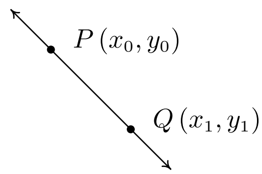
Chapter 1

Linear and Quadratic Functions

1.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated below.



To give a sense of the 'steepness' of the line, we recall we can compute the **slope** of the line using the formula below.

Equation 1.1. The **slope** m of the line containing the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Equation 1.1 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure.[1] Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

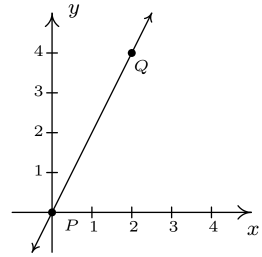
Example 1.1.1. Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

1. $P(0, 0)$, $Q(2, 4)$
2. $P(-1, 2)$, $Q(3, 4)$
3. $P(-2, 3)$, $Q(2, -3)$
4. $P(-3, 2)$, $Q(4, 2)$
5. $P(2, 3)$, $Q(2, -1)$
6. $P(2, 3)$, $Q(2.1, -1)$

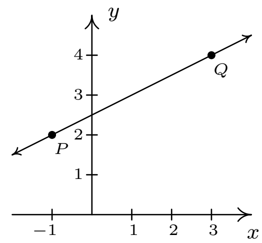
Solution. In each of these examples, we apply the slope formula, Equation 1.1.

[1] See www.mathforum.org or www.mathworld.wolfram.com for discussions on this topic.

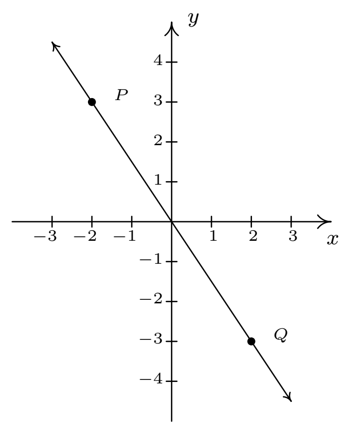
1. $m = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$



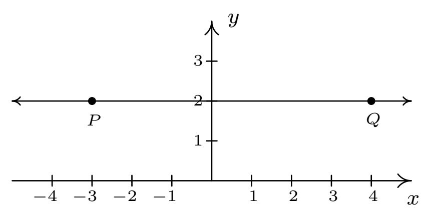
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



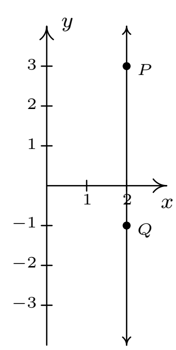
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



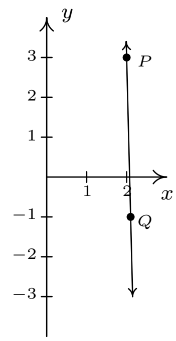
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined

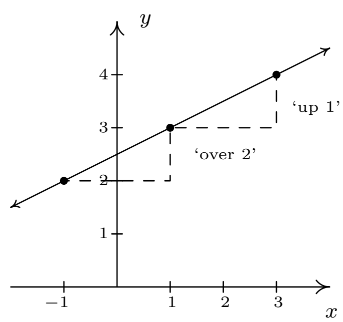


6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



$\square$

A few comments about Example 1.1.1 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a vertical line.[2] Second, the larger the slope is in absolute value, the steeper the line. You may recall from Intermediate Algebra that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 1.1.1, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown below.



Using more formal notation, given points (x_0, y_0) and (x_1, y_1), we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'. Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of y with respect to x. Rates of change abound in the 'real world,' as the next example illustrates.

Example 1.1.2. At 6 AM, it is 24°F; at 10 AM, it is 32°F.

1. Find the slope of the line containing the points $(6, 24)$ and $(10, 32)$.

[2] Some authors use the unfortunate moniker 'no slope' when a slope is undefined. It's easy to confuse the notions of 'no slope' with 'slope of 0'. For this reason, we will describe slopes of vertical lines as 'undefined'.

2. Interpret your answer to the first part in terms of temperature and time.

3. Predict the temperature at noon.

SOLUTION.

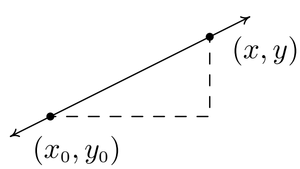
1. For the slope, we have $m = \frac{32-24}{10-6} = \frac{8}{4} = 2$.

2. Since the values in the numerator correspond to the temperatures in °F, and the values in the denominator correspond to time in hours, we can interpret the slope as $2 = \dfrac{2}{1} = \dfrac{2^\circ\,\text{F}}{1\,\text{hour}}$, or 2°F per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of 2°F per hour.

3. Noon is two hours after 10 AM. Assuming a temperature increase of 2°F per hour, in two hours the temperature should rise 4°F. Since the temperature at 10 AM is 32°F, we would expect the temperature at noon to be $32 + 4 = 36$°F.

Now it may well happen that in the previous scenario, at noon the temperature is only 33°F. This doesn't mean our calculations are incorrect. Rather, it means that the temperature change throughout the day isn't a constant 2°F per hour. Mathematics is often used to describe, or **model**, real world phenomena. Mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world. □

In Section **??**, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated below.



We have

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \\ y &= m(x - x_0) + y_0. \end{aligned}$$

We have just derived the **point-slope form** of a line.[3]

[3]We can also understand this equation in terms of applying transformations to the function $I(x) = x$. See the exercises.

Equation 1.2. The **point-slope form** of the line with slope m containing the point (x_0, y_0) is the equation $y = m(x - x_0) + y_0$.

Example 1.1.3. Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution. In order to use Equation 1.2 we need to find the slope of the line in question. So we use Equation 1.1 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ y &= -\frac{2}{3}(x - (-1)) + 3 \\ y &= -\frac{2}{3}x - \frac{2}{3} + 3 \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically, as we did in Section **??**.

□

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in Intermediate Algebra. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the y-intercept of the line $y = mx + b$ is $(0, b)$.

Equation 1.3. The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$ which matches our formula for a horizontal line given in Section **??**. The formula given in Equation 1.3 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (why?) and so we have finally reached a good point to introduce **linear functions**.

DEFINITION 1.1. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

DEFINITION 1.2. A **constant function** is a function of the form

$$f(x) = b,$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Recall that to graph a function, f, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope m and y-intercept $(0, b)$; the graph of a constant function is a horizontal line (with slope $m = 0$) and a y-intercept of $(0, b)$. Now think back to Section **??**, specifically Definition **??** concerning increasing, decreasing and constant functions. A line with positive slope was called an increasing line because a linear function with $m > 0$ is an increasing function. Similarly, a line with a negative slope was called a decreasing line because a linear function with $m < 0$ is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

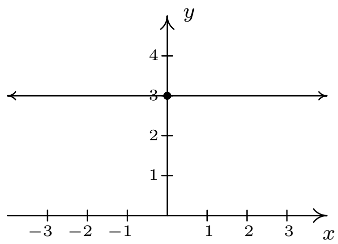
EXAMPLE 1.1.4. Graph the following functions. Identify the slope and y-intercept.

1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3 - 2x}{4}$
4. $f(x) = \dfrac{x^2 - 4}{x - 2}$

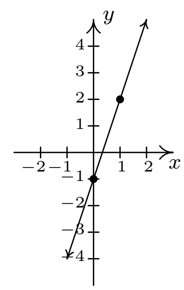
SOLUTION.

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$.

2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 1.3 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our y-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$.



$f(x) = 3$



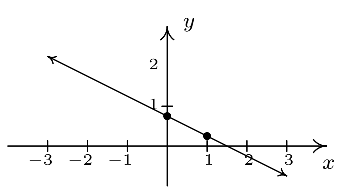
$f(x) = 3x - 1$

3. At first glance, the function $f(x) = \frac{3-2x}{4}$ does not fit the form in Definition 1.1 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} = -\frac{1}{2}x + \frac{3}{4}$. We identify $m = -\frac{1}{2}$ and $b = \frac{3}{4}$. Hence, our graph is a line with a slope of $-\frac{1}{2}$ and a y-intercept of $\left(0, \frac{3}{4}\right)$. Plotting an additional point, we can choose $(1, f(1))$ to get $\left(1, \frac{1}{4}\right)$.

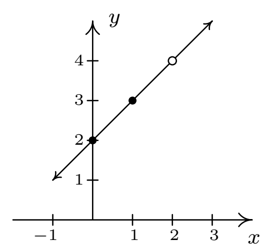
4. If we simplify the expression for f, we get

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x + 2.$$

If we were to state $f(x) = x + 2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case, f has big problems when $x = 2$, and as such, the domain of f is $(-\infty, 2) \cup (2, \infty)$. To indicate this, we write $f(x) = x + 2$, $x \neq 2$. So, except at $x = 2$, we graph the line $y = x + 2$. The slope $m = 1$ and the y-intercept is $(0, 2)$. A second point on the graph is $(1, f(1)) = (1, 3)$. Since our function f is not defined at $x = 2$, we put an open circle at the point that would be on the line $y = x + 2$ when $x = 2$, namely $(2, 4)$.



$f(x) = \dfrac{3 - 2x}{4}$



$f(x) = \dfrac{x^2 - 4}{x - 2}$

□

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 1.1, since some algebraic manipulations may be needed to rewrite a given function to match 'the form.' Keep in mind that the domains of linear and constant functions are all real numbers, $(-\infty, \infty)$, and so while $f(x) = \frac{x^2-4}{x-2}$ simplified to a formula $f(x) = x + 2$, f is not considered a linear function since its domain excludes $x = 2$. However, we would consider

$$f(x) = \frac{2x^2+2}{x^2+1}$$

to be a constant function since its domain is all real numbers (why?) and

$$f(x) = \frac{2x^2+2}{x^2+1} = \frac{2\cancel{(x^2+1)}}{\cancel{(x^2+1)}} = 2$$

The following example uses linear functions to model some basic economic relationships.

Example 1.1.5. The cost, C, in dollars, to produce x PortaBoy game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(10)$.
2. How many PortaBoys can be produced for \$15,000?
3. Explain the significance of the restriction on the domain, $x \geq 0$.
4. Find and interpret $C(0)$.
5. Find and interpret the slope of the graph of $y = C(x)$.

Solution.

1. To find $C(10)$, we replace every occurrence of x with 10 in the formula for $C(x)$ to get $C(10) = 80(10) + 150 = 950$. Since x represents the number of PortaBoys produced, and $C(x)$ represents the cost, in dollars, $C(10) = 950$ means it costs \$950 to produce 10 PortaBoys for the local retailer.

2. To find how many PortaBoys can be produced for \$15,000, we set the cost, $C(x)$, equal to 15000, and solve for x

$$\begin{aligned} C(x) &= 15000 \\ 80x + 150 &= 15000 \\ 80x &= 14850 \\ x &= \frac{14850}{80} = 185.625 \end{aligned}$$

Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for \$15,000.

3. The restriction $x \geq 0$ is the applied domain, as discussed in Section **??**. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.[4]

4. To find $C(0)$, we replace every occurrence of x with 0 in the formula for $C(x)$ to get $C(0) = 80(0) + 150 = 150$. This means it costs \$150 to produce 0 PortaBoys. The \$150 is often called the **fixed** or **start-up** cost of this venture. (What might contribute to this cost?)

5. If we were to graph $y = C(x)$, we would be graphing the portion of the line $y = 80x + 150$ for $x \geq 0$. We recognize the slope, $m = 80$. Like any slope, we can interpret this as a rate of change. In this case, $C(x)$ is the cost in dollars, while x measures the number of PortaBoys so
$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1\,\text{PortaBoy}}.$$
In other words, the cost is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for this venture. □

The next example asks us to find a linear function to model a related economic problem.

EXAMPLE 1.1.6. The local retailer in Example 1.1.5 has determined that the number of PortaBoy game systems sold in a week, x, is related to the price of each system, p, in dollars. When the price was \$220, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at \$190 a piece.

1. Find a linear function which fits this data. Use the weekly sales, x, as the independent variable and the price p, as the dependent variable.

