+2 & \text{expand} \\
2x^3-x^2-x & = & 0 & \\
x(2x+1)(x-1) & = & 0 & \text{factor} \\
x & = & -\frac{1}{2},\ 0,\ 1 &
\end{array}$$

   Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x = 1$ does not satisfy the original equation and must be discarded. Our solutions are $x = -\frac{1}{2}$ and $x = 0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation − namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on x, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 18 in Section 1.2.

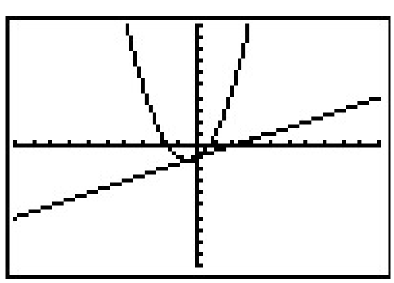
$$
\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1} - \frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right) - x(x-1) + 1(2(x-1))}{2(x-1)} &\geq 0 \qquad \text{get a common denominator} \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 \qquad \text{expand}
\end{aligned}
$$

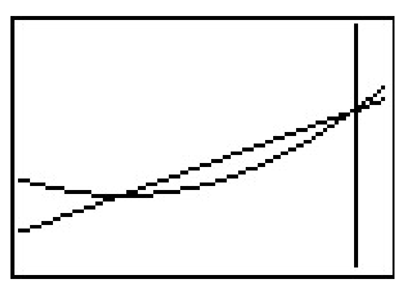
Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of r are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.



We are interested in where $r(x) \geq 0$. We find $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0, 1)$ and $(1, \infty)$. We add to these intervals the zeros of r, $-\frac{1}{2}$ and 0, to get our final solution: $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. Geometrically, if we set $f(x) = \frac{x^3-2x+1}{x-1}$ and $g(x) = \frac{1}{2}x - 1$, the solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above $(>)$ the graph of $g(x)$. We obtain the graphs below.





The 'Intersect' command confirms that the graphs cross when $x = -\frac{1}{2}$ and $x = 0$. It is clear from the calculator that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $\left(-\infty, -\frac{1}{2}\right)$ as well as on $(0, \infty)$. According to the calculator, our solution is then $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$

which **almost** matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, and, even though it isn't shown on the calculator, there is a hole[1] in the graph of $y = f(x)$ when $x = 1$ which is why $x = 1$ needs to be excluded from our final answer. □

Our next example deals with the **average cost** function of PortaBoy Game systems from Example **??** in Section **??**.

Example 1.3.2. Given a cost function $C(x)$, which returns the total cost of producing x products, the **average cost** function, $AC(x) = \frac{C(x)}{x}$, computes the cost per item. Recall that the cost C, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function $AC(x)$. Determine an appropriate applied domain for AC.
2. Find and interpret $AC(10)$.
3. Solve $AC(x) < 100$ and interpret.
4. Determine the behavior of $AC(x)$ as $x \to \infty$ and interpret.

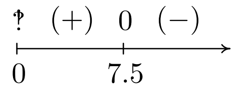
Solution.

1. From $AC(x) = \frac{C(x)}{x}$, we obtain $AC(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $AC(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.
2. We find $AC(10) = \frac{80(10)+150}{10} = 95$, so the average cost to produce 10 game systems is \$95 per system.
3. Solving $AC(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned} \frac{80x+150}{x} &< 100 \\ \frac{80x+150}{x} - 100 &< 0 \\ \frac{80x+150-100x}{x} &< 0 \qquad \text{common denominator} \\ \frac{150-20x}{x} &< 0 \end{aligned}$$

If we take the left hand side to be a rational function $r(x)$, we need to keep in mind the the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.

[1] There is no asymptote at $x = 1$ since the graph is well behaved near $x = 1$. According to Theorem 1.1, there must be a hole there.

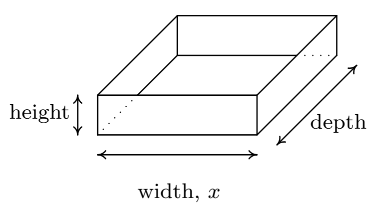


In the context of the problem, x represents the number of PortaBoy games systems produced and $AC(x)$ is the average cost to produce each system. Solving $AC(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

4. We can apply Theorem 1.2 to $AC(x)$ and we find $y = 80$ is a horizontal asymptote to the graph of $y = AC(x)$. To more precisely determine the behavior of $AC(x)$ as $x \to \infty$, we first use long division[2] and rewrite $AC(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^+$, which means $AC(x) \approx 80 +$ very small $(+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $AC(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $AC(x) > 80$ for all $x > 0$. This means the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example **??**, we realize \$80 is the variable cost per system − the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively counterbalance the fixed cost. $\square$

Our next example is another classic 'box with no top' problem.

