Chapter 1

Functions

1.1 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 1.1. A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

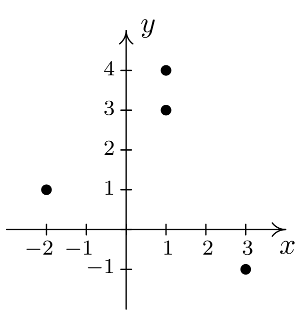
Example 1.1.1. Which of the following relations describe y as a function of x?

1. $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
2. $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$

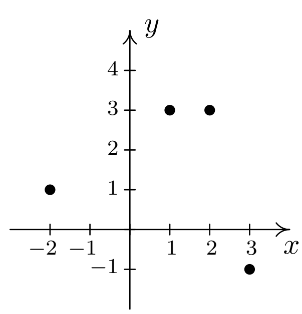
Solution. A quick scan of the points in R_1 reveals that the x-coordinate 1 is matched with two **different** y-coordinates: namely 3 and 4. Hence in R_1, y is **not** a function of x. On the other hand, every x-coordinate in R_2 occurs only once which means each x-coordinate has only one corresponding y-coordinate. So, R_2 **does** represent y as a function of x. □

Note that in the previous example, the relation R_2 contained two different points with the same $\boldsymbol{y}$-coordinates, namely $(1, 3)$ and $(2, 3)$. Remember, in order to say y is a function of x, we just need to ensure the same x-coordinate isn't used in more than one point.[1]

To see what the function concept means geometrically, we graph R_1 and R_2 in the plane.



The graph of R_1



The graph of R_2

The fact that the x-coordinate 1 is matched with two different y-coordinates in R_1 presents itself graphically as the points $(1, 3)$ and $(1, 4)$ lying on the same vertical line, $x = 1$. If we turn

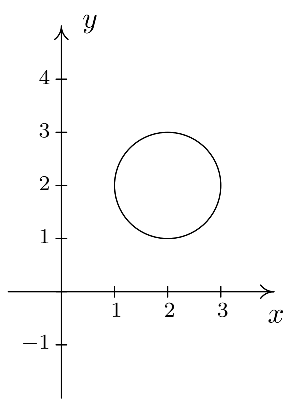
[1] We will have occasion later in the text to concern ourselves with the concept of x being a function of y. In this case, R_1 represents x as a function of y; R_2 does not.

our attention to the graph of R_2, we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

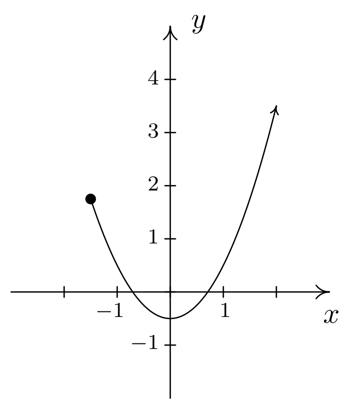
THEOREM 1.1. **The Vertical Line Test:** A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of 'function' as well as the concept of 'graph'.

EXAMPLE 1.1.2. Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of R

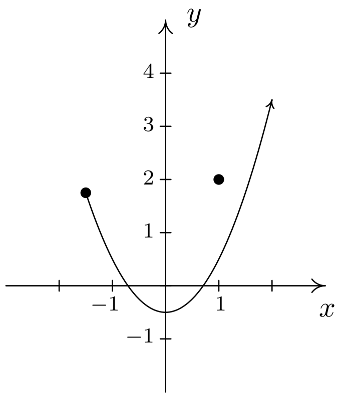


The graph of S

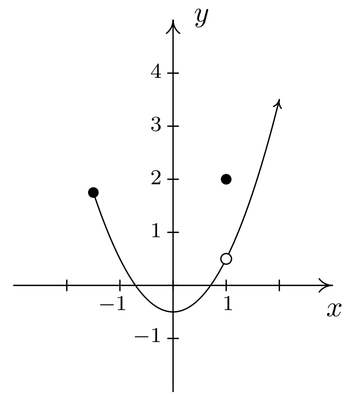
SOLUTION. Looking at the graph of R, we can easily imagine a vertical line crossing the graph more than once. Hence, R does **not** represent y as a function of x. However, in the graph of S, every vertical line crosses the graph at most once, and so S **does** represent y as a function of x. $\square$

In the previous test, we say that the graph of the relation R **fails** the Vertical Line Test, whereas the graph of S **passes** the Vertical Line Test. Note that in the graph of R there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is **one** vertical line that fits the bill, as the next example illustrates.