2. Find a suitable applied domain.

3. Interpret the slope.

4. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

5. What would the weekly sales be if the price were set at \$150 per system?

SOLUTION.

1. We recall from Section **??** the meaning of 'independent' and 'dependent' variable. Since x is to be the independent variable, and p the dependent variable, we treat x as the input variable and p as the output variable. Hence, we are looking for a function of the form $p(x) = mx + b$. To determine m and b, we use the fact that 20 PortaBoys were sold during the week the price was 220 dollars and 40 units were sold when the price was 190 dollars.

[4] Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgivable in some textbooks but not to us.

Using function notation, these two facts can be translated as $p(20) = 220$ and $p(40) = 190$. Since m represents the rate of change of p with respect to x, we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

We now have determined $p(x) = -1.5x + b$. To determine b, we can use our given data again. Using $p(20) = 220$, we substitute $x = 20$ into $p(x) = 1.5x + b$ and set the result equal to 220: $-1.5(20) + b = 220$. Solving, we get $b = 250$. Hence, we get $p(x) = -1.5x + 250$. We can check our formula by computing $p(20)$ and $p(40)$ to see if we get 220 and 190, respectively. Incidentally, this equation is sometimes called the **price-demand**[5] equation for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so $x \geq 0$. However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Another constraint, then, is that the price, $p(x) \geq 0$. Solving $-1.5x + 250 \geq 0$ results in $-1.5x \geq -250$ or $x \leq \frac{500}{3} = 166.\overline{6}$. Since x represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for p is $[0, 166]$.

3. The slope $m = -1.5$, once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

4. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150)+250 = 25$. That is, the price would have to be \$25.

5. If the price of a PortaBoy were set at \$150, we have $p(x) = 150$, or, $-1.5x+250 = 150$. Solving, we get $-1.5x = -100$ or $x = 66.\overline{6}$. This means you would be able to sell 66 PortaBoys a week if the price were \$150 per system. □

Not all real-world phenomena can be modeled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following:

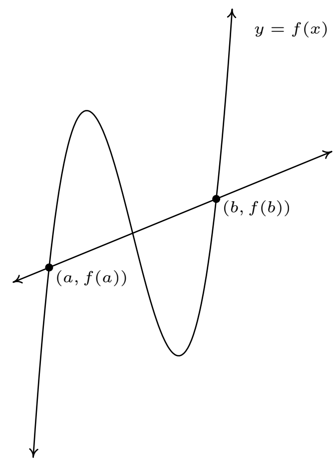
DEFINITION 1.3. Let f be a function defined on the interval $[a, b]$. The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of $y = f(x)$, the average rate of change over $[a, b]$ is the slope of the line which connects $(a, f(a))$ and $(b, f(b))$. This is called the **secant line** through these

[5] Or simply the **demand** equation

points. For that reason, some textbooks use the notation m_{sec} for the average rate of change of a function. Note that for a linear function $m = m_{\text{sec}}$, or in other words, its rate of change over an interval is the same as its average rate of change.



The graph of $y = f(x)$ and its secant line through $(a, f(a))$ and $(b, f(b))$

The interested reader may question the adjective 'average' in the phrase 'average rate of change.' In the figure above, we can see that the function changes wildly on $[a, b]$, yet the slope of the secant line only captures a snapshot of the action at a and b. This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 miles. Your average speed is $\frac{100\,\text{miles}}{2\,\text{hours}} = 50\,\text{miles per hour}$. However, it is entirely possible that at the start of your journey, you traveled 25 miles per hour, then sped up to 65 miles per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rates of change**, and this is one of the central themes of Calculus.[6]

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

EXAMPLE 1.1.7. The **revenue** of selling x units at a price p per unit is given by the formula $R = xp$. Suppose we are in the scenario of Examples 1.1.5 and 1.1.6.

1. Find and simplify an expression for the weekly revenue R as a function of weekly sales, x.
2. Find and interpret the average rate of change of R over the interval $[0, 50]$.
3. Find and interpret the average rate of change of R as x changes from 50 to 100 and compare that to your result in part 2.

[6] Here we go again...

4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

Solution.

1. Since $R = xp$, we substitute $p(x) = -1.5x + 250$ from Example 1.1.6 to get

$$R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$$

2. Using Definition 1.3, we get the average rate of change is

$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$

   Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases \$175.

3. The wording of this part is slightly different than that in Definition 1.3, but its meaning is to find the average rate of change of R over the interval $[50, 100]$. To find this rate of change, we compute

$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$

   In other words, for each additional PortaBoy sold, the revenue increases by \$25. Note while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)

4. Translating the English to the mathematics, we are being asked to find the average rate of change of R over the interval $[100, 150]$. We find

$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$

   This means that we are losing \$125 dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?)

□

We close this section with a new look at difference quotients, first introduced in Section **??**. If we wish to compute the average rate of change of a function f over the interval $[x, x + h]$, then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus.[7]

[7] So, we are not torturing you with these for nothing.

1.1.1 Exercises

1. Find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

   (a) $m = \frac{1}{7}$, $P(-1, 4)$
   (b) $m = -\sqrt{2}$, $P(0, -3)$
   (c) $m = -5$, $P(\sqrt{3}, 2\sqrt{3})$
   (d) $m = 678$, $P(-1, -12)$

2. Find the slope-intercept form of the line which passes through the given points.

   (a) $P(0, 0)$, $Q(-3, 5)$
   (b) $P(-1, -2)$, $Q(3, -2)$
   (c) $P(5, 0)$, $Q(0, -8)$
   (d) $P(3, -5)$, $Q(7, 4)$

3. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

   (a) Find a linear function F that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert 20°C into Fahrenheit.
   (b) Find a linear function C that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert 110°F into Celsius.
   (c) Is there a temperature n such that $F(n) = C(n)$?

4. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find a linear function that represents her total weekly pay in terms of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

5. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.

6. Economic forces beyond anyone's control have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 1.1.5 with this new cost function.

7. In response to the economic forces in the exercise above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Examples 1.1.6 and 1.1.7 with this new data. What difficulties do you encounter?

8. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40°F$ outside and only 5 times per hour if it's $70°F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20°F$ outside. What is the applied domain of this function? Why?

9. (Parallel Lines) Recall from Intermediate Algebra that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In the exercises below, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

(a) $y = 3x + 2,\ P(0,0)$

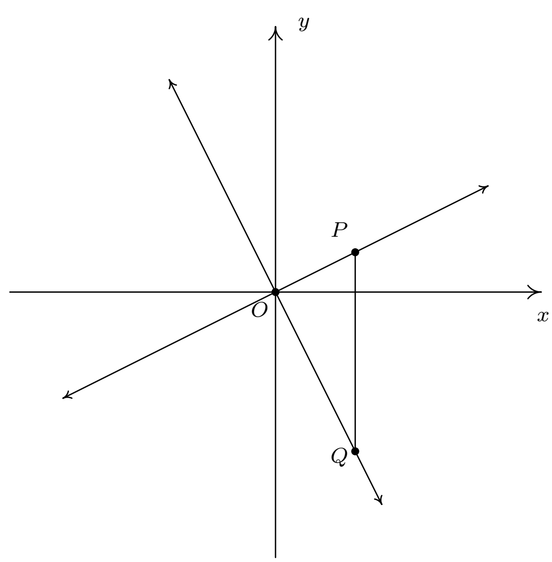
(b) $y = -6x + 5,\ P(3,2)$

10. (Perpendicular Lines) Recall from Intermediate Algebra that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope m_1 and the other has slope m_2 then $m_1 \cdot m_2 = -1$. (You will be guided through a proof of this result in the next exercise.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume $m_1 \neq 0$ and $m_2 \neq 0$. In the exercises below, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

    (a) $y = \frac{1}{3}x + 2,\ P(0,0)$

    (b) $y = -6x + 5,\ P(3,2)$

11. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0,0)$, $P(1,m_1)$ and $Q(1,m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

12. The function defined by $I(x) = x$ is called the Identity Function.
    (a) Discuss with your classmates why this name makes sense.
    (b) Show that the point-slope form of a line (Equation 1.2) can be obtained from I using a sequence of the transformations defined in Section **??**.

13. Compute the average rate of change of the given function over the specified interval.
    (a) $f(x) = x^3,\ [-1, 2]$
    (b) $f(x) = \frac{1}{x},\ [1, 5]$
    (c) $f(x) = \sqrt{x},\ [0, 16]$
    (d) $f(x) = x^2,\ [-3, 3]$
    (e) $f(x) = \frac{x+4}{x-3},\ [5, 7]$
    (f) $f(x) = 3x^2 + 2x - 7,\ [-4, 2]$

14. Compute the average rate of change of the given function over the interval $[x, x+h]$. Here we assume $[x, x+h]$ is in the domain of each function.
    (a) $f(x) = x^3$
    (b) $f(x) = \frac{1}{x}$
    (c) $f(x) = \frac{x+4}{x-3}$
    (d) $f(x) = 3x^2 + 2x - 7$

15. With the help of your classmates find several "real-world" examples of rates of change that are used to describe non-linear phenomena.

1.1.2 Answers

1. (a) $y - 4 = \frac{1}{7}(x+1)$
   $y = \frac{1}{7}x + \frac{29}{7}$
   (b) $y + 3 = -\sqrt{2}(x - 0)$
   $y = -\sqrt{2}x - 3$
   (c) $y - 2\sqrt{3} = -5(x - \sqrt{3})$
   $y = -5x + 7\sqrt{3}$
   (d) $y + 12 = 678(x+1)$
   $y = 678x + 666$

2. (a) $y = -\frac{5}{3}x$
   (b) $y = -2$
   (c) $y = \frac{8}{5}x - 8$
   (d) $y = \frac{9}{4}x - \frac{47}{4}$

3. (a) $F(C) = \frac{9}{5}C + 32$
   (b) $C(F) = \frac{5}{9}F - \frac{160}{9}$
   (c) $F(-40) = -40 = C(-40)$.

4. $W(x) = 200 + .05x$, She must make \$5500 in weekly sales.

5. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

8. $N(T) = -\frac{2}{15}T + \frac{43}{3}$

   Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of 107.5° on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than -60° so that the applied domain is $[-60, 107.5]$.

9. (a) $y = 3x$
   (b) $y = -6x + 20$

10. (a) $y = -3x$
    (b) $y = \frac{1}{6}x + \frac{3}{2}$

13. (a) $\dfrac{2^3 - (-1)^3}{2 - (-1)} = 3$
    (b) $\dfrac{\frac{1}{5} - \frac{1}{1}}{5 - 1} = -\dfrac{1}{5}$
    (c) $\dfrac{\sqrt{16} - \sqrt{0}}{16 - 0} = \dfrac{1}{4}$
    (d) $\dfrac{3^2 - (-3)^2}{3 - (-3)} = 0$
    (e) $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7 - 5} = -\dfrac{7}{8}$
    (f) $\dfrac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2 - (-4)} = -4$

14. (a) $3x^2 + 3xh + h^2$
    (b) $\dfrac{-1}{x(x+h)}$
    (c) $\dfrac{-7}{(x-3)(x+h-3)}$
    (d) $6x + 3h + 2$

1.2 Defining Functions (Word Problems)

Example 1.2.1. Suppose you have two numbers which add to 12.

(i) If one of the numbers is 2, what is the sum of the squares of the two numbers?

Solution: If one of the numbers is 2, then the other number must be $12 - 2 = 10$. Each of their squares would be $2^2 = 4$ and $10^2 = 100$. So the sum of their squares would be $2^2 + 10^2$, which is 104.

(ii) If one of the numbers is x, then find a function $f(x)$ for the sum of the squares of the two numbers.

Solution: If one of the numbers is x, then the other number must be $12 - x$. Each of their squares would be x^2 and $(12 - x)^2$. So the sum of their squares would be $f(x) = x^2 + (12 - x)^2$.

Example 1.2.2. The sum of two numbers is 12. Find a function $f(x)$ which computes the sum of the cubes of the two numbers, where x is one of the two numbers.