EXAMPLE 1.3.3. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let x denote the width of the box, in centimeters. Refer to the figure below.



1. Express the height h in centimeters as a function of the width x and state the applied domain.
2. Solve $h(x) \geq x$ and interpret.
3. Find and interpret the behavior of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.
4. Express the surface area S of the box as a function of x and state the applied domain.
5. Use a calculator to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

[2] In this case, long division amounts to term-by-term division.

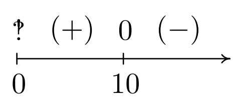
Solution.

1. We are told the volume of the box is 1000 cubic centimeters and that x represents the width, in centimeters. From geometry, we know Volume = width $\times$ height $\times$ depth. Since the base of the box is to be a square, the width and the depth are both x centimeters. Using h for the height, we have $1000 = x^2 h$, so that $h = \frac{1000}{x^2}$. Using function notation,[3] $h(x) = \frac{1000}{x^2}$ As for the applied domain, in order for there to be a box at all, $x > 0$, and since every such choice of x will return a positive number for the height h we have no other restrictions and conclude our domain is $(0, \infty)$.

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality and use a sign diagram.

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator} \end{aligned}$$

We consider the left hand side of the inequality as our rational function $r(x)$. We see r is undefined at $x = 0$, but, as in the previous example, the applied domain of the problem is $x > 0$, so we are considering only the behavior of r on $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, which is $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the following diagram.



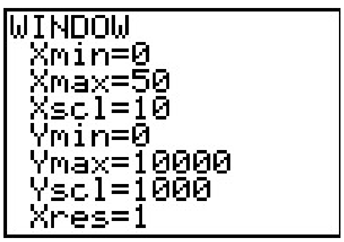
We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, h represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.

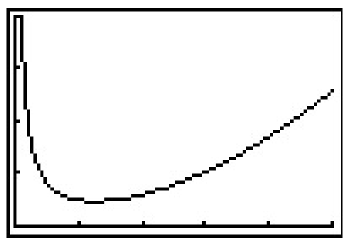
3. As $x \to 0^+$, $h(x) = \frac{1000}{x^2} \to \infty$. This means the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimeters. As $x \to \infty$, we find $h(x) \to 0^+$, which means to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger the smaller the height becomes.

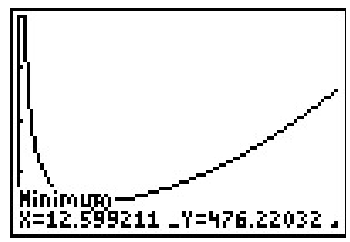
[3]That is, $h(x)$ means 'h of x', not 'h times x' here.

4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by h. We get the surface area, $S = x^2 + 4xh$. To get S as a function of x, we substitute $h = \frac{1000}{x^2}$ to obtain $S = x^2 + 4x\left(\frac{1000}{x^2}\right)$. Hence, as a function of x, $S(x) = x^2 + \frac{4000}{x}$. The domain of S is the same as h, namely $(0, \infty)$, for the same reasons as above.

5. A first attempt at the graph of $y = S(x)$ on the calculator may lead to frustration. Chances are good that the first window chosen to view the graph will suggest $y = S(x)$ has the x-axis as a horizontal asymptote. From the formula $S(x) = x^2 + \frac{4000}{x}$, however, we get $S(x) \approx x^2$ as $x \to \infty$, so $S(x) \to \infty$. Readjusting the window, we find S does possess a relative minimum at $x \approx 12.60$. As far as we can tell,[4] this is the only relative extremum, and so it is the absolute minimum as well. This means the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.







□

In many instances in the sciences, rational functions are encountered as a result of fundamental natural laws which are typically a result of assuming certain basic relationships between variables. These basic relationships are summarized in the definition below.

DEFINITION 1.4. Suppose x, y, and z are variable quantities. We say

- y **varies directly with** (or is **directly proportional to**) x if there is a constant k such that $y = kx$.
- y **varies inversely with** (or is **inversely proportional to**) x if there is a constant k such that $y = \frac{k}{x}$.
- z **varies jointly with** (or is **jointly proportional to**) x and y if there is a constant k such that $z = kxy$.