EXAMPLE 1.1.3. Use the Vertical Line Test to determine which of the following relations describes y as a function of x.

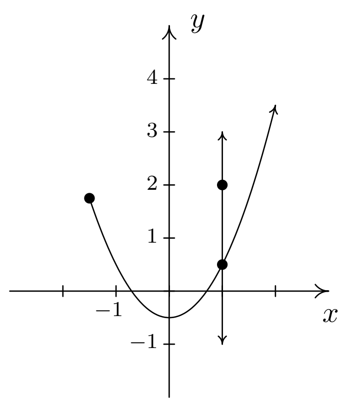


The graph of S_1



The graph of S_2

SOLUTION. Both S_1 and S_2 are slight modifications to the relation S in the previous example whose graph we determined passed the Vertical Line Test. In both S_1 and S_2, it is the addition of the point $(1, 2)$ which threatens to cause trouble. In S_1, there is a point on the curve with x-coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below.) Hence, the graph of S_1 fails the Vertical Line Test, so y is **not** a function of x here. However, in S_2 notice that the point with x-coordinate 1 on the curve has been omitted, leaving an 'open circle' there. Hence, the vertical line $x = 1$ crosses the graph of S_2 only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we have that the graph of S_2 passes the Vertical Line Test. Thus it describes y as a function of x.



S_1 and the line $x = 1$

□

Suppose a relation F describes y as a function of x. The sets of x- and y-coordinates are given special names.

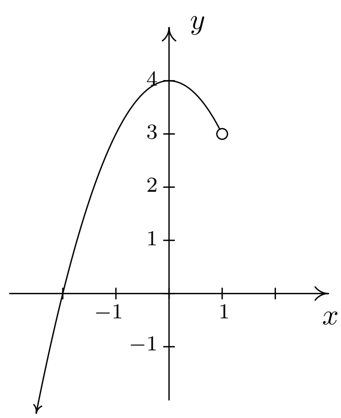
DEFINITION 1.2. Suppose F is a relation which describes y as a function of x.

- The set of the x-coordinates of the points in F is called the **domain** of F.
- The set of the y-coordinates of the points in F is called the **range** of F.

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

EXAMPLE 1.1.4. Find the domain and range of the following functions

1. $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$
2. G is the function graphed below:

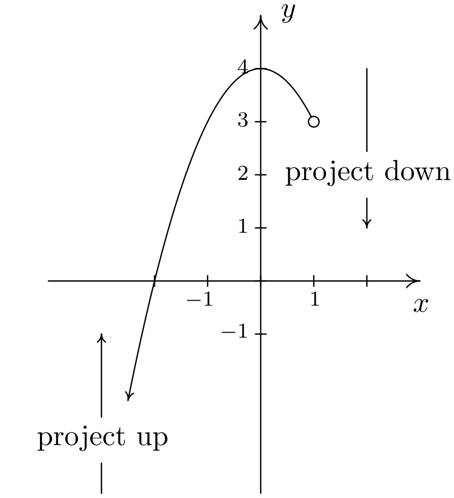


The graph of G

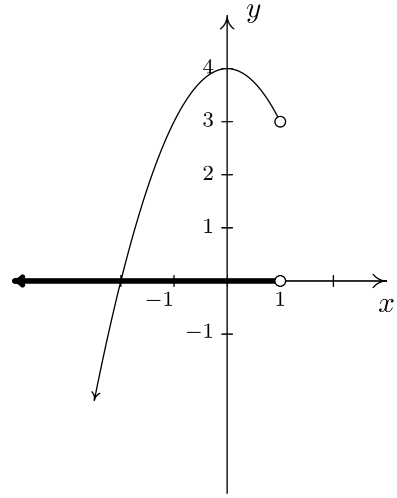
SOLUTION. The domain of F is the set of the x-coordinates of the points in F: $\{-3, 0, 4, 5\}$ and the range of F is the set of the y-coordinates: $\{1, 2\}$.[2]

To determine the domain and range of G, we need to determine which x and y values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the x-axis and determining the portion of the x-axis that gets covered. This is called **projecting** the curve to the x-axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ isn't on the graph, but all points on the curve leading up to that point are on the curve.

