

CHAPTER 1

SEQUENCES, SERIES AND THE BINOMIAL THEOREM

1.1 SEQUENCES

When we first introduced a function as a special type of relation in Section ??, we did not put any restrictions on the domain of the function. All we said was that the set of x -coordinates of the points in the function F is called the domain, and it turns out that any subset of the real numbers, regardless of how weird that subset may be, can be the domain of a function. As our exploration of functions continued beyond Section ??, we saw fewer and fewer functions with ‘weird’ domains. It is worth your time to go back through the text to see that the domains of the polynomial, rational, exponential, logarithmic and algebraic functions discussed thus far have fairly predictable domains which almost always consist of just a collection of intervals on the real line. This may lead some readers to believe that the only important functions in a College Algebra text have domains which consist of intervals and everything else was just introductory nonsense. In this section, we introduce **sequences** which are an important class of functions whose domains are the set of natural numbers.¹ Before we get too far ahead of ourselves, let’s look at what the term ‘sequence’ means mathematically. Informally, we can think of a sequence as an infinite list of numbers. For example, consider the sequence

$$\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \dots \quad (1)$$

As usual, the periods of ellipsis, \dots , indicate that the proposed pattern continues forever. Each of the numbers in the list is called a **term**, and we call $\frac{1}{2}$ the ‘first term’, $-\frac{3}{4}$ the ‘second term’, $\frac{9}{8}$ the ‘third term’ and so forth. In numbering them this way, we are setting up a function, which we’ll call a per tradition, between the natural numbers and the terms in the sequence.

n	$a(n)$
1	$\frac{1}{2}$
2	$-\frac{3}{4}$
3	$\frac{9}{8}$
4	$-\frac{27}{16}$
\vdots	\vdots

In other words, $a(n)$ is the n^{th} term in the sequence. We formalize these ideas in our definition of a sequence and introduce some accompanying notation.

DEFINITION 1.1. A **sequence** is a function a whose domain is the natural numbers. The value $a(n)$ is often written as a_n and is called the **n^{th} term** of the sequence. The sequence itself is usually denoted $\{a_n\}_{n=1}^{\infty}$.

Applying the notation provided in Definition 1.1 to the sequence given (1), we have $a_1 = \frac{1}{2}$, $a_2 = -\frac{3}{4}$, $a_3 = \frac{9}{8}$ and so forth. Now suppose we wanted to know a_{117} , that is, the 117th term in the sequence. While the pattern of the sequence is apparent, it would benefit us greatly to have an explicit formula

¹Recall that this is the set $\{1, 2, 3, \dots\}$.

for a_n . Unfortunately, there is no general algorithm that will produce a formula for every sequence, so any formulas we do develop will come from that greatest of teachers, experience. In other words, it is time for an example.

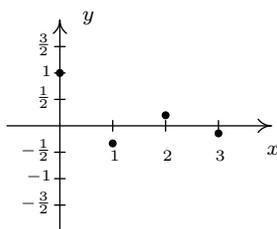
EXAMPLE 1.1.1. Write the first four terms of the following sequences.

- | | |
|--|---|
| 1. $a_n = \frac{5^{n-1}}{3^n}, n \geq 1$ | 4. $\left\{ \frac{1 + (-1)^j}{j} \right\}_{j=2}^{\infty}$ |
| 2. $b_k = \frac{(-1)^k}{2k+1}, k \geq 0$ | 5. $a_1 = 7, a_{n+1} = 2 - a_n, n \geq 1$ |
| 3. $\{2n - 1\}_{n=1}^{\infty}$ | 6. $f_0 = 1, f_n = n \cdot f_{n-1}, n \geq 1$ |

SOLUTION.

- Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4 . Since the notation a_1 means the same thing as $a(1)$, we obtain our first term by replacing every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.
- For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3 . Proceeding as before, replacing in this case the variable k with the appropriate whole number, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. (This sequence is called an **alternating** sequence since the signs alternate between $+$ and $-$. The reader is encouraged to think what component of the formula is producing this effect.)
- From $\{2n - 1\}_{n=1}^{\infty}$, we have that $a_n = 2n - 1, n \geq 1$. We get $a_1 = 1, a_2 = 3, a_3 = 5$ and $a_4 = 7$. (The first four terms are the first four odd natural numbers. The reader is encouraged to examine whether or not this pattern continues indefinitely.)
- Proceeding as in the previous problem, we set $a_j = \frac{1+(-1)^j}{j}, j \geq 2$. We find $a_2 = 1, a_3 = 0, a_4 = \frac{1}{2}$ and $a_5 = 0$.
- To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.
- As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$. \square