Solution: If x is one of two numbers which add to 12, then the other number must be $12 - x$. So their cubes would be x^3 and $(12 - x)^3$, making

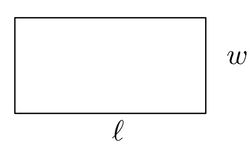
$$f(x) = x^3 + (12 - x)^3.$$

To find a function it helps to remember the following facts:

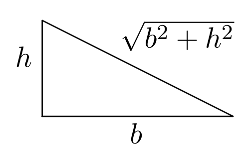
- The distance between (x_1, y_1) and (x_2, y_2) is $\boxed{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$

- The midpoint between (x_1, y_1) and (x_2, y_2) is $\boxed{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).}$

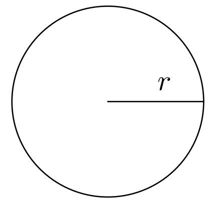
- A rectangle with dimensions l and w
has area $\boxed{l \cdot w}$
and perimeter $\boxed{2l + 2w}$.



- A right triangle with dimensions b and h
has area $\boxed{\frac{1}{2}bh}$
and hypothenuse $\boxed{\sqrt{b^2 + h^2}}$

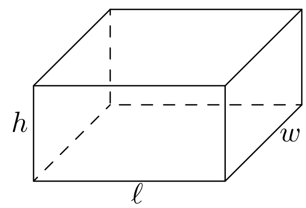


▸ A circle with radius r has area $\boxed{\pi r^2}$ and circumference $\boxed{2\pi r}$

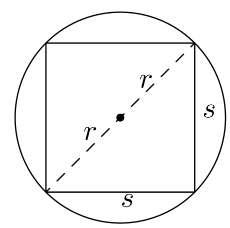


▸ A rectangular box with dimensions l, w, and h has volume $\boxed{l \cdot w \cdot h}$ and surface area $\boxed{2lw + 2lh + 2hw}$.

The surface area with no top is $\boxed{lw + 2lh + 2wh}$.



EXAMPLE 1.2.3. A square of side s is inscribed inside a circle of radius r.



(i) Find a formula for the diameter of the circle in terms of r.

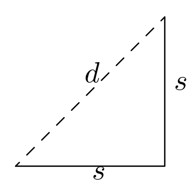
SOLUTION: The diameter of a circle is twice the length of the radius. So $d = 2 \cdot r$.

(i) Find a formula for the length of the side of the square in terms of the diameter, then the radius.

SOLUTION: Using the Pythagorean theorem, we find that

$$d^2 = s^2 + s^2 = 2s^2$$

We can solve for the side length to get



$$\begin{aligned} s^2 &= \frac{d^2}{2} \\ \sqrt{s^2} &= \sqrt{\frac{d^2}{2}} = \frac{\sqrt{d^2}}{\sqrt{2}} = \frac{d}{\sqrt{2}} \\ s &= \frac{d}{\sqrt{2}} \end{aligned}$$

(Note that $\sqrt{s^2} = s$ and $\sqrt{d^2} = d$ because s and d represent distances and distances are always positive) Now, we can find the side length in terms of the radius:

$$s = \frac{d}{\sqrt{2}} = \frac{(2r)}{\sqrt{2}} = \frac{(2r)}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2r\sqrt{2}}{2} = r\sqrt{2}.$$

(iv) Find a function $P(r)$ which computes the perimeter of the inscribed square as a function of the radius r.

SOLUTION: The perimeter of a square is $P = 4s$, and using what we found s to be above, we get

$$P(r) = 4s = 4(r\sqrt{2}) = 4\sqrt{2} \cdot r.$$

(v) Find a function $A(r)$ which computes the area of the inscribed square as a function of the radius r.

SOLUTION: The area of a square is $A = s^2$, and using what we found s to be above, we get

$$A(s) = s^2 = (r\sqrt{2})^2 = r^2(\sqrt{2})^2 = 2r^2$$

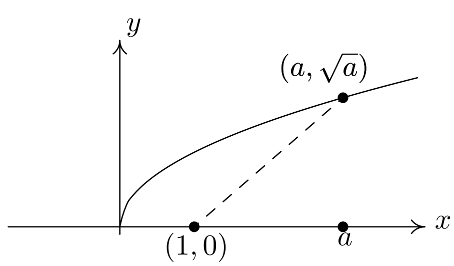
EXAMPLE 1.2.4. Let $f(x) = \sqrt{x}$.

(i) Find the function $d(a)$, which computes the distance from the point $(1,0)$ to the point on the graph of $y = f(x)$ whose x-coordinate is a.

SOLUTION: To find this function, we need to convert this statement into a mathematical expression, so

$$d(a) = \begin{cases} \text{the distance from the point } (1,0) \text{ to the point} \\ \text{on the graph of } y = f(x) \text{ whose } x\text{-coordinate is } a. \end{cases}$$

To find the distance between two points, we need to know the coordinates of the two points, then we can use the distance formula. We know one point is $(1,0)$. The other point is on the graph $y = f(x) = \sqrt{x}$ and has x-coordinate equalling a. Then the y-coordinate for that point would be $y = f(a) = \sqrt{a}$, making the second point be $(a, \sqrt{a})$.



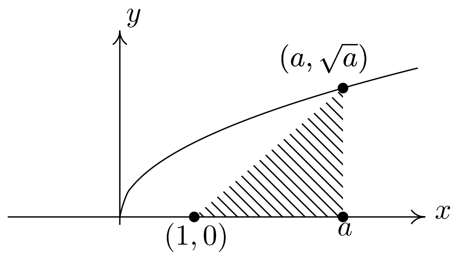
Now, our function is

$$d(a) = \text{the distance from the point } (1,0) \text{ to the point } (a, \sqrt{a}).$$

We can use the distance formula to express this distance and get that

$$d(a) = \sqrt{(1-a)^2 + (0-\sqrt{a})^2} = \sqrt{(1-a)^2 + a} = \sqrt{1 - 2a + a^2 + a} = \sqrt{1 - a + a^2}.$$

(ii) Find a function $A(a)$ which computes the area of a triangle whose vertices are $(0,0)$, $(a,0)$, and (a,b), where (a,b) is on the graph of $y = f(x)$.



SOLUTION: To compute the area of a triangle, we must use the formula

$$A = \frac{1}{2}(\text{base}) \times (\text{height})$$

Using the picture, we can determine that

$$\text{base } = a, \text{ height } = b$$

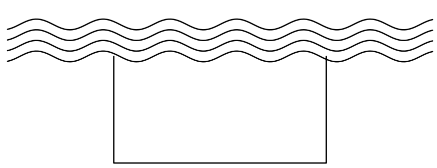
So the formula for the area is

$$A = \frac{1}{2}ab.$$

Now, we need to reduce this formula to a single variable a. So we must find a relationship between a and b (an equation) which we can solve for b and substitute. As in part (i) the point (a, b) lies on the graph of $y = f(x) = \sqrt{x}$, it must be so that $b = f(a) = \sqrt{a}$. This is our relationship. Now we can substitute $(\sqrt{b})$ in for b in our area equation to find area completely in terms of a:

$$A(a) = \frac{1}{2}a(\sqrt{a}) = \frac{1}{2}a\sqrt{a}.$$

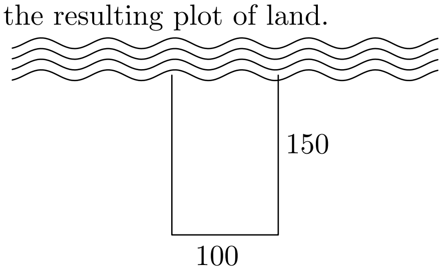
EXAMPLE 1.2.5. You own land along side a river, which you intend to fence. You have 400 feet of fence which you intend to fence three sides of a rectangular plot, allowing the river to compose the final side.



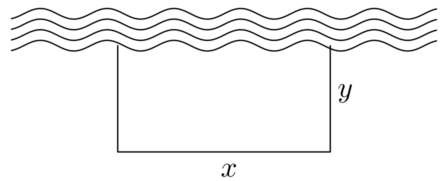
(i) If the side parallel to the river is to be 100 feet, find the area of the resulting plot of land.

SOLUTION: If we use all of the fencing, then we would use 100 feet along the side parallel to the river. Of the remaining 300 feet $(400 - 100)$ we can split it in half and put 150 feet $(\frac{400-100}{2})$ on each side. This would make the enclosed area be

$$A = 100 \times 150 = 15000$$



(ii) Let x be the length of fence parallel to the river. Find the function $A(x)$ which computes the area of the enclosed rectangular area.



SOLUTION: To find the area of this rectangular region, we start with the equation for the area of a rectangle. Let x and y be the dimensions of the rectangle as depicted. Then

$$A = x \cdot y.$$

To make A be a function of x, we need to be able to compute y in terms of x. Notice in the example before, to figure out the value of y, we first figured out how the fence was being allocated. There are two lengths of y and a length of x which would be needing fence, and if we use all of the fence, this means that

$$x + 2 \cdot y = 400$$

If we solve for y, we find that

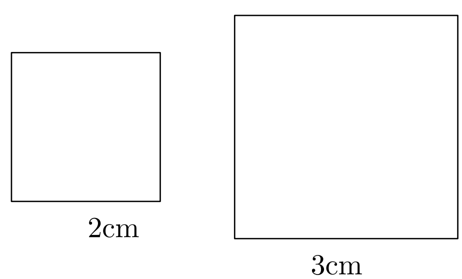
$$\begin{aligned} 2y &= 400 - x \\ y &= \tfrac{1}{2}(400 - x) \end{aligned}$$

Using this computed value for y in terms of x, we can substitute in the equation for area to get

$$A = A(x) = x \cdot \left(\frac{1}{2}(400 - x)\right) = x \cdot (200 - \frac{1}{2}x) = 200x - \frac{1}{2}x^2.$$

EXAMPLE 1.2.6. A piece of wire which is 20cm long is to be cut in to two smaller wires.

(i) If one length is 8cm (and thus the other is 12cm) and each is bent into the shape of a square, what would be the enclosed area?

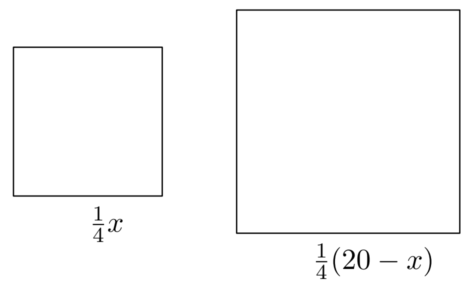


SOLUTION: One needs to recognize that the cut length of wire becomes the perimeter of the resulting shape. If the perimeter of the first square is 8cm, then the side length would be one fourth of 8cm, or 2cm. So the area of the first square would be $2^2 = 4\text{cm}^2$. The second square will have perimeter 12cm, which means its side length would be 3cm, so its area would be $3^2 = 9\text{cm}^2$. So the total area enclosed in both squares would be $9 + 4 = 13\text{cm}^2$.

(ii) Let x be one of the cut wire lengths. Find the function $A(x)$ which computes the enclosed area if each length of wire is shaped into a square.

SOLUTION: If one of our cut lengths is x, then the other must be $20 - x$. If the piece of length x is shaped into a square, that means the square's perimeter is x. So its side length is $\frac{x}{4}$. Thus its enclosed area would be

$$\left(\frac{x}{4}\right)^2 = \frac{x^2}{16}.$$

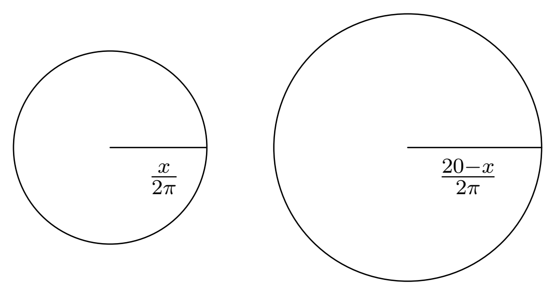


The other piece is of length $20 - x$, meaning the square into which it is shaped will have a side length of $\frac{20-x}{4} = 5 - \frac{x}{4}$. So the enclosed area of this square would be $\left(5 - \frac{1}{4}x\right)^2$. This means that the total enclosed area by both squares ($A(x)$) would be

$$A(x) = \frac{x^2}{16} + \left(5 - \frac{1}{4}x\right)^2.$$

(iii) Let x be one of the cut wire lengths. Find the function $B(x)$ which computes the enclosed area if each length of wire is shaped into a circle.