The constant k in the above definitions is called the **constant of proportionality**.

[4] without Calculus, that is...

EXAMPLE 1.3.4. Translate the following into mathematical equations using Definition 1.4.

1. Hooke's Law: The force F exerted on a spring is directly proportional the extension x of the spring.

2. Boyle's Law: At a constant temperature, the pressure P of an ideal gas is inversely proportional to its volume V.

3. The volume V of a right circular cone varies jointly with the height h of the cone and the square of the radius r of the base.

4. Ohm's Law: The current I through a conductor between two points is directly proportional to the voltage V between the two points and inversely proportional to the resistance R between the two points.

5. Newton's Law of Universal Gravitation: Suppose two objects, one of mass m and one of mass M, are positioned so that the distance between their centers of mass is r. The gravitational force F exerted on the two objects varies directly with the product of the two masses and inversely with the square of the distance between their centers of mass.

SOLUTION.

1. Applying the definition of direct variation, we get $F = kx$ for some constant k.

2. Since P and V are inversely proportional, we write $P = \frac{k}{V}$.

3. There is a bit of ambiguity here. It's clear the volume and height of the cone is represented by the quantities V and h, respectively, but does r represent the radius of the base or the square of the radius of the base? It is the former. Usually, if an algebraic operation is specified (like squaring), it is meant to be expressed in the formula. We apply Definition 1.4 to get $V = khr^2$.

4. Even though the problem doesn't use the phrase 'varies jointly', the fact that the current I is given as relating to two different quantities implies this. Since I varies directly with V but inversely with R, we write $I = \frac{kV}{R}$.

5. We write the product of the masses mM and the square of the distance as r^2. We have F varies directly with mM and inversely with r^2, so that $F = \frac{kmM}{r^2}$. $\square$

1.3.1 Exercises

1. Solve each rational equation. Be sure to check for extraneous solutions.

   (a) $\dfrac{x}{5x+4} = 3$

   (b) $\dfrac{3x-1}{x^2+1} = 1$

   (c) $\dfrac{1}{x+3} + \dfrac{1}{x-3} = \dfrac{x^2-3}{x^2-9}$

   (d) $\dfrac{2x+17}{x+1} = x+5$

   (e) $\dfrac{x^2-2x+1}{x^3+x^2-2x} = 1$

   (f) $\dfrac{-x^3+4x}{x^2-9} = 4x$

2. Solve each rational inequality. Express your answer using interval notation.

   (a) $\dfrac{1}{x+2} \geq 0$

   (b) $\dfrac{x-3}{x+2} \leq 0$

   (c) $\dfrac{x}{x^2-1} > 0$

   (d) $\dfrac{4x}{x^2+4} \leq 0$

   (e) $\dfrac{4x}{x^2-4} \geq 0$

   (f) $\dfrac{x^2-x-12}{x^2+x-6} > 0$

   (g) $\dfrac{3x^2-5x-2}{x^2-9} < 0$

   (h) $\dfrac{x^3+2x^2+x}{x^2-x-2} \geq 0$

   (i) $\dfrac{x^2+5x+6}{x^2-1} > 0$

   (j) $\dfrac{3x-1}{x^2+1} \leq 1$

   (k) $\dfrac{2x+17}{x+1} > x+5$

   (l) $\dfrac{-x^3+4x}{x^2-9} \geq 4x$

   (m) $\dfrac{1}{x^2+1} < 0$

   (n) $\dfrac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \geq x$

   (o) $\dfrac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$

3. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[5]

   (a) Find an expression for the volume V of the can based on the height h and the base radius r.

   (b) Find an expression for the surface area S of the can based on the height h and the base radius r. (Hint: The top and bottom of the can are circles of radius r and the side of the can is really just a rectangle that has been bent into a cylinder.)

   (c) Using the fact that $V = 33.6$, write S as a function of r and state its applied domain.

   (d) Use your graphing calculator to find the dimensions of the can which has minimal surface area.

4. In Exercise **??** in Section **??**, the population of Sasquatch in Portage County was modeled by the function $P(t) = \frac{150t}{t+15}$, where $t = 0$ represents the year 1803. When were there fewer than 100 Sasquatch in the county?

[5]According to www.dictionary.com, there are different values given for this conversion. We will stick with 33.6in^3 for this problem.

5. The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by $C(p) = \frac{1770p}{100-p}$ where $0 \leq p < 100$.

   (a) Find and interpret $C(25)$ and $C(95)$.