Some remarks about Example 1.1.1 are in order. We first note that since sequences are functions, we can graph them in the same way we graph functions. For example, if we wish to graph the sequence $\{b_k\}_{k=0}^{\infty}$ from Example 1.1.1, we graph the equation $y = b(k)$ for the values $k \geq 0$. That is, we plot the points $(k, b(k))$ for the values of k in the domain, $k = 0, 1, 2, \dots$. The resulting collection of points is the graph of the sequence. Note that we do not connect the dots in a pleasing fashion as we are used to doing, because the domain is just the whole numbers in this case, not a collection of intervals of real numbers. If you feel a sense of nostalgia, you should see Section ??.



$$\text{Graphing } y = b_k = \frac{(-1)^k}{2k + 1}, k \geq 0$$

Speaking of $\{b_k\}_{k=0}^{\infty}$, the astute and mathematically minded reader will correctly note that this technically isn't a sequence, since according to Definition 1.1, sequences are functions whose domains are the *natural* numbers, not the *whole* numbers, as is the case with $\{b_k\}_{k=0}^{\infty}$. In other words, to satisfy Definition 1.1, we need to shift the variable k so it starts at $k = 1$ instead of $k = 0$. To see how we can do this, it helps to think of the problem graphically. What we want is to shift the graph of $y = b(k)$ to the right one unit, and thinking back to Section ??, we can accomplish this by replacing k with $k - 1$ in the definition of $\{b_k\}_{k=0}^{\infty}$. Specifically, let $c_k = b_{k-1}$ where $k - 1 \geq 0$. We get $c_k = \frac{(-1)^{k-1}}{2(k-1)+1} = \frac{(-1)^{k-1}}{2k-1}$, where now $k \geq 1$. We leave to the reader to verify that $\{c_k\}_{k=1}^{\infty}$ generates the same list of numbers as does $\{b_k\}_{k=0}^{\infty}$, but the former satisfies Definition 1.1, while the latter does not. Like so many things in this text, we acknowledge that this point is pedantic and join the vast majority of authors who adopt a more relaxed view of Definition 1.1 to include any function which generates a list of numbers which can be then be matched up with the natural numbers.² Finally, we wish to note the sequences in parts 5 and 6 are examples of sequences described **recursively**. In each instance, an initial value of the sequence is given which is then followed by a **recursion equation** – a formula which enables us to use known terms of the sequence to determine other terms. The terms of the sequence in part 6 are given a special name: f_n is called ***n*-factorial**. We will study factorials in greater detail in Section 1.4. The world famous [Fibonacci Numbers](#) are defined recursively and are explored in the exercises. While none of the sequences worked out to be the sequence in (1), they do give us some insight into what kinds of patterns to look for. Two patterns in particular are given in the next definition.

²Math fans will delight to know we are basically talking about the 'countably infinite' subsets of the real number line when we do this.

DEFINITION 1.2. **Arithmetic and Geometric Sequences:** Suppose $\{a_n\}_{n=k}^{\infty}$ is a sequence^a

- If there is a number d so that $a_{n+1} = a_n + d$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **arithmetic sequence**. The number d is called the **common difference**.
- If there is a number r so that $a_{n+1} = ra_n$ for all $n \geq k$, then $\{a_n\}_{n=k}^{\infty}$ is called an **geometric sequence**. The number r is called the **common ratio**.

^aNote that we have adjusted for the fact that not all ‘sequences’ begin at $n = 1$.

Both arithmetic and geometric sequences are defined in terms of recursion equations. In English, an arithmetic sequence is one in which we proceed from one term to the next by always *adding* the fixed number d . The name ‘common difference’ comes from a slight rewrite of the recursion equation from $a_{n+1} = a_n + d$ to $a_{n+1} - a_n = d$. Analogously, a geometric sequence is one in which we proceed from one term to the next by always *multiplying* by the same fixed number r . If $r \neq 0$, we can rearrange the recursion equation to get $\frac{a_{n+1}}{a_n} = r$, hence the name ‘common ratio.’ Some sequences are arithmetic, some are geometric and some are neither as the next example illustrates.³

EXAMPLE 1.1.2. Determine if the following sequences are arithmetic, geometric or neither. If arithmetic, find the common difference d ; if geometric, find the common ratio r .