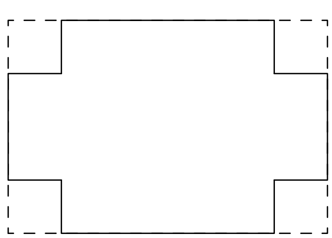
SOLUTION: Here, if the wire of length x is shaped into a circle, then its *circumference* is x. So its radius for the first circle can be determined by $2\pi r = x$, so $r = \dfrac{x}{2\pi}$. This means the area of this circle would be $\pi\left(\dfrac{x}{2\pi}\right)^2$. Similarly, the area of the second circle is computed to be $\pi\left(\dfrac{20-x}{2\pi}\right)^2$. So the area function $B(x)$ would be



$$B(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \pi\left(\frac{20-x}{2\pi}\right)^2 = \frac{x^2 + (20-x)^2}{4\pi}.$$

(Try to simplify to this!)

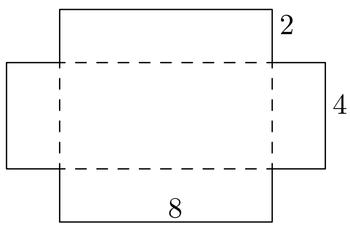
EXAMPLE 1.2.7. An open-top box is created by taking a 8×12 in^2 piece of cardboard and cutting squares from the corners, then folding up the sides.

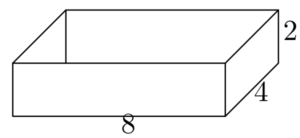


(i) If you cut out 2×2 squares from each corner, what would be the volume of the resulting box?

SOLUTION: If we cut out squares of side length 2, then the lengths which become the length and width of the base become $8 - 2(2) = 4$in and $12 - 2(2) = 8$in. The resulting height of the box is the same as the cut length, 2in. So the volume of this box would be

$$\text{Volume } = 2 \times 4 \times 8 = 64\text{in}^3.$$

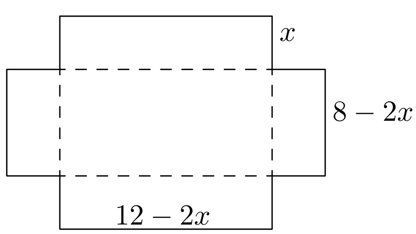


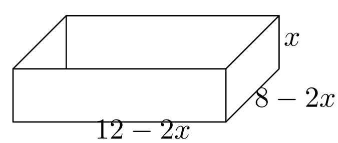


(ii) Let x be the length of the side of a cut-out square. Find the function $V(x)$ which computes the resulting volume of the box.

SOLUTION: To compute the volume of this box, we need to know the length, width, and height in terms of x. As noted above, the height would be precisely x. The way we computed our other dimensions above is to subtract twice x from each of the original dimensions, so our length would be $12 - 2x$ and the width would be $8 - 2x$. Therefore, the volume of this box would be

$$V(x) = x(12 - 2x)(8 - 2x).$$





1.2.1 Exercises

1.2.2 Answers

1.3 Quadratic Functions

You may recall studying quadratic equations in Intermediate Algebra. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

DEFINITION 1.4. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

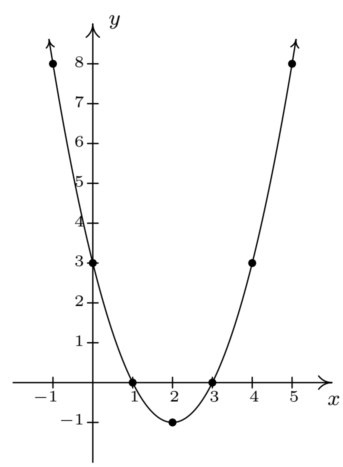
where a, b, and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

EXAMPLE 1.3.1. Graph each of the following quadratic functions. Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing, or constant and find the relative and absolute extrema, if they exist.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = -2(x-3)^2 + 1$.

SOLUTION.

1. To find the zeros of f, we set $f(x) = 0$ and solve the equation $x^2 - 4x + 3 = 0$. Factoring gives us $(x-3)(x-1) = 0$ so that $x = 3$ or $x = 1$. The x-intercepts are then $(1, 0)$ and $(3, 0)$. To find the y-intercept, we set $x = 0$ and find that $y = f(0) = 3$. Hence, the y-intercept is $(0, 3)$. Plotting additional points, we get



$f(x) = x^2 - 4x + 3$

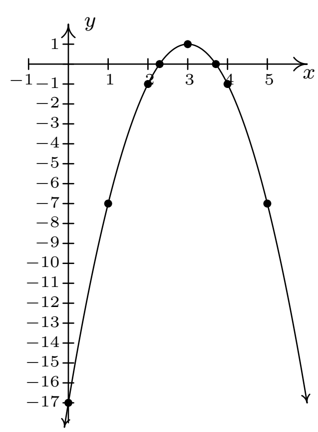
From the graph, we see the domain is $(-\infty, \infty)$ and the range is $[-1, \infty)$. The function f is increasing on $[2, \infty)$ and decreasing on $(-\infty, 2]$. A relative minimum occurs at the point $(2, -1)$ and the value -1 is both the relative and absolute minimum of f.

2. Note that the formula for $g(x)$ doesn't match the form given in Definition 1.4. However, if we took the time to expand $g(x) = -2(x-3)^2 + 1$, we would get $g(x) = -2x^2 + 12x - 17$ which

does match with Definition 1.4. When we find the zeros of g, we can use either formula, since both are equivalent. Using the formula which was given to us, we get

$$\begin{aligned}
g(x) &= 0 \\
-2(x-3)^2+1 &= 0 \\
-2(x-3)^2 &= -1 \\
(x-3)^2 &= \frac{1}{2} && \text{divide by } -2 \\
x-3 &= \pm\sqrt{\frac{1}{2}} && \text{extract square roots} \\
x-3 &= \pm\frac{\sqrt{2}}{2} && \text{rationalize the denominator} \\
x &= 3 \pm \frac{\sqrt{2}}{2} \\
x &= \frac{6 \pm \sqrt{2}}{2} && \text{get a common denominator}
\end{aligned}$$

Hence, we have two x-intercepts: $\left(\frac{6+\sqrt{2}}{2}, 0\right)$ and $\left(\frac{6-\sqrt{2}}{2}, 0\right)$. (The inquisitive reader may wonder what we would have done had we chosen to set the expanded form of $g(x)$ equal to zero. Since $-2x^2+12x-17$ does not factor nicely, we would have had to resort to other methods, which are reviewed later in this section, to solve $-2x^2+12x-17=0$.) To find the y-intercept, we set $x=0$ and get $g(0)=-17$. Our y-intercept is then $(0,-17)$. Plotting some additional points, we get



$g(x) = -2(x-3)^2+1$

The domain of g is $(-\infty,\infty)$ and the range is $(-\infty,1]$. The function g is increasing on $(-\infty,3]$ and decreasing on $[3,\infty)$. The relative maximum occurs at the point $(3,1)$ with 1 being both the relative and absolute maximum value of g. □

Hopefully the previous examples have reminded you of some of the basic characteristics of the graphs of quadratic equations. First and foremost, the graph of $y = ax^2 + bx + c$ where a, b, and c are real numbers with $a \neq 0$ is called a **parabola**. If the coefficient of x^2, a, is positive, the parabola opens upwards; if a is negative, it opens downwards, as illustrated below.[1]

Graphs of $y = ax^2 + bx + c$.

The point at which the relative minimum (if $a > 0$) or relative maximum (if $a < 0$) occurs is called the **vertex** of the parabola. Note that each of the parabolas above is symmetric about the dashed vertical line which contains its vertex. This line is called the **axis of symmetry** of the parabola. As you may recall, there are two ways to quickly find the vertex of a parabola, depending on which form we are given. The results are summarized below.

EQUATION 1.4. **Vertex Formulas for Quadratic Functions**: Suppose a, b, c, h, and k are real numbers with $a \neq 0$.

- If $f(x) = a(x-h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

EXAMPLE 1.3.2. Use Equation 1.4 to find the vertex of the graphs in Example 1.3.1.

SOLUTION.

1. The formula $f(x) = x^2 - 4x + 3$ is in the form $f(x) = ax^2 + bx + c$. We identify $a = 1$, $b = -4$, and $c = 3$, so that
$$-\frac{b}{2a} = -\frac{-4}{2(1)} = 2,$$
and
$$f\left(-\frac{b}{2a}\right) = f(2) = -1,$$
so the vertex is $(2, -1)$ as previously stated.

[1]We will justify the role of a in the behavior of the parabola later in the section.

2. We see that the formula $g(x) = -2(x-3)^2+1$ is in the form $g(x) = a(x-h)^2+k$. We identify $a = -2$, $x-h$ as $x-3$ (so $h = 3$), and $k = 1$ and get the vertex $(3, 1)$, as required. □

The formula $f(x) = a(x-h)^2 + k$, $a \neq 0$ in Equation 1.4 is sometimes called the **standard form** of a quadratic function; the formula $f(x) = ax^2 + bx + c$, $a \neq 0$ is sometimes called the **general form** of a quadratic function.

To see why the formulas in Equation 1.4 produce the vertex, let us first consider a quadratic function in standard form. If we consider the graph of the equation $y = a(x-h)^2 + k$ we see that when $x = h$, we get $y = k$, so (h, k) is on the graph. If $x \neq h$, then $x - h \neq 0$ and so $(x-h)^2$ is a positive number. If $a > 0$, then $a(x-h)^2$ is positive, and so $y = a(x-h)^2 + k$ is always a number larger than k. That means that when $a > 0$, (h, k) is the lowest point on the graph and thus the parabola must open upwards, making (h, k) the vertex. A similar argument shows that if $a < 0$, (h, k) is the highest point on the graph, so the parabola opens downwards, and (h, k) is also the vertex in this case. Alternatively, we can apply the machinery in Section **??**. The vertex of the parabola $y = x^2$ is easily seen to be the origin, $(0, 0)$. We leave it to the reader to convince oneself that if we apply any of the transformations in Section **??** (shifts, reflections, and/or scalings) to $y = x^2$, the vertex of the resulting parabola will always be the point the graph corresponding to $(0, 0)$. To obtain the formula $f(x) = a(x-h)^2 + k$, we start with $g(x) = x^2$ and first define $g_1(x) = ag(x) = ax^2$. This is results in a vertical scaling and/or reflection.[2] Since we multiply the output by a, we multiply the y-coordinates on the graph of g by a, so the point $(0, 0)$ remains $(0, 0)$ and remains the vertex. Next, we define $g_2(x) = g_1(x-h) = a(x-h)^2$. This induces a horizontal shift right or left h units[3] moves the vertex, in either case, to $(h, 0)$. Finally, $f(x) = g_2(x) + k = a(x-h)^2 + k$ which effects a vertical shift up or down k units[4] resulting in the vertex moving from $(h, 0)$ to (h, k).

To verify the vertex formula for a quadratic function in general form, we complete the square to convert the general form into the standard form.[5]

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{complete the square} \\
&= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{factor; get a common denominator}
\end{aligned}$$

[2] Just a scaling if $a > 0$. If $a < 0$, there is a reflection involved.

[3] Right if $h > 0$, left if $h < 0$.

[4] Up if $k > 0$, down if $k < 0$

[5] Actually, we could also take the standard form, $f(x) = a(x-h)^2 + k$, expand it, and compare the coefficients of it and the general form to deduce the result. However, we will have another use for the completed square form of the general form of a quadratic, so we'll proceed with the conversion.

Comparing this last expression with the standard form, we identify $(x-h)$ as $\left(x+\frac{b}{2a}\right)$ so that $h=-\dfrac{b}{2a}$. Instead of memorizing the value $k=\frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right)=\frac{4ac-b^2}{4a}$. As such, we have derived the vertex formula for the general form as well. Note that the value a plays the exact same role in both the standard and general equations of a quadratic function $-$ it is the coefficient of x^2 in each. No matter what the form, if $a>0$, the parabola opens upwards; if $a<0$, the parabola opens downwards.