[2]When listing numbers in a set, we list each number only once, in increasing order.

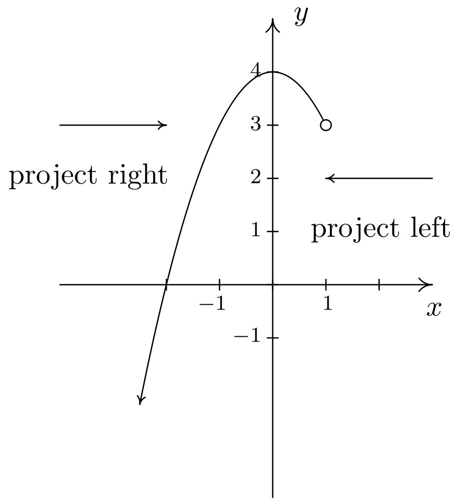


The graph of G

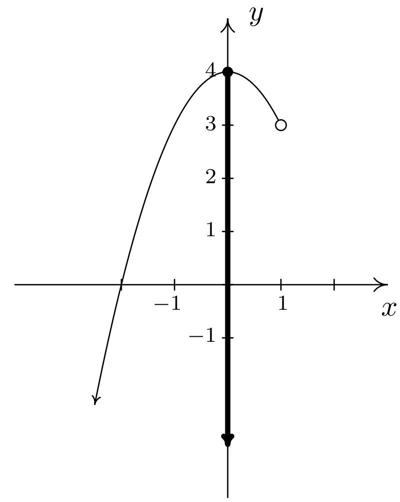


The graph of G

We see from the figure that if we project the graph of G to the x-axis, we get all real numbers less than 1. Using interval notation, we write the domain of G is $(-\infty, 1)$. To determine the range of G, we project the curve to the y-axis as follows:



The graph of G



The graph of G

Note that even though there is an open circle at $(1, 3)$, we still include the y value of 3 in our range, since the point $(-1, 3)$ is on the graph of G. We see that the range of G is all real numbers less than or equal to 4, or, in interval notation: $(-\infty, 4]$. □

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section **??** may or may not describe y as a function of x. The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section **??**.)

EXAMPLE 1.1.5. Determine which equations represent y as a function of x:

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $x^2 y = 1 - 3y$

SOLUTION. For each of these equations, we solve for y and determine whether each choice of x will determine only one corresponding value of y.

1.
$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1 - x^3} \qquad \text{extract square roots} \\ y &= \pm\sqrt{1 - x^3} \end{aligned}$$

   If we substitute $x = 0$ into our equation for y, we get: $y = \pm\sqrt{1 - 0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does **not** represent y as a function of x.

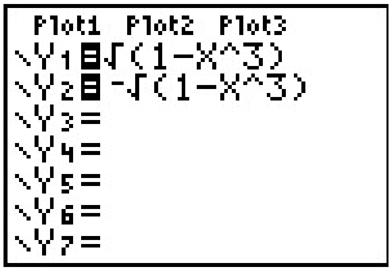
2.
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

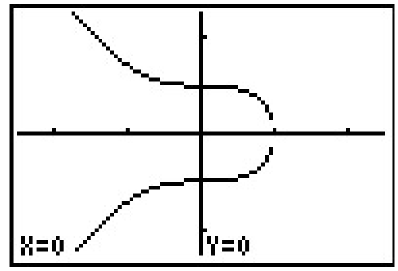
   For every choice of x, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of y. Hence, this equation describes y as a function of x.

3.
$$\begin{aligned} x^2 y &= 1 - 3y \\ x^2 y + 3y &= 1 \\ y\left(x^2 + 3\right) &= 1 \qquad \text{factor} \\ y &= \frac{1}{x^2 + 3} \end{aligned}$$

For each choice of x, there is only one value for y, so this equation describes y as a function of x. □

Of course, we could always use our graphing calculator to verify our responses to the previous example. For example, if we wanted to verify that the first equation does not represent y as a function of x, we could enter the equation for y into the calculator as indicated below and graph. Note that we need to enter both solutions – the positive and the negative square root – for y. The resulting graph clearly fails the Vertical Line Test, so does not represent y as a function of x.