1. $a_n = \frac{5^{n-1}}{3^n}, n \geq 1$

3. $\{2n - 1\}_{n=1}^{\infty}$

2. $b_k = \frac{(-1)^k}{2k + 1}, k \geq 0$

4. $\frac{1}{2}, -\frac{3}{4}, \frac{9}{8}, -\frac{27}{16}, \dots$

SOLUTION. A good rule of thumb to keep in mind when working with sequences is “When in doubt, write it out!” Writing out the first several terms can help you identify the pattern of the sequence should one exist.

1. From Example 1.1.1, we know that the first four terms of this sequence are $\frac{1}{3}, \frac{5}{9}, \frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no ‘common difference’ and we have established that the sequence is *not* arithmetic. To investigate whether or not it is geometric, we compute the ratios of successive terms. The first three ratios

$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$

suggest that the sequence is geometric. To prove it, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n .

$$\frac{a_{n+1}}{a_n} = \frac{5^{(n+1)-1}}{3^{n+1}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$

³Sequences which are both arithmetic and geometric are discussed in the Exercises.

This sequence is geometric with common ratio $r = \frac{5}{3}$.

- Again, we have Example 1.1.1 to thank for providing the first four terms of this sequence: $1, -\frac{1}{3}, \frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Hence, the sequence is not arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since there is no ‘common ratio,’ we conclude the sequence is not geometric, either.
- As we saw in Example 1.1.1, the sequence $\{2n - 1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \dots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To verify this, we find

$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$

This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

- We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we assume that the pattern of ratios continue in this fashion and conclude that the sequence is geometric. \square

We are now one step away from determining an explicit formula for the sequence given in (1). We know that it is a geometric sequence and our next result gives us the explicit formula we require.

EQUATION 1.1. Formulas for Arithmetic and Geometric Sequences:

- An arithmetic sequence with first term a and common difference d is given by

$$a_n = a + (n - 1)d, \quad n \geq 1$$

- A geometric sequence with first term a and common ratio $r \neq 0$ is given by

$$a_n = ar^{n-1}, \quad n \geq 1$$

While the formal proofs of the formulas in Equation 1.1 require the techniques set forth in Section 1.3, we attempt to motivate them here. According to Definition 1.2, given an arithmetic sequence with first term a and common difference d , the way we get from one term to the next is by adding d . Hence, the terms of the sequence are: $a, a+d, a+2d, a+3d, \dots$. We see that to reach the n th term, we add d to a exactly $(n-1)$ times, which is what the formula says. The derivation of the formula for geometric series follows similarly. Here, we start with a and go from one term to the next by multiplying by r . We get a, ar, ar^2, ar^3 and so forth. The n th term results from multiplying a by r exactly $(n-1)$ times. We note here that the reason $r = 0$ is excluded from Equation 1.1 is to avoid an instance of 0^0 which is an indeterminate form.⁴ With Equation 1.1 in place, we finally

⁴See the footnotes on page ?? in Section ?? and page ?? of Section ??.

have the tools required to find an explicit formula for the n th term of the sequence given in (1). We know from Example 1.1.2 that it is geometric with common ratio $r = -\frac{3}{2}$. The first term is $a = \frac{1}{2}$ so by Equation 1.1 we get $a_n = ar^{n-1} = \frac{1}{2} \left(-\frac{3}{2}\right)^{n-1}$ for $n \geq 1$. After a touch of simplifying, we get $a_n = \frac{(-3)^{n-1}}{2^n}$ for $n \geq 1$. Note that we can easily check our answer by substituting in values of n and seeing that the formula generates the sequence given in (1). We leave this to the reader. Our next example gives us more practice finding patterns.

EXAMPLE 1.1.3. Find an explicit formula for the n^{th} term of the following sequences.

1. 0.9, 0.09, 0.009, 0.0009, ...
2. $\frac{2}{5}, 2, -\frac{2}{3}, -\frac{2}{7}, \dots$
3. $1, -\frac{2}{7}, \frac{4}{13}, -\frac{8}{19}, \dots$

SOLUTION.