Now that we have the completed square form of the general form of a quadratic function, it is time to remind ourselves of the **quadratic formula**. In a function context, it gives us a means to find the zeros of a quadratic function in general form.

EQUATION 1.5. **The Quadratic Formula:** If a, b, c are real numbers with $a \neq 0$, then the solutions to $ax^2+bx+c=0$ are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Assuming the conditions of Equation 1.5, the solutions to $ax^2+bx+c=0$ are precisely the zeros of $f(x)=ax^2+bx+c$. We have shown an equivalent formula for f is

$$f(x)=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

Hence, an equation equivalent to $ax^2+bx+c=0$ is

$$a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}=0.$$

Solving gives

$$
\begin{aligned}
a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} &= 0 \\
a\left(x+\frac{b}{2a}\right)^2 &= -\frac{4ac-b^2}{4a} \\
\frac{1}{a}\left[a\left(x+\frac{b}{2a}\right)^2\right] &= \frac{1}{a}\left(\frac{b^2-4ac}{4a}\right) \\
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\
x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}} \qquad \text{extract square roots} \\
x+\frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\
x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2-4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand b^2-4ac. It turns out that the quantity b^2-4ac plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name and is discussed below.

DEFINITION 1.5. If a, b, c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2+bx+c=0$ is the quantity b^2-4ac.

THEOREM 1.1. **Discriminant Trichotomy:** Let a, b, and c be real numbers with $a \neq 0$.

- If $b^2-4ac<0$, the equation $ax^2+bx+c=0$ has no real solutions.
- If $b^2-4ac=0$, the equation $ax^2+bx+c=0$ has exactly one real solution.
- If $b^2-4ac>0$, the equation $ax^2+bx+c=0$ has exactly two real solutions.

The proof of Theorem 1.1 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

EXAMPLE 1.3.3. The **profit** function for a product is defined by the equation Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. Recall from Example 1.1.7 that the weekly revenue, in dollars, made by selling x PortaBoy Game Systems is given by $R(x) = -1.5x^2 + 250x$. The cost, in dollars, to produce x PortaBoy Game Systems is given in Example 1.1.5 as $C(x) = 80x + 150$, $x \geq 0$.

1. Determine the weekly profit function, $P(x)$.
2. Graph $y = P(x)$. Include the x- and y-intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of P.
4. Interpret the vertex of the graph of $y = P(x)$.
5. Recall the weekly price-demand equation for PortaBoys is: $p(x) = -1.5x + 250$, where $p(x)$ is the price per PortaBoy, in dollars, and x is the weekly sales. What should the price per system be in order to maximize profit?

SOLUTION.

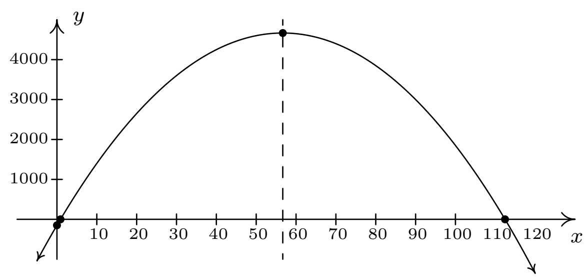
1. To find the profit function $P(x)$, we subtract

$$P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150.$$

2. To find the x-intercepts, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 1.5. Identifying $a = -1.5$, $b = 170$, and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \end{aligned}$$

We get two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the y-intercept, we set $x = 0$ and find $y = P(0) = -150$ for a y-intercept of $(0, -150)$. To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ is in the general form of a quadratic function and appeal to Equation 1.4. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ and find our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$. To sketch a reasonable graph, we approximate the x-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. (Note that in order to get the x-intercepts and the vertex to show up in the same picture, we had to scale the x-axis differently than the y-axis. This results in the left-hand x-intercept and the y-intercept being uncomfortably close to each other and to the origin in the picture.)



3. The zeros of P are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. Since P represents the weekly profit, $P(x) = 0$ means the weekly profit is \$0. Sometimes, these values of x are called the 'break-even' points of the profit function, since these are places where the revenue equals the cost; in other words we gave sold enough product to recover the cost spent to make the product. More importantly, we see from the graph that as long as x is between 0.89 and 112.44, the graph $y = P(x)$ is above the x-axis, meaning $y = P(x) > 0$ there. This means that for these values of x, a profit is being made. Since x represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.

4. From the graph, we see the maximum value of P occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, x represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude we will make a maximum profit of \$4666.50 if we sell 57 game systems.

5. In the previous part, we found we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. The price should be set at \$164.50. □

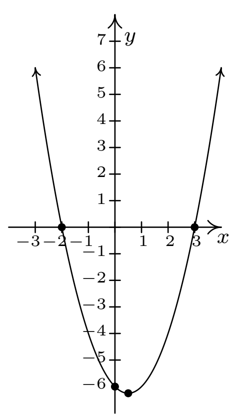
We conclude this section with a more complicated absolute value function.

Example 1.3.4. Graph $f(x) = |x^2 - x - 6|$.

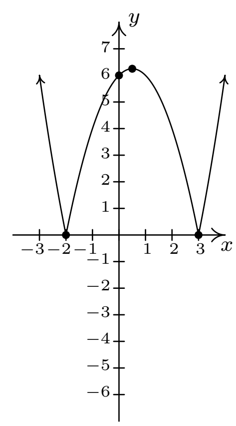
Solution. Using the definition of absolute value, Definition **??**, we have

$$f(x) = \begin{cases} -\left(x^2 - x - 6\right), & \text{if} \quad x^2 - x - 6 < 0 \\ x^2 - x - 6, & \text{if} \quad x^2 - x - 6 \geq 0 \end{cases}$$

The trouble is that we have yet to develop any analytic techniques to solve nonlinear inequalities such as $x^2 - x - 6 < 0$. You won't have to wait long; this is one of the main topics of Section 1.4. Nevertheless, we can attack this problem graphically. To that end, we graph $y = g(x) = x^2 - x - 6$ using the intercepts and the vertex. To find the x-intercepts, we solve $x^2 - x - 6 = 0$. Factoring gives $(x-3)(x+2) = 0$ so $x = -2$ or $x = 3$. Hence, $(-2, 0)$ and $(3, 0)$ are x-intercepts. The y-intercept is found by setting $x = 0$, $(0, -6)$. To find the vertex, we find $x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}$, and $y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = -\frac{25}{4} = -6.25$. Plotting, we get the parabola seen below on the left. To obtain points on the graph of $y = f(x) = |x^2 - x - 6|$, we can take points on the graph of $g(x) = x^2 - x - 6$ and apply the absolute value to each of the y values on the parabola. We see from the graph of g that for $x \leq -2$ or $x \geq 3$, the y values on the parabola are greater than or equal to zero (since the graph is on or above the x-axis), so the absolute value leaves this portion of the graph alone. For x between -2 and 3, however, the y values on the parabola are negative. For example, the point $(0, -6)$ on $y = x^2 - x - 6$ would result in the point $(0, |-6|) = (0, -(-6)) = (0, 6)$ on the graph of $f(x) = |x^2 - x - 6|$. Proceeding in this manner for all points with x-coordinates between -2 and 3 results in the graph seen above on the right.



$y = g(x) = x^2 - x - 6$



$y = f(x) = |x^2 - x - 6|$

□

If we take a step back and look at the graphs of g and f in the last example, we notice that to obtain the graph of f from the graph of g, we reflect a *portion* of the graph of g about the x-axis. We can see this analytically by substituting $g(x) = x^2 - x - 6$ into the formula for $f(x)$ and calling to mind Theorem **??** from Section **??**.

$$f(x) = \begin{cases} -g(x), & \text{if} \quad g(x) < 0 \\ g(x), & \text{if} \quad g(x) \geq 0 \end{cases}$$

The function f is defined so that when $g(x)$ is negative (i.e., when its graph is below the x-axis), the graph of f is its refection across the x-axis. This is a general template to graph functions of the form $f(x) = |g(x)|$. From this perspective, the graph of $f(x) = |x|$ can be obtained by reflection the portion of the line $g(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.

1.3.1 Exercises

1. Graph each of the following quadratic functions. Find the x- and y-intercepts of each graph, if any exist. If it is given in the general form, convert it into standard form. Find the domain and range of each function and list the intervals on which the function is increasing or decreasing. Identify the vertex and the axis of symmetry and determine whether the vertex yields a relative and absolute maximum or minimum.

   (a) $f(x) = x^2 + 2$
   (b) $f(x) = -(x + 2)^2$
   (c) $f(x) = x^2 - 2x - 8$
   (d) $f(x) = -2(x + 1)^2 + 4$
   (e) $f(x) = 2x^2 - 4x - 1$
   (f) $f(x) = -3x^2 + 4x - 7$
   (g) $f(x) = -3x^2 + 5x + 4$
   (h) [6] $f(x) = x^2 - \dfrac{1}{100}x - 1$

2. Graph $f(x) = |1 - x^2|$

3. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$

4. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

5. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0t + s_0$ where h is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

   (a) What is the applied domain of this function?
   (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.
   (c) Come up with a scenario in which $s_0 < 0$.
   (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?
   (e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?
   (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

6. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$. How many cookies should they bake in order to maximize their profit?

[6] We have already seen the graph of this function. It was used as an example in Section **??** to show how the graphing calculator can be misleading.

7. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[7] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

8. What is the largest rectangular area one can enclose with 14 inches of string?

9. Solve the following quadratic equations for the indicated variable.

   (a) $x^2 - 10y^2 = 0$ for x

   (b) $y^2 - 4y = x^2 - 4$ for x

   (c) $x^2 - mx = 1$ for x

   (d) $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)

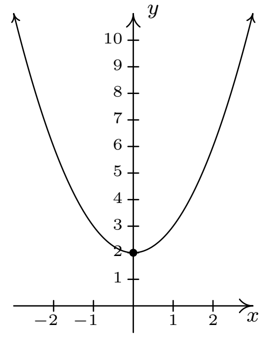
   (e) $y^2 - 3y = 4x$ for y

   (f) $y^2 - 4y = x^2 - 4$ for y

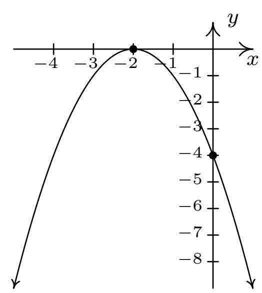
[7] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise **??** in Section **??** what shape a free hanging cable makes.