1.1.1 Exercises

1. Determine which of the following relations represent y as a function of x. Find the domain and range of those relations which are functions.

   (a) $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$

   (b) $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

   (c) $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

   (d) $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

   (e) $\{(x,y) : x \text{ is an odd integer, and } y \text{ is an even integer}\}$

   (f) $\{(x,1) : x \text{ is an irrational number}\}$

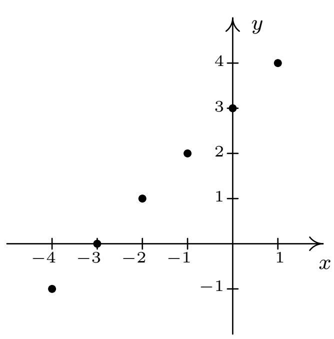
   (g) $\{(1,0),\ (2,1),\ (4,2),\ (8,3),\ (16,4),\ (32,5),\ \ldots\}$

   (h) $\{\ldots,\ (-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9),\ \ldots\}$

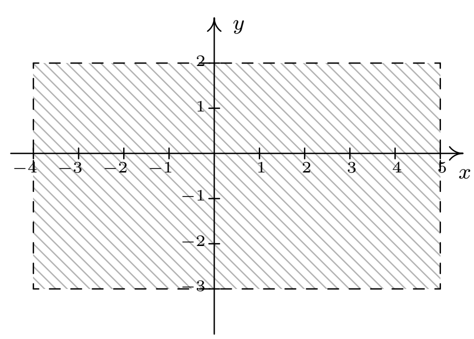
   (i) $\{(-2,y) : -3 < y < 4\}$

   (j) $\{(x,3) : -2 \le x < 4\}$

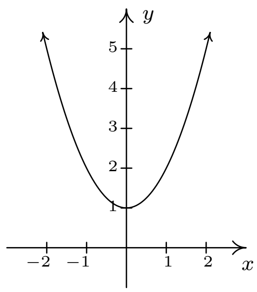
2. Determine which of the following relations represent y as a function of x. Find the domain and range of those relations which are functions.



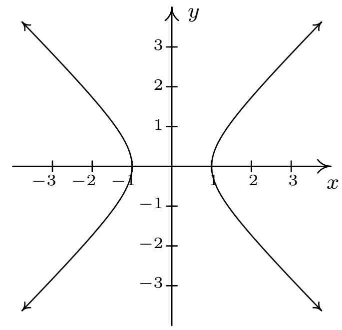
(a)



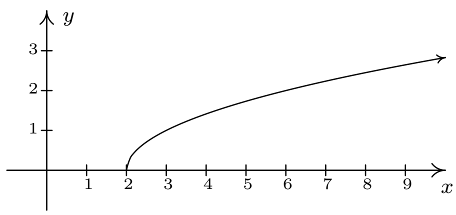
(b)


y
5
4
3
2
1
−2 −1 1 2
x

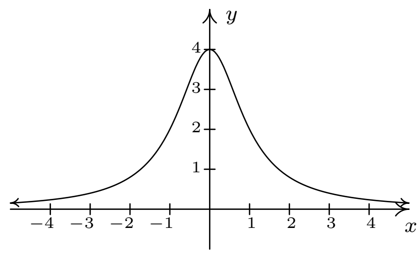
(c)


y
3
2
1
−3 −2 −1 1 2 3
x
−1
−2
−3

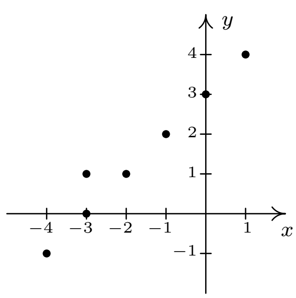
(d)


y
3
2
1
1 2 3 4 5 6 7 8 9
x

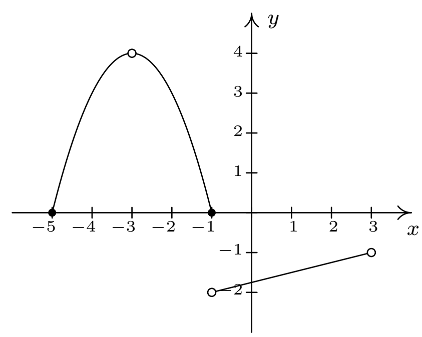
(e)