   (b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?

   (c) What percentage of the Ippizuti fish can you remove for \$40000?

6. Translate the following into mathematical equations.

   (a) At a constant pressure, the temperature T of an ideal gas is directly proportional to its volume V. (This is Charles's Law)

   (b) The frequency of a wave f is inversely proportional to the wavelength of the wave λ.

   (c) The density d of a material is directly proportional to the mass of the object m and inversely proportional to its volume V.

   (d) The square of the orbital period of a planet P is directly proportional to the cube of the semi-major axis of its orbit a. (This is Kepler's Third Law of Planetary Motion)

   (e) The drag of an object traveling through a fluid D varies jointly with the density of the fluid ρ and the square of the velocity of the object ν.

   (f) Suppose two electric point charges, one with charge q and one with charge Q, are positioned r units apart. The electrostatic force F exerted on the charges varies directly with the product of the two charges and inversely with the square of the distance between the charges. (This is Coulomb's Law)

7. According to this webpage, the frequency f of a vibrating string is given by $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$ where T is the tension, μ is the linear mass[6] of the string and L is the length of the vibrating part of the string. Express this relationship using the language of variation.

8. According to the Centers for Disease Control and Prevention www.cdc.gov, a person's Body Mass Index B is directly proportional to his weight W in pounds and inversely proportional to the square of his height h in inches.

   (a) Express this relationship as a mathematical equation.

   (b) If a person who was 5 feet, 10 inches tall weighed 235 pounds had a Body Mass Index of 33.7, what is the value of the constant of proportionality?

   (c) Rewrite the mathematical equation found in part 8a to include the value of the constant found in part 8b and then find your Body Mass Index.

9. We know that the circumference of a circle varies directly with its radius with 2π as the constant of proportionality. (That is, we know $C = 2\pi r$.) With the help of your classmates, compile a list of other basic geometric relationships which can be seen as variations.

[6] Also known as the linear density. It is simply a measure of mass per unit length.

1.3.2 Answers

1. (a) $x = -\frac{6}{7}$
   (b) $x = 1,\ x = 2$
   (c) $x = -1$
   (d) $x = -6,\ x = 2$
   (e) No solution
   (f) $x = 0,\ x = \pm 2\sqrt{2}$

2. (a) $(-2, \infty)$
   (b) $(-2, 3]$
   (c) $(-1, 0) \cup (1, \infty)$
   (d) $(-\infty, 0]$
   (e) $(-2, 0] \cup (2, \infty)$
   (f) $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
   (g) $\left(-3, -\frac{1}{3}\right) \cup (2, 3)$
   (h) $(-1, 0] \cup (2, \infty)$
   (i) $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$
   (j) $(-\infty, 1] \cup [2, \infty)$
   (k) $(-\infty, -6) \cup (-1, 2)$
   (l) $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$
   (m) No solution
   (n) $[-3, 0) \cup (0, 4) \cup [5, \infty)$
   (o) $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$

3. (a) $V = \pi r^2 h$.
   (b) $S = 2\pi r^2 + 2\pi r h$
   (c) $S(r) = 2\pi r^2 + \frac{67.2}{r}$, Domain $r > 0$
   (d) $r \approx 1.749$ in. and $h \approx 3.498$ in.

4. $P(30) = 100$ so before 1903 there were fewer than 100 Sasquatch in Portage County.

5. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.
   (b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.
   (c) For \$40000 you could remove about 95.76% of the fish.

6. (a) $T = kV$
   (b) [7] $f = \dfrac{k}{\lambda}$.
   (c) $d = \dfrac{km}{V}$
   (d) $P^2 = ka^3$
   (e) [8] $D = k\rho\nu^2$.
   (f) [9] $F = \dfrac{kqQ}{r^2}$

7. Rewriting $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$ as $f = \dfrac{\frac{1}{2}\sqrt{T}}{L\sqrt{\mu}}$ we see that the frequency f varies directly with the square root of the tension and varies inversely with the length and the square root of the linear mass.

8. (a) $B = \dfrac{kW}{h^2}$
   (b) [10] $k = 702.68$
   (c) $B = \dfrac{703W}{h^2}$

[7] The character λ is the lower case Greek letter 'lambda.'
[8] Note: The characters ρ and ν are the lower case Greek letters 'rho' and 'nu,' respectively.
[9] Note the similarity to this formula and Newton's Law of Universal Gravitation as discussed in Example 5.
[10] The CDC uses 703.