1.3.2 Answers

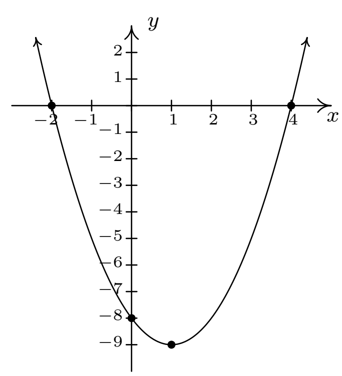
1. (a) $f(x) = x^2 + 2$
   No x-intercepts
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



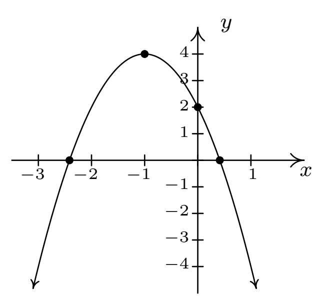
   (b) $f(x) = -(x + 2)^2$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$



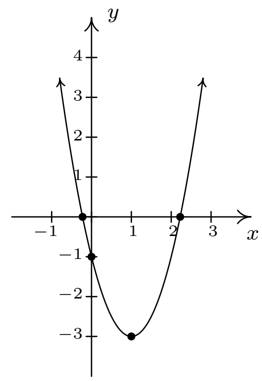
   (c) $f(x) = x^2 - 2x - 8 = (x - 1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$



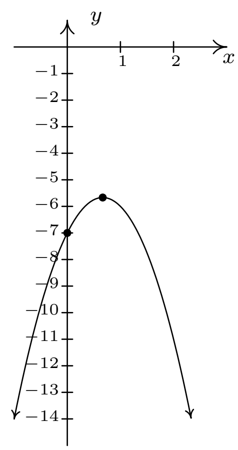
   (d) $f(x) = -2(x + 1)^2 + 4$
   x-intercepts $(-1 - \sqrt{2}, 0)$ and $(-1 + \sqrt{2}, 0)$
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 4])$
   Increasing on $(-\infty, -1]$
   Decreasing on $[-1, \infty)$
   Vertex $(-1, 4)$ is a maximum
   Axis of symmetry $x = -1$



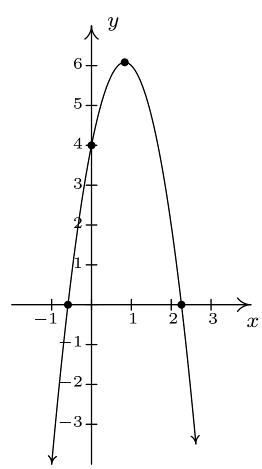
(e) $f(x) = 2x^2 - 4x - 1 = 2(x-1)^2 - 3$
x-intercepts $\left(-1 - \frac{\sqrt{6}}{2}, 0\right)$ and $\left(-1 + \frac{\sqrt{6}}{2}, 0\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$
Increasing on $[1, \infty)$
Decreasing on $(-\infty, 1]$
Vertex $(1, -3)$ is a minimum
Axis of symmetry $x = 1$



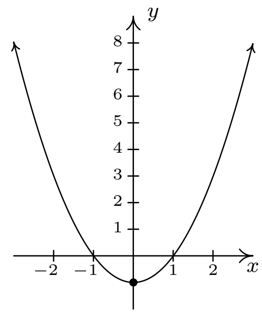
(f) $f(x) = -3x^2 + 4x - 7 = -3(x - \frac{2}{3})^2 - \frac{17}{3}$
No x-intercepts
y-intercept $(0, -7)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, -\frac{17}{3}]$
Increasing on $(-\infty, \frac{2}{3}]$
Decreasing on $[\frac{2}{3}, \infty)$
Vertex $(\frac{2}{3}, -\frac{17}{3})$ is a maximum
Axis of symmetry $x = \frac{2}{3}$



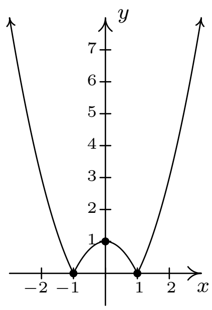
(g) $f(x) = -3x^2 + 5x + 4 = -3\left(x - \frac{5}{6}\right)^2 + \frac{73}{12}$
x-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a minimum
Axis of symmetry $x = \frac{5}{6}$



(h) $f(x) = x^2 - \frac{1}{100}x - 1 = \left(x - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
x-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[8]
Axis of symmetry $x = \frac{1}{200}$



2. $y = |1 - x^2|$



3. $\left(\dfrac{3-\sqrt{7}}{2}, \dfrac{-1+\sqrt{7}}{2}\right), \left(\dfrac{3+\sqrt{7}}{2}, \dfrac{-1-\sqrt{7}}{2}\right)$

5. (a) The applied domain is $[0, \infty)$.

   (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.

   (e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.

   (f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

6. 495 cookies

7. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

8. The largest rectangle has area 12.25in^2.

[8]You'll need to use your calculator to zoom in far enough to see that the vertex is not the y-intercept.

9. (a) $x = \pm y\sqrt{10}$

   (b) $x = \pm(y-2)$

   (c) $x = \dfrac{m \pm \sqrt{m^2+4}}{2}$

   (d) $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

   (e) $y = \dfrac{3 \pm \sqrt{16x+9}}{2}$

   (f) $y = 2 \pm x$

1.4 Inequalities

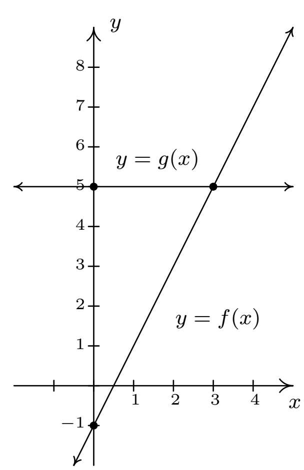
In this section, not only do we develop techniques for solving various classes of inequalities analytically, we also look at them graphically. The next example motivates the core ideas.

Example 1.4.1. Let $f(x) = 2x - 1$ and $g(x) = 5$.

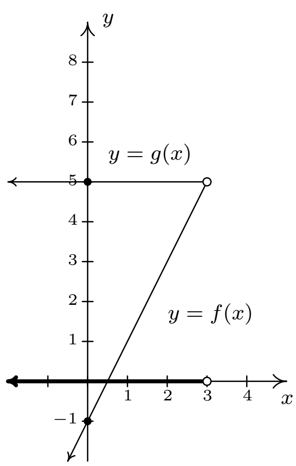
1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) > g(x)$.
4. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

Solution.

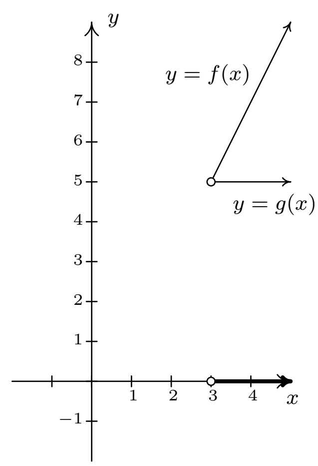
1. To solve $f(x) = g(x)$, we replace $f(x)$ with $2x - 1$ and $g(x)$ with 5 to get $2x - 1 = 5$. Solving for x, we get $x = 3$.
2. The inequality $f(x) < g(x)$ is equivalent to $2x - 1 < 5$. Solving gives $x < 3$ or $(-\infty, 3)$.
3. To find where $f(x) > g(x)$, we solve $2x - 1 > 5$. We get $x > 3$, or $(3, \infty)$.
4. To graph $y = f(x)$, we graph $y = 2x - 1$, which is a line with a y-intercept of $(0, -1)$ and a slope of 2. The graph of $y = g(x)$ is $y = 5$ which is a horizontal line through $(0, 5)$.



To see the connection between the graph and the algebra, we recall the Fundamental Graphing Principle for Functions in Section **??**: the point (a, b) is on the graph of f if and only if $f(a) = b$. In other words, a generic point on the graph of $y = f(x)$ is $(x, f(x))$, and a generic point on the graph of $y = g(x)$ is $(x, g(x))$. When we seek solutions to $f(x) = g(x)$, we are looking for values x whose y values on the graphs of f and g are the same. In part 1, we found $x = 3$ is the solution to $f(x) = g(x)$. Sure enough, $f(3) = 5$ and $g(3) = 5$ so that the point $(3, 5)$ is on both graphs. We say the graphs of f and g **intersect** at $(3, 5)$. In part 2, we set $f(x) < g(x)$ and solved to find $x < 3$. For $x < 3$, the point $(x, f(x))$ is **below** $(x, g(x))$ since the y values on the graph of f are less than the y values on the graph of g there. Analogously, in part 3, we solved $f(x) > g(x)$ and found $x > 3$. For $x > 3$, note that the graph of f is **above** the graph of g, since the y values on the graph of f are greater than the y values on the graph of g for those values of x.



$f(x) < g(x)$



$f(x) > g(x)$

□

The preceding example demonstrates the following, which is a consequence of the Fundamental Graphing Principle for Functions.

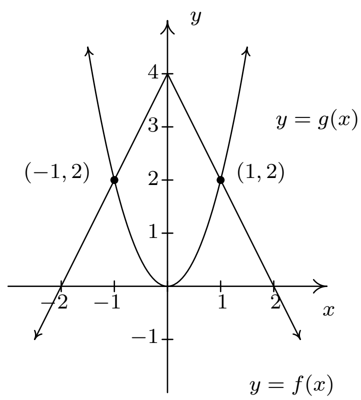
Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions.

- The solutions to $f(x) = g(x)$ are precisely the x values where the graphs of $y = f(x)$ and $y = g(x)$ intersect.
- The solutions to $f(x) < g(x)$ are precisely the x values where the graph of $y = f(x)$ is **below** the graph of $y = g(x)$.
- The solutions to $f(x) > g(x)$ are precisely the x values where the graph of $y = f(x)$ **above** the graph of $y = g(x)$.

The next example turns the tables and furnishes the graphs of two functions and asks for solutions to equations and inequalities.

Example 1.4.2. The graphs of f and g are below. The graph of $y = f(x)$ resembles the upside down $\vee$ shape of an absolute value function while the graph of $y = g(x)$ resembles a parabola. Use these graphs to answer the following questions.

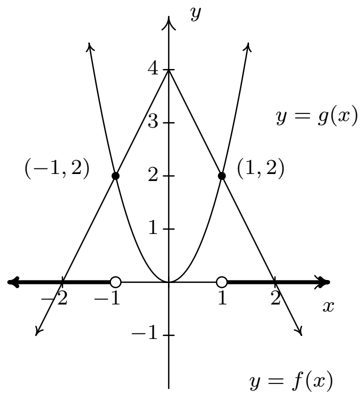


1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) \geq g(x)$.

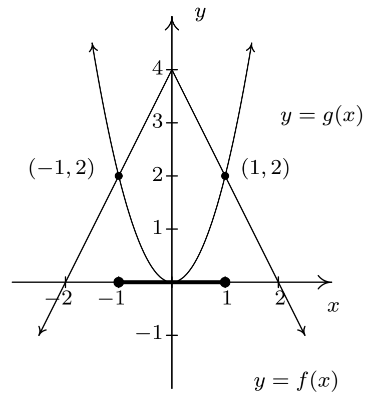
Solution.

1. To solve $f(x) = g(x)$, we look for where the graphs of f and g intersect. These appear to be at the points $(-1, 2)$ and $(1, 2)$, so our solutions to $f(x) = g(x)$ are $x = -1$ and $x = 1$.
2. To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.

3. To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. We solved the former equation and found $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.



$f(x) < g(x)$



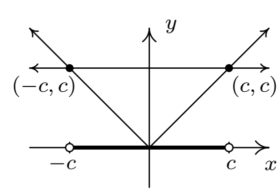
$f(x) \geq g(x)$

□

We now turn our attention to solving inequalities involving the absolute value. We have the following theorem from Intermediate Algebra to help us.

> THEOREM 1.2. **Inequalities Involving the Absolute Value:** Let c be a real number.
>
> - For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
> - For $c \leq 0$, $|x| < c$ has no solution.
> - For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
> - For $c < 0$, $|x| > c$ is true for all real numbers.

As with Theorem **??** in Section **??**, we could argue Theorem 1.2 using cases. However, in light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know the graphs intersect when $|x| = c$, which, from Section **??**, we know happens when $x = c$ or $x = -c$. Graphing, we get



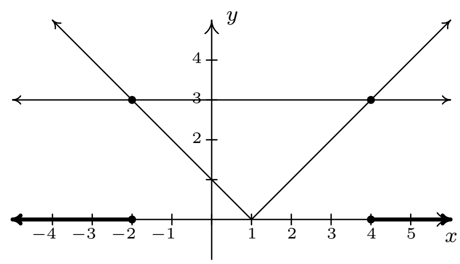
We see the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in Theorem 1.2 can be shown similarly.

EXAMPLE 1.4.3. Solve the following inequalities analytically; check your answers graphically.

1. $|x - 1| \geq 3$
2. $4 - 3|2x + 1| > -2$
3. $2 < |x - 1| \leq 5$
4. $|x + 1| \geq \dfrac{x + 4}{2}$

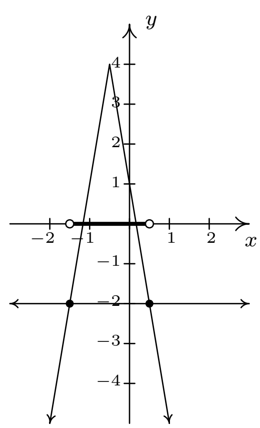
SOLUTION.