1. Although this sequence may seem strange, the reader can verify it is actually a geometric sequence with common ratio $r = 0.1 = \frac{1}{10}$. With $a = 0.9 = \frac{9}{10}$, we get $a_n = \frac{9}{10} \left(\frac{1}{10}\right)^{n-1}$ for $n \geq 0$. Simplifying, we get $a_n = \frac{9}{10^n}$, $n \geq 1$. There is more to this sequence than meets the eye and we shall return to this example in the next section.
2. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions. We have $\frac{2}{5}, \frac{2}{1}, -\frac{2}{3}, -\frac{2}{7}, \dots$. If we associate the negative ‘-’ of the last two terms with the denominators we get $\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, \frac{2}{-7}, \dots$. This tells us that we can tentatively sketch out the formula for the sequence as $a_n = \frac{2}{d_n}$ where d_n is the sequence of denominators. Looking at the denominators 5, 1, -3, -7, ..., we find that they go from one term to the next by subtracting 4 which is the same as adding -4. This means we have an arithmetic sequence on our hands. Using Equation 1.1 with $a = 5$ and $d = -4$, we get the n th denominator by the formula $d_n = 5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.
3. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting c_n and d_n denote the sequence of numerators and denominators, respectively, we have $a_n = \frac{c_n}{d_n}$. After some experimentation,⁵ we choose to write the first term as a fraction and associate the negatives ‘-’ with the numerators. This yields $\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \dots$. The numerators form the sequence 1, -2, 4, -8, ... which is geometric with $a = 1$ and $r = -2$, so we get $c_n = (-2)^{n-1}$, for $n \geq 1$. The denominators 1, 7, 13, 19, ... form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $d_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$. We obtain our formula for $a_n = \frac{c_n}{d_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out. \square

⁵Here we take ‘experimentation’ to mean a frustrating guess-and-check session.

While the last problem in Example 1.1.3 was neither geometric nor arithmetic, it did resolve into a combination of these two kinds of sequences. If handed the sequence $2, 5, 10, 17, \dots$, we would be hard-pressed to find a formula for a_n if we restrict our attention to these two archetypes. We said before that there is no general algorithm for finding the explicit formula for the n th term of a given sequence, and it is only through experience gained from evaluating sequences from explicit formulas that we learn to begin to recognize number patterns. The pattern $1, 4, 9, 16, \dots$ is rather recognizable as the squares, so the formula $a_n = n^2$, $n \geq 1$ may not be too hard to determine. With this in mind, it's possible to see $2, 5, 10, 17, \dots$ as the sequence $1 + 1, 4 + 1, 9 + 1, 16 + 1, \dots$, so that $a_n = n^2 + 1$, $n \geq 1$. Of course, since we are given only a small *sample* of the sequence, we shouldn't be too disappointed to find out this isn't the *only* formula which generates this sequence. For example, consider the sequence defined by $b_n = -\frac{1}{4}n^4 + \frac{5}{2}n^3 - \frac{31}{4}n^2 + \frac{25}{2}n - 5$, $n \geq 1$. The reader is encouraged to verify that it also produces the terms $2, 5, 10, 17$. In fact, it can be shown that given any finite sample of a sequence, there are infinitely many explicit formulas all of which generate those same finite points. This means that there will be infinitely many correct answers to some of the exercises in this section.⁶ Just because your answer doesn't match ours doesn't mean it's wrong. As always, when in doubt, write your answer out. As long as it produces the same terms in the same order as what the problem wants, your answer is correct.

We would be remiss to close this section without mention of the utility of sequences in everyday life. Indeed, sequences play a major role in the Mathematics of Finance, as we have already seen with Equation ?? in Section ?. Recall that if we invest P dollars at an annual percentage rate r and compound the interest n times per year, the formula for A_k , the amount in the account after k compounding periods, is $A_k = P \left(1 + \frac{r}{n}\right)^k = \left[P \left(1 + \frac{r}{n}\right)\right] \left(1 + \frac{r}{n}\right)^{k-1}$, $k \geq 1$. We now spot this as a geometric sequence with first term $P \left(1 + \frac{r}{n}\right)$ and common ratio $\left(1 + \frac{r}{n}\right)$. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an **annuity** and will be discussed in the next section once we have developed more mathematical machinery.