y
4
3
2
1
−4 −3 −2 −1 1 2 3 4
x

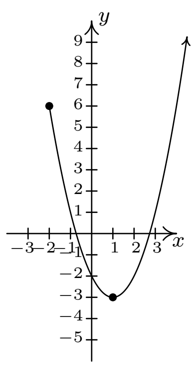
(f)


y
4
3
2
1
−4 −3 −2 −1 1
x
−1

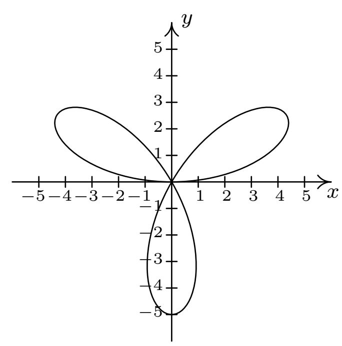
(g)


y
4
3
2
1
−5 −4 −3 −2 −1 1 2 3
x
−1
−2

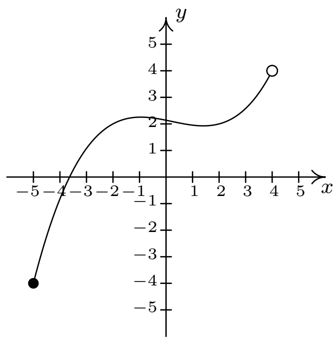
(h)


y
9
8
7
6
5
4
3
2
1
−3 −2 −1 1 2 3
x
−1
−2
−3
−4
−5

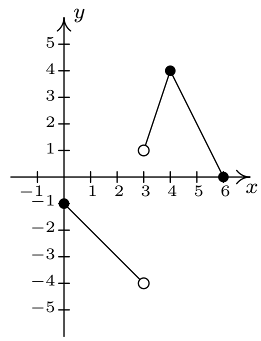
(i)


y
5
4
3
2
1
−5 −4 −3 −2 −1 1 2 3 4 5
x
−1
−2
−3
−4
−5

(j)



(k)



(l)

3. Determine which of the following equations represent y as a function of x.

(a) $y = x^3 - x$
(b) $y = \sqrt{x-2}$
(c) $x^3y = -4$
(d) $x^2 - y^2 = 1$
(e) $y = \dfrac{x}{x^2-9}$
(f) $x = -6$
(g) $x = y^2 + 4$
(h) $y = x^2 + 4$
(i) $x^2 + y^2 = 4$
(j) $y = \sqrt{4 - x^2}$
(k) $x^2 - y^2 = 4$
(l) $x^3 + y^3 = 4$
(m) $2x + 3y = 4$
(n) $2xy = 4$

4. Explain why the height h of a Sasquatch is a function of its age N in years. Given that a Sasquatch is 2 feet tall at birth, experiences growth spurts at ages 3, 23 and 57, and lives to be about 150 years old with a maximum height of 9 feet, sketch a rough graph of the height function.

5. Explain why the population P of Sasquatch in a given area is a function of time t. What would be the range of this function?

6. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

7. The process given in Example **??** for determining whether an equation of a relation represents y as a function of x breaks down if we cannot solve the equation for y in terms of x. However, that does not prevent us from proving that an equation which fails to represent y as a function of x actually fails to do so. What we really need is two points with the same x-coordinate and different y-coordinates which both satisfy the equation so that the graph of the relation would fail the Vertical Line Test **??**. Discuss with your classmates how you might find such points for the relations given below.