1. To solve $|x - 1| \geq 3$, we seek solutions to $|x - 1| > 3$ as well as solutions to $|x - 1| = 3$. From Theorem 1.2, $|x - 1| > 3$ is equivalent to $x - 1 < -3$ or $x - 1 > 3$. From Theorem **??**, $|x - 1| = 3$ is equivalent to $x - 1 = -3$ or $x - 1 = 3$. Combining these equations with the inequalities, we solve $x - 1 \leq -3$ or $x - 1 \geq 3$. Our answer is $x \leq -2$ or $x \geq 4$, which, in interval notation is $(-\infty, -2] \cup [4, \infty)$. Graphically, we have

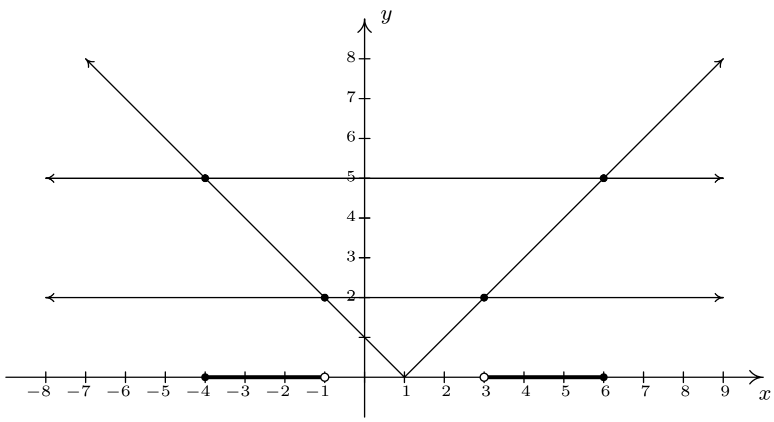


We see the graph of $y = |x - 1|$ (the $\vee$) is above the horizontal line $y = 3$ for $x < -2$ and $x > 4$, and, hence, this is where $|x - 1| > 3$. The two graphs intersect when $x = -2$ and $x = 4$, and so we have graphical confirmation of our analytic solution.

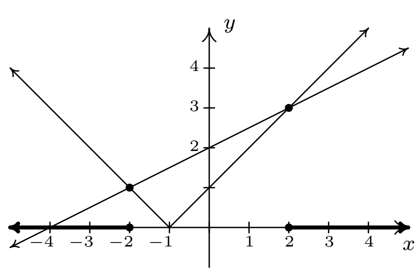
2. To solve $4 - 3|2x + 1| > -2$ analytically, we first isolate the absolute value before applying Theorem 1.2. To that end, we get $-3|2x + 1| > -6$ or $|2x + 1| < 2$. Rewriting, we now have $-2 < 2x + 1 < 2$ so that $-\frac{3}{2} < x < \frac{1}{2}$. In interval notation, we write $\left(-\frac{3}{2}, \frac{1}{2}\right)$. Graphically we see the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$.



3. Rewriting the compound inequality $2 < |x-1| \leq 5$ as '$2 < |x-1|$ and $|x-1| \leq 5$' allows us to solve each piece using Theorem 1.2. The first inequality, $2 < |x-1|$ can be re-written as $|x-1| > 2$ and so $x-1 < -2$ or $x-1 > 2$. We get $x < -1$ or $x > 3$. Our solution to the first inequality is then $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \leq 5$, we combine results in Theorems **??** and 1.2 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4, 6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, and so we take what's called the 'set theoretic intersection' of $(-\infty, -1) \cup (3, \infty)$ with $[-4, 6]$ to obtain $[-4, -1) \cup (3, 6]$. Graphically, we see the graph of $y = |x-1|$ is 'between' the horizontal lines $y = 2$ and $y = 5$ for x values between -4 and -1 as well as those between 3 and 6. Including the x values where $y = |x-1|$ and $y = 5$ intersect, we get

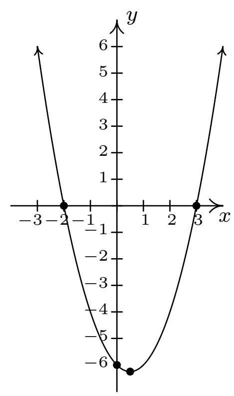


We need to exercise some special caution when solving $|x+1| \geq \frac{x+4}{2}$. When variables are both inside and outside of the absolute value, it's usually best to refer to the definition of absolute value, Definition **??**, to remove the absolute values and proceed from there. To that end, we have $|x+1| = -(x+1)$ if $x < -1$ and $|x+1| = x+1$ if $x \geq -1$. We break the inequality into cases, the first case being when $x < -1$. For these values of x, our inequality becomes $-(x+1) \geq \frac{x+4}{2}$. Solving, we get $-2x-2 \geq x+4$, so that $-3x \geq 6$, which means $x \leq -2$. Since all of these solutions fall into the category $x < -1$, we keep them all. For the second case, we assume $x \geq -1$. Our inequality becomes $x+1 \geq \frac{x+4}{2}$, which gives $2x+2 \geq x+4$ or $x \geq 2$. Since all of these values of x are greater than or equal to -1, we accept all of these solutions as well. Our final answer is $(-\infty, -2] \cup [2, \infty)$.



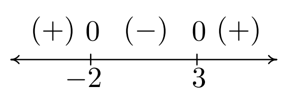
□

We now turn our attention to quadratic inequalities. In the last example of Section 1.3, we needed to determine the solution to $x^2 - x - 6 < 0$. We will now re-visit this problem using some of the techniques developed in this section not only to reinforce our solution in Section 1.3, but to also help formulate a general analytic procedure for solving all quadratic inequalities. If we consider $f(x) = x^2 - x - 6$ and $g(x) = 0$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = g(x) = 0$ (the x-axis.) We've provided the graph again for reference.



$y = x^2 - x - 6$

We can see that the graph of f does dip below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. We can schematically represent this with the **sign diagram** below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2, 3)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. An important property[1] of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of **all** of the

[1]We will give this property a name in Chapter **??** and revisit this concept then.

function values on a given interval by testing the function at just **one** value in the interval. This gives us the following.

Steps for Solving a Quadratic Inequality

1. Rewrite the inequality, if necessary, as a quadratic function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the intervals which correspond to the correct sign to solve the inequality.

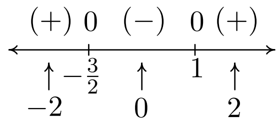
Example 1.4.4. Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

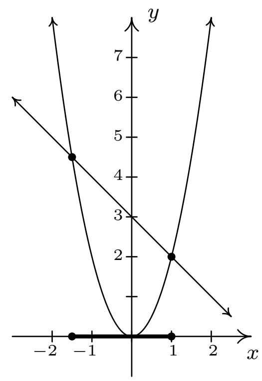
1. $2x^2 \leq 3 - x$
2. $x^2 > 2x + 1$
3. $9x^2 + 4 \leq 12x$
4. $2x - x^2 \geq |x - 1| - 1$

Solution.

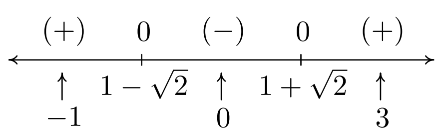
1. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$,or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$, and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[2] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$ (so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$ (so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$); and, $f(2) = 7$ (which means $(+)$ is placed above $(1, \infty)$). Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $\left(-\frac{3}{2}, 1\right)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so or final answer is $\left[-\frac{3}{2}, 1\right]$. To check our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the two graphs intersect (the solutions to $g(x) = h(x)$.) The graphs of g and h are given on the right with the sign chart on the left.

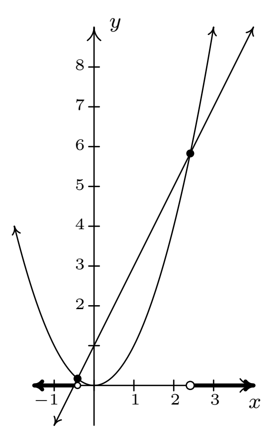
[2] We have to choose something in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.





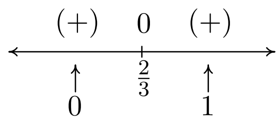
2. Once again, we re-write $x^2 > 2x + 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$, and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$. To check the inequality $x^2 > 2x + 1$ graphically, we set $g(x) = x^2$ and $h(x) = 2x + 1$. We are looking for the x values where the graph of g is above the graph of h. As before we present the graphs on the right and the sign chart on the left.

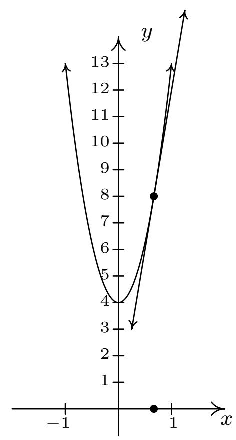




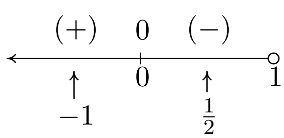
3. To solve $9x^2 + 4 \leq 12x$, as before, we solve $9x^2 - 12x + 4 \leq 0$. Setting $f(x) = 9x^2 - 12x + 4 = 0$, we find the only one zero of f, $x = \frac{2}{3}$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 1$ as test values. We find $f(0) = 4 > 0$ and $f(1) = 1 > 0$. Since we are looking for solutions to $9x^2 - 12x + 4 \leq 0$, we are looking for

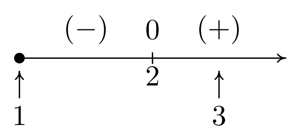
x values where $9x^2 - 12x + 4 < 0$ as well as where $9x^2 - 12x + 4 = 0$. Looking at our sign diagram, there are no places where $9x^2 - 12x + 4 < 0$ (there are no $(-)$), so our solution is only $x = \frac{2}{3}$ (where $9x^2 - 12x + 4 = 0$). We write this as $\left\{\frac{2}{3}\right\}$. Graphically, we solve $9x^2 + 4 \leq 12x$ by graphing $g(x) = 9x^2 + 4$ and $h(x) = 12x$. We are looking for the x values where the graph of g is below the graph of h (for $9x^2 + 4 < 12x$) and where the two graphs intersect ($9x^2 + 4 = 12x$). We see the line and the parabola touch at $\left(\frac{2}{3}, 8\right)$, but the parabola is always above the line otherwise.[3]





4. To solve our last inequality, $2x - x^2 \geq |x - 1| - 1$, we re-write the absolute value using cases. For $x < 1$, $|x - 1| = -(x - 1) = 1 - x$, so we get $2x - x^2 \geq 1 - x - 1$, or $x^2 - 3x \leq 0$. Finding the zeros of $f(x) = x^2 - 3x$, we get $x = 0$ and $x = 3$. However, we are only concerned with the portion of the number line where $x < 1$, so the only zero that we concern ourselves with is $x = 0$. This divides the interval $x < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $x = -1$ and $x = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Solving $x^2 - 3x \leq 0$ for $x < 1$ gives us $[0, 1)$. Next, we turn our attention to the case $x \geq 1$. Here, $|x - 1| = x - 1$, so our original inequality becomes $2x - x^2 \geq x - 1 - 1$, or $x^2 - x - 2 \leq 0$. Setting $g(x) = x^2 - x - 2$, we find the zeros of g to be $x = -1$ and $x = 2$. Of these, only $x = 2$ lies in the region $x \geq 1$, so we ignore $x = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $x = 1$ and $x = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. To solve $g(x) \leq 0$, we have $[1, 2)$.

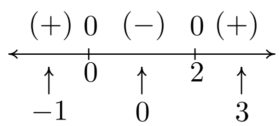


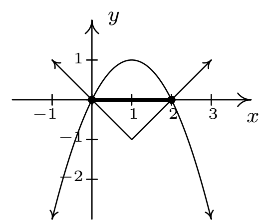


Combining these into one sign diagram, we get our solution is $[0, 2]$. Graphically, to check $2x - x^2 \geq |x - 1| - 1$, we set $h(x) = 2x - x^2$ and $i(x) = |x - 1| - 1$ and look for the x values

[3] In this case, we say the line $y = 12x$ is **tangent** to $y = 9x^2 + 4$ at $\left(\frac{2}{3}, 8\right)$. Finding tangent lines to arbitrary functions is the stuff of legends, I mean, Calculus.

where the graph of h is above the the graph of i (the solution of $h(x) > i(x)$) as well as the x-coordinates of the intersection points of both graphs (where $h(x) = i(x)$). The combined sign chart is given on the left and the graphs are on the right.