1.1.1 EXERCISES

1. Write out the four terms of the following sequences.

(a) $a_n = 2^n - 1$, $n \geq 0$

(b) $d_j = (-1)^{\frac{j(j+1)}{2}}$, $j \geq 1$

(c) $\{5k - 2\}_{k=1}^{\infty}$

(d) $\left\{\frac{n^2+1}{n+1}\right\}_{n=0}^{\infty}$

(e) $\left\{\frac{x^n}{n^2}\right\}_{n=1}^{\infty}$

(f) $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty}$

(g) $a_1 = 3$, $a_{n+1} = a_n - 1$, $n \geq 1$

(h) $d_0 = 12$, $d_m = \frac{d_{m-1}}{100}$, $m \geq 1$

(i) $b_1 = 2$, $b_{k+1} = 3b_k + 1$, $k \geq 1$

(j) $c_0 = -2$, $c_j = \frac{c_{j-1}}{(j+1)(j+2)}$, $m \geq 1$

(k) $a_1 = 117$, $a_{n+1} = \frac{1}{a_n}$, $n \geq 1$

(l) $s_0 = 1$, $s_{n+1} = x^{n+1} + s_n$, $n \geq 0$

⁶For more on this, see [When Every Answer is Correct: Why Sequences and Number Patterns Fail the Test](#) by Professor Don White of Kent State University.

- (m) $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$ (This is the famous [Fibonacci Sequence](#))
2. Find an explicit formula for the n^{th} term of the following sequences. Use the formulas in Equation 1.1 as needed.
- | | | |
|---|---|---------------------------------------|
| (a) $3, 5, 7, 9, \dots$ | (d) $1, \frac{2}{3}, \frac{1}{3}, \frac{4}{27}, \dots$ | (g) $0.9, 0.99, 0.999, 0.9999, \dots$ |
| (b) $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$ | (e) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$ | (h) $27, 64, 125, 216, \dots$ |
| (c) $1, \frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \dots$ | (f) $x, -\frac{x^3}{3}, \frac{x^5}{5}, -\frac{x^7}{7}, \dots$ | (i) $1, 0, 1, 0, \dots$ |
3. Find a sequence which is both arithmetic and geometric. (Hint: Start with $a_n = c$ for all n .)
4. Show that a geometric sequence can be transformed into an arithmetic sequence by taking the natural logarithm of the terms.
5. Thomas Robert Malthus is credited with saying, “The power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison with the second.” (See this [webpage](#) for more information.) Discuss this quote with your classmates from a sequences point of view.
6. This classic problem involving sequences shows the power of geometric sequences. Suppose that a wealthy benefactor agrees to give you one penny today and then double the amount she gives you each day for 30 days. So, for example, you get two pennies on the second day and four pennies on the third day. How many pennies do you get on the 30th day? What is the total dollar value of the gift you have received?
7. Research the terms ‘arithmetic mean’ and ‘geometric mean.’ With the help of your classmates, show that a given term of a arithmetic sequence $a_k, k \geq 2$ is the arithmetic mean of the term immediately preceding, a_{k-1} it and immediately following it, a_{k+1} . State and prove an analogous result for geometric sequences.
8. Discuss with your classmates how the results of this section might change if we were to examine sequences of other mathematical things like complex numbers or matrices. Find an explicit formula for the n^{th} term of the sequence $i, -1, -i, 1, i, \dots$. List out the first four terms of the matrix sequences we discussed in Exercise ?? in Section ??.

1.1.2 ANSWERS

- | | |
|--------------------------------------|--|
| 1. (a) $0, 1, 3, 7$ | (e) $x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}$ |
| (b) $-1, -1, 1, 1$ | (f) $0, \ln(2), \frac{\ln(3)}{3}, \frac{\ln(4)}{4}$ |
| (c) $3, 8, 13, 18$ | (g) $3, 2, 1, 0$ |
| (d) $1, 1, \frac{5}{3}, \frac{5}{2}$ | (h) $12, 0.12, 0.0012, 0.000012$ |

$$f(x) = \sum_{k=0}^n a_k x^k$$

for real numbers a_k , $k = 0, 1, \dots, n$. The reader is invited to compare this with what is given in Definition ???. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the i th row R_i of a matrix $A = [a_{ij}]_{m \times n}$ and the j th column C_j of a matrix $B = [b_{ij}]_{n \times r}$ as

$$R_i \cdot C_j = \sum_{k=1}^n a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition ???. Our next example gives us practice with this new notation.

EXAMPLE 1.2.1.

1. Find the following sums.

$$(a) \sum_{k=1}^4 \frac{13}{100^k}$$

$$(b) \sum_{n=1}^5 \frac{(-1)^{n+1}}{n} (x-1)^n$$

2. Write the following sums using summation notation.

$$(a) 1 + 3 + 5 + \dots + 117$$

$$(b) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{117}$$

$$(c) 0.9 + 0.09 + 0.009 + \dots + \underbrace{0.\underbrace{0 \dots 0}_n 9}_{n-1 \text{ zeros}}$$

SOLUTION.

1. (a) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned} \sum_{k=1}^4 \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313 \end{aligned}$$

- (b) We proceed as in part (a), replacing the index n , but *not* the variable x , with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned} \sum_{n=1}^5 \frac{(-1)^{n+1}}{n} (x-1)^n &= \frac{(-1)^{1+1}}{1} (x-1)^1 + \frac{(-1)^{2+1}}{2} (x-1)^2 + \frac{(-1)^{3+1}}{3} (x-1)^3 \\ &\quad + \frac{(-1)^{4+1}}{4} (x-1)^4 + \frac{(-1)^{5+1}}{5} (x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} \end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section 1.1.

- (a) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. We get a formula for the n th term of the sequence using Equation 1.1 to get $a_n = 1 + (n-1)2 = 2n-1$, $n \geq 1$. At this stage, we have the formula for the terms, namely $2n-1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n-1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + \dots + 117 = \sum_{n=1}^{59} (2n-1)$$

- (b) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives ‘-’ with the numerators to get

$$\frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \dots + \frac{1}{117}$$

The numerators, 1, -1, etc. can be described by the geometric sequence¹ $c_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence² $d_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \frac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{117} = \sum_{n=1}^{117} \frac{(-1)^{n-1}}{n}$$

- (c) Thanks to Example 1.1.3, we know that one formula for the n^{th} term is $a_n = \frac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation. To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n . Hence, n is

¹This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.

²It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.

the upper limit of summation. Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.³ We choose k and get

$$0.9 + 0.09 + 0.009 + \dots + \underbrace{0.\underbrace{0\cdots 0}_{n-1 \text{ zeros}}9}_{n-1 \text{ zeros}} = \sum_{k=1}^n \frac{9}{10^k}$$

□

The following theorem presents some general properties of summation notation. While we shall not have much need of these properties in Algebra, they do play a great role in Calculus. Moreover, there is much to be learned by thinking about why the properties hold. We invite the reader to prove these results. To get started, remember, “When in doubt, write it out!”

THEOREM 1.1. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- $\sum_{n=m}^p (a_n \pm b_n) = \sum_{n=m}^p a_n \pm \sum_{n=m}^p b_n$
- $\sum_{n=m}^p c a_n = c \sum_{n=m}^p a_n$, for any real number c .
- $\sum_{n=m}^p a_n = \sum_{n=m}^j a_n + \sum_{n=j+1}^p a_n$, for any natural number $m \leq j < j+1 \leq p$.
- $\sum_{n=m}^p a_n = \sum_{n=m+r}^{p+r} a_{n-r}$, for any whole number r .

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S , we write it out in two different ways

$$\begin{aligned} S &= a & + & (a+d) & + & \dots & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S &= (a+(n-1)d) & + & (a+(n-2)d) & + & \dots & + & (a+d) & + & a \end{aligned}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a + (n-1)d) + (2a + (n-1)d) + \dots + (2a + (n-1)d) + (2a + (n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a + (n-1)d)$ so we get $2S = n(2a + (n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

³To see why, try writing the summation using ‘ n ’ as the index.

$$S = \frac{n}{2}(2a + (n-1)d)$$

If we rewrite the quantity $2a + (n-1)d$ as $a + (a + (n-1)d) = a_1 + a_n$, we get the formula

$$S = n \left(\frac{a_1 + a_n}{2} \right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS , we get

$$\begin{aligned} S &= a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} \\ rS &= \quad \quad ar + ar^2 + \dots + ar^{n-2} + ar^{n-1} + ar^n \end{aligned}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a(1-r^n)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a \left(\frac{1-r^n}{1-r} \right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \dots + a}_{n \text{ times}} = na$$

Our results are summarized below.

EQUATION 1.2. Sums of Arithmetic and Geometric Sequences:

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^n a_k = n \left(\frac{a_1 + a_n}{2} \right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

1. $S = \sum_{k=1}^n a_k = \frac{a_1 - a_{n+1}}{1-r} = a \left(\frac{1-r^n}{1-r} \right)$, if $r \neq 1$.