□

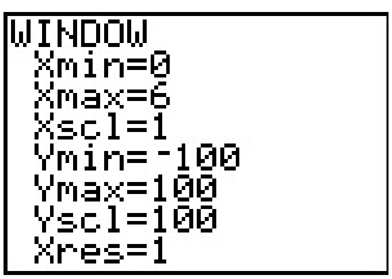
It is quite possible to encounter inequalities where the analytical methods developed so far will fail us. In this case, we resort to using the graphing calculator to approximate the solution, as the next example illustrates.

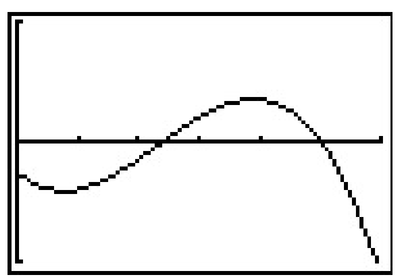
Example 1.4.5. Suppose the revenue R, in thousands of dollars, from producing and selling x hundred LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $x \geq 0$, while the cost, in thousands of dollars, to produce x hundred LCD TVs is given by $C(x) = 200x + 25$ for $x \geq 0$. How many TVs, to the nearest TV, should be produced to make a profit?

Solution. Recall that profit = revenue − cost. If we let P denote the profit, in thousands of dollars, which results from producing and selling x hundred TVs then

$$P(x) = R(x) - C(x) = \left(-5x^3 + 35x^2 + 155x\right) - (200x + 25) = -5x^3 + 35x^2 - 45x - 25,$$

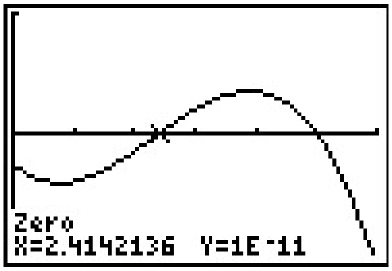
where $x \geq 0$. If we want to make a profit, then we need to solve $P(x) > 0$; in other words, $-5x^3 + 35x^2 - 45x - 25 > 0$. We have yet to discuss how to go about finding the zeros of P, let alone making a sign diagram for such an animal,[4] as such we resort to the graphing calculator. After finding a suitable window, we get

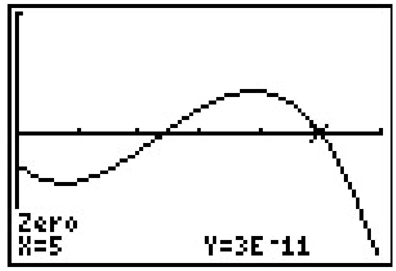




We are looking for the x values for which $P(x) > 0$, that is, where the graph of P is above the x-axis. We make use of the 'Zero' command and find two x-intercepts.

[4]The procedure, as we shall see in Chapter **??** is identical to what we have developed here.





We remember that x denotes the number of TVs in **hundreds**, so if we are to find our solution using the calculator, we need our answer to two decimal places. The zero[5] $2.414\ldots$ corresponds to $241.4\ldots$ TVs. Since we can't make a fractional part of a TV, we round this up to 242 TVs.[7] The other zero seems dead on at 5, which corresponds to 500 TVs. Hence to make a profit, we should produce (and sell) between 242 and 499 TVs, inclusive. □

Our last example in the section demonstrates how inequalities can be used to describe regions in the plane, as we saw earlier in Section **??**.

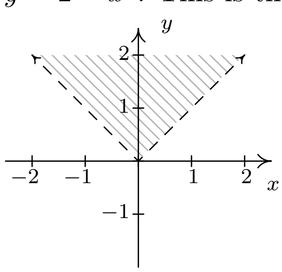
EXAMPLE 1.4.6. Sketch the following relations.

1. $R = \{(x, y) : y > |x|\}$.
2. $S = \{(x, y) : y \leq 2 - x^2\}$.
3. $T = \{(x, y) : |x| < y \leq 2 - x^2\}$.

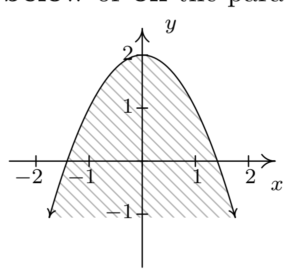
SOLUTION.

1. The relation R consists of all points (x, y) whose y-coordinate is greater than $|x|$. If we graph $y = |x|$, then we want all of the points in the plane **above** the points on the graph. Dotting the graph of $y = |x|$ as we have done before to indicate the points on the graph itself are not in the relation, we get the shaded region below on the left.

2. For a point to be in S, its y-coordinate must be less than or equal to the y-coordinate on the parabola $y = 2 - x^2$. This is the set of all points **below** or **on** the parabola $y = 2 - x^2$.



The graph of R



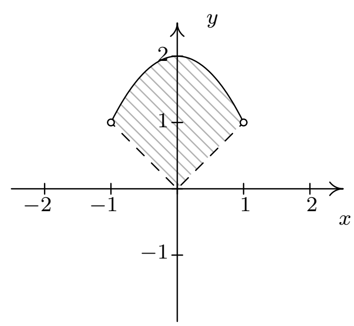
The graph of S

[5] Note the y-coordinates of the points here aren't registered as 0. They are expressed in Scientific Notation. For instance, $1E-11$ corresponds to 0.00000000001, which is pretty close in the calculator's eyes[6] to 0.

[6] but not a Mathematician's

[7] Notice that $P(241) < 0$ and $P(242) > 0$ so we need to round up to 242 in order to make a profit.

3. Finally, the relation T takes the points whose y-coordinates satisfy both the conditions in R and S. So we shade the region between $y = |x|$ and $y = 2 - x^2$, keeping those points on the parabola, but not the points on $y = |x|$. To get an accurate graph, we need to find where these two graphs intersect, so we set $|x| = 2 - x^2$. Proceeding as before, breaking this equation into cases, we get $x = -1, 1$. Graphing yields



The graph of T

$\square$

1.4.1 Exercises

1. Solve the inequality. Express your answer in interval form.

(a) $|3x-5| \leq 4$
(b) $|7x+2| > 10$
(c) $1 < |2x-9| \leq 3$
(d) $|-2x+1| \geq x+5$
(e) $|x+3| \geq |6x+9|$
(f) $x^2+2x-3 \geq 0$
(g) $16x^2+8x+1 > 0$
(h) $x^2+9 < 6x$
(i) $9x^2+16 \geq 24x$
(j) $x^2+4 \leq 4x$
(k) $x^2+1 < 0$
(l) $3x^2 \leq 11x+4$
(m) $x > x^2$
(n) $2x^2-4x-1 > 0$
(o) $5x+4 \leq 3x^2$
(p) $2 \leq |x^2-9| < 9$
(q) $x^2 \leq |4x-3|$
(r) $x^2+x+1 \geq 0$

2. Prove the second, third and fourth parts of Theorem 1.2.

3. If a slingshot is used to shoot a marble straight up into the air from 2 meters above the ground with an initial velocity of 30 meters per second, for what values of time t will the marble be over 35 meters above the ground? (Refer to Exercise 5 in Section 1.3 for assistance if needed.) Round your answers to two decimal places.

4. What temperature values in degrees Celsius are equivalent to the temperature range $50°F$ to $95°F$? (Refer to Exercise 3 in Section 1.1 for assistance if needed.)

5. The surface area S of a cube with edge length x is given by $S(x) = 6x^2$ for $x > 0$. Suppose the cubes your company manufactures are supposed to have a surface area of exactly 42 square centimeters, but the machines you own are old and cannot always make a cube with the precise surface area desired. Write an inequality using absolute value that says the surface area of a given cube is no more than 3 square centimeters away (high or low) from the target of 42 square centimeters. Solve the inequality and express your answer in interval form.

6. Sketch the following relations.

(a) $R = \{(x,y) : y \leq x-1\}$
(b) $R = \left\{(x,y) : y > x^2+1\right\}$
(c) $R = \{(x,y) : -1 < y \leq 2x+1\}$
(d) $R = \left\{(x,y) : x^2 \leq y < x+2\right\}$
(e) $R = \{(x,y) : |x|-4 < y < 2-x\}$
(f) $R = \left\{(x,y) : x^2 < y \leq |4x-3|\right\}$

7. Suppose f is a function, L is a real number and ε is a positive number. Discuss with your classmates what the inequality $|f(x)-L| < \varepsilon$ means algebraically and graphically.[8]

[8] Understanding this type of inequality is really important in Calculus.

1.4.2 Answers

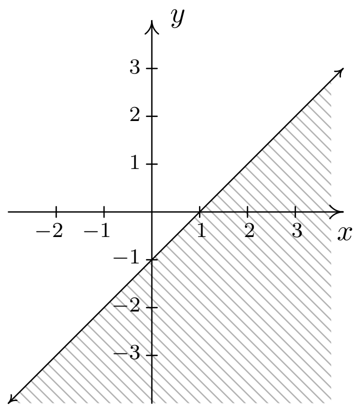
1. (a) $\left[\frac{1}{3}, 3\right]$
   (b) $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
   (c) $[3, 4) \cup (5, 6]$
   (d) $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
   (e) $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
   (f) $(-\infty, -3] \cup [1, \infty)$
   (g) $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
   (h) No solution
   (i) $(-\infty, \infty)$
   (j) $\{2\}$
   (k) No solution
   (l) $\left[-\frac{1}{3}, 4\right]$
   (m) $(0, 1)$
   (n) $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
   (o) $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$
   (p) $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup \left[-\sqrt{7}, 0\right) \cup \left(0, \sqrt{7}\right] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$
   (q) $\left[-2 - \sqrt{7}, -2 + \sqrt{7}\right] \cup [1, 3]$
   (r) $(-\infty, \infty)$

3. $1.44 < t < 4.68$

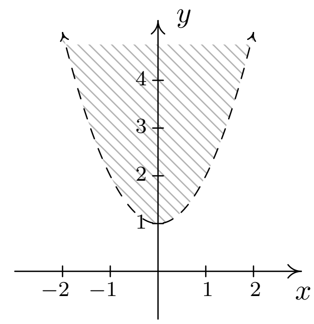
4. From our previous work $C(F) = \frac{5}{9}(F - 32)$ so $50 \leq F \leq 95$ becomes $10 \leq C \leq 35$.

5. The surface area could go as low as 39cm^2 or as high as 45cm^2 so we have $39 \leq S(x) \leq 45$. Using absolute value and the fact that $S(x) = 6x^2$ we get $|6x^2 - 42| \leq 3$. Solving this inequality yields $\left[\sqrt{\frac{13}{2}}, \sqrt{\frac{15}{2}}\right]$.

6. (a)



   (b)



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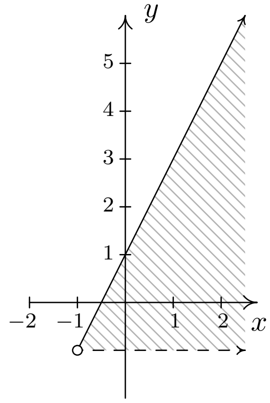

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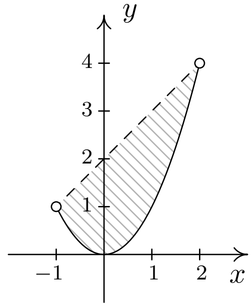
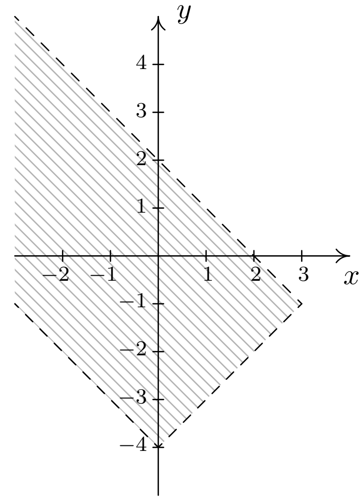


(e)



(f)