2. $S = \sum_{k=1}^n a_k = \sum_{k=1}^n a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 1.2, formal proofs require the machinery in Section 1.3. An application of the arithmetic sum formula which proves useful in Calculus results in formula for the sum of the first n natural numbers. The natural numbers themselves are a sequence⁴ 1, 2, 3, ... which is arithmetic with $a = d = 1$. Applying Equation 1.2,

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

So, for example, the sum of the first 100 natural numbers is $\frac{100(101)}{2} = 5050$.⁵

We close this section with a peek into Calculus by considering *infinite* sums, called **series**. Consider the number $0.\bar{9}$. We can write this number as

$$0.\bar{9} = 0.9999\dots = 0.9 + 0.09 + 0.009 + 0.0009 + \dots$$

From Example 1.2.1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\dots9}_{n \text{ nines}} = .9 + 0.09 + 0.009 + \dots 0.\underbrace{0\dots0}_{n-1 \text{ zeros}}9 = \sum_{k=1}^n \frac{9}{10^k}$$

Using Equation 1.2, we have

$$\sum_{k=1}^n \frac{9}{10^k} = \frac{9}{10} \left(\frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} \right) = 1 - \frac{1}{10^{n+1}}$$

It stands to reason that $0.\bar{9}$ is the same value of $1 - \frac{1}{10^{n+1}}$ as $n \rightarrow \infty$. Our knowledge of exponential expressions from Section ?? tells us that $\frac{1}{10^{n+1}} \rightarrow 0$ as $n \rightarrow \infty$, so $1 - \frac{1}{10^{n+1}} \rightarrow 1$. We have just argued that $0.\bar{9} = 1$, which may cause some distress for some readers.⁶ Any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We end this section with a theorem concerning geometric series.

THEOREM 1.2. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + \dots = \sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + \dots$ is not defined.

The justification of the result in Theorem 1.2 comes from taking the formula in Equation 1.2 for the sum of the first n terms of a geometric sequence and examining the formula as $n \rightarrow \infty$. Assuming $|r| < 1$ means $-1 < r < 1$, so $r^n \rightarrow 0$ as $n \rightarrow \infty$. Hence as $n \rightarrow \infty$,

⁴This is the identity function on the natural numbers!

⁵There is an interesting anecdote which says that the famous mathematician [Carl Friedrich Gauss](#) was given this problem in primary school and devised a very clever solution.

⁶To make this more palatable, it is usually accepted that $0.\bar{3} = \frac{1}{3}$ so that $0.\bar{9} = 3(0.\bar{3}) = 3(\frac{1}{3}) = 1$. Feel better?

$$\sum_{k=1}^n ar^{k-1} = a \left(\frac{1-r^n}{1-r} \right) \rightarrow \frac{a}{1-r}$$

As to what goes wrong when $|r| \geq 1$, we leave that to Calculus as well, but will explore some cases in the exercises.

1.2.1 EXERCISES

1. Find the following sums.

$$(a) \sum_{g=4}^9 (5g + 3)$$

$$(c) \sum_{j=0}^5 2^j$$

$$(e) \sum_{i=1}^4 \frac{1}{4}(i^2 + 1)$$

$$(b) \sum_{k=3}^8 \frac{1}{k}$$

$$(d) \sum_{k=0}^2 (3k - 5)x^k$$

$$(f) \sum_{n=1}^{100} (-1)^n$$

2. Rewrite the sum using summation notation.

$$(a) 8 + 11 + 14 + 17 + 20$$

$$(e) 2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5}$$

$$(b) 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8$$

$$(f) -\ln(3) + \ln(4) - \ln(5) + \dots + \ln(20)$$

$$(c) x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$(g) 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$$

$$(d) 1 + 2 + 4 + \dots + 2^{29}$$

$$(h) \frac{1}{2}(x-5) + \frac{1}{4}(x-5)^2 + \frac{1}{6}(x-5)^3 + \frac{1}{8}(x-5)^4$$

3. Find the sum of the first 10 terms of the following sequences.

$$(a) a_n = 3 + 5n$$

$$(b) b_n = \left(\frac{1}{2}\right)^n$$

$$(c) c_n = -2n + \left(\frac{5}{3}\right)^n$$

4. Express the following repeating decimals as a fraction of integers.

$$(a) 0.\overline{7}$$

$$(b) 0.\overline{13}$$

$$(c) 10.\overline{159}$$

$$(d) -5.\overline{867}$$

5. Prove the properties listed in Theorem 1.1.

6. Discuss with your classmates the problems which arise in trying to find the sum of the following geometric series. When in doubt, write them out!

$$(a) \sum_{k=1}^{\infty} 2^{k-1}$$

$$(b) \sum_{k=1}^{\infty} (1.0001)^{k-1}$$

$$(c) \sum_{k=1}^{\infty} (-1)^{k-1}$$

1.2.2 ANSWERS

1.

$$A_3 = A_2(1+i) + P = P(1+i) \left(1 + \frac{1}{1+i}\right) (1+i) + P = P(1+i)^2 \left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1+i)$, and when we add the fourth payment, we factor out $P(1+i)^3$ to get

$$A_4 = P(1+i)^3 \left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3}\right)$$

This pattern continues so that at the end of the k th compounding, we get

$$A_k = P(1+i)^{k-1} \left(1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1+i}$. Using Equation 1.2, we get

$$1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^{k-1}} = 1 \left(\frac{1 - \frac{1}{(1+i)^k}}{1 - \frac{1}{1+i}} \right) = \frac{(1+i)(1 - (1+i)^{-k})}{i}$$

Hence, we get

$$A_k = P(1+i)^{k-1} \left(\frac{(1+i)(1 - (1+i)^{-k})}{i} \right) = \frac{P((1+i)^k - 1)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an **ordinary annuity**.

EQUATION 1.3. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P((1+i)^{nt} - 1)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 1.3 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made at the *beginning* of the compounding period instead of at the end, the annuity is called an **annuity-due**. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

EXAMPLE 1.3.1. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.

2. How many years will it take for the annuity to grow to \$100,000?

SOLUTION.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50 \left((1 + 0.005)^{(12)(30)} - 1 \right)}{0.005} \approx 50225.75$$

Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t . We isolate the exponential and take the natural logarithm of both sides of the equation.

$$\begin{aligned} 100000 &= \frac{50 \left((1 + 0.005)^{12t} - 1 \right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln \left((1.005)^{12t} \right) &= \ln(11) \\ 12t \ln(1.005) &= \ln(11) \\ t &= \frac{\ln(11)}{12 \ln(1.005)} \approx 40.06 \end{aligned}$$

This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. \square

A **mortgage** is like a backwards annuity. Here's an example.

EXAMPLE 1.3.2. A family takes out a 30 year \$100,000 mortgage to buy a house. The interest rate is 6% per year and they will repay the loan in 360 equal monthly payments. To compute the monthly payment imagine that the family has taken 360 loans

$$100000 = L_1 + L_2 + \cdots + L_{360}$$

and they will repay the k th loan at the end of the k th month with the monthly payment of a dollars. The monthly interest rate is $6\%/12 = 0.005$ so the amount owed on the k th loan after k months is $(1.005)^k L_k$. This is the amount of the k th payment so $a = (1.005)^k L_k$ so $L_k = a(1.005)^{-k}$ so

$$100000 = L_1 + L_2 + \cdots + L_{360} = a(1.005)^{-1} + a(1.005)^{-2} + \cdots + a(1.005)^{-360}.$$

Using Sigma notation and Theorem ?? with $r = (1.005)^{-1}$ this may be written

$$100000 = \sum_{k=1}^{360} a(1.005)^{-k} = a \frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1}$$

To evaluate the expression on the right multiply top and bottom by 1.005 to get

$$\frac{(1.005)^{-361} - (1.005)^{-1}}{(1.005)^{-1} - 1} = \frac{(1.005)^{-360} - 1}{1 - 1.005} = \frac{1 - (1.005)^{-360}}{0.005} = 166.79$$

so $100000 = 166.79a$ so the monthly payment is

$$a = 100000/166.79 = 599.56.$$

1.3.1 EXERCISES

1. If monthly payments of \$300 are made to an ordinary annuity with an APR of 2.5% compounded monthly what is the value of the annuity after 17 years?
2. Show that the formula for the future value of an annuity due is

$$A = P(1 + i) \left[\frac{(1 + i)^{nt} - 1}{i} \right]$$

1.3.2 ANSWERS

1. \$76,163.67