(a) $x^3 + y^3 - 3xy = 0$

(b) $x^4 = x^2 + y^2$

(c) $y^2 = x^3 + 3x^2$

(d) $(x^2 + y^2)^2 = x^3 + y^3$

1.1.2 Answers

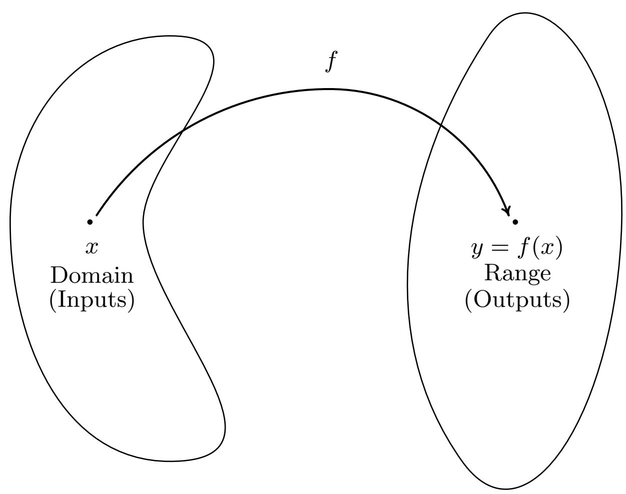
1. (a) Function, domain $= \{-3, -2, -1, 0, 1, 2, 3\}$, range $= \{0, 1, 4, 9\}$.
   (b) Not a function.
   (c) Function, domain $= \{-7, -3, 3, 4, 5, 6\}$, range $= \{0, 4, 5, 6, 9\}$
   (d) Function, domain $= \{1, 4, 9, 16, 25, 36, \ldots\} = \{x : x \text{ is a perfect square}\}$,
   range $= \{2, 4, 6, 8, 10, 12, \ldots\} = \{y : y \text{ is a positive even integer}\}$
   (e) Not a function
   (f) Function, domain $= \{x : x \text{ is irrational}\}$, range $= \{1\}$.
   (g) Function, domain $= \{x : x = 2^n \text{ for some whole number } n\}$, range $= \{y : y \text{ is any whole number}\}$,
   (h) Function, domain $= \{x : x \text{ is any integer}\}$, range $= \{y : y = n^2 \text{ for some integer } n\}$.
   (i) Not a function.
   (j) Function, domain $= [-2, 4)$, range $= \{3\}$.

2. (a) Function, domain $= \{-4,\ -3,\ -2,\ -1,\ 0,\ 1\}$, range $= \{-1,\ 0,\ 1,\ 2,\ 3,\ 4\}$
   (b) Not a function
   (c) Function, domain $= (-\infty, \infty)$, range $= [1, \infty)$
   (d) Not a function
   (e) Function, domain $= [2, \infty)$, range $= [0, \infty)$
   (f) Function, domain $= (-\infty, \infty)$, range $= (0, 4]$
   (g) Not a function
   (h) Function, domain $= [-5, -3) \cup (-3, 3)$, range $= (-2, -1) \cup [0, 4)$
   (i) Function, domain $= [-2, \infty)$, range $= [-3, \infty)$
   (j) Not a function
   (k) Function, domain $= [-5, 4)$, range $= [-4, 4)$
   (l) Function, domain $= [0, 3) \cup (3, 6]$, range $= (-4, -1] \cup [0, 4]$

3. (a) Function
   (b) Function
   (c) Function
   (d) Not a function
   (e) Function
   (f) Not a function
   (g) Not a function
   (h) Function
   (i) Not a function
   (j) Function
   (k) Not a function
   (l) Function
   (m) Function
   (n) Function

1.2 Function Notation

In Definition **??**, we described a function as a special kind of relation – one in which each x-coordinate is matched with only one y-coordinate. In this section, we focus more on the **process** by which the x is matched with the y. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In this case, the parentheses here do not indicate multiplication, as they do elsewhere in algebra. This could cause confusion if the context is not clear. In other words, $f(x)$ is the **output** which results by applying the **process** f to the **input** x. This relationship is typically visualized using a diagram similar to the one below.



The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. multiply by 3
2. add 4

If we choose 5 as our input, in step 1 we multiply by 3 to get $(5)(3) = 15$. In step 2, we add 4 to our result from step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying step 1 produces $3x$. Following with step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we **replace** the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

EXAMPLE 1.2.1. Suppose a function g is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

SOLUTION. Starting with 5, step 1 gives $5+4=9$. Continuing with step 2, we get $(3)(9)=27$. To find a formula for $g(x)$, we start with our input x. Step 1 produces $x+4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x+4)=3x+12$. Hence, $g(x)=3x+12$. We can check our formula by replacing x with 5 to get $g(5)=3(5)+12=15+12=27\checkmark$. $\square$

Most of the functions we will encounter in College Algebra will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other math courses.

EXAMPLE 1.2.2. For $f(x)=-x^2+3x+4$, find and simplify

1. $f(-1)$, $f(0)$, $f(2)$
2. $f(2x)$, $2f(x)$
3. $f(x+2)$, $f(x)+2$, $f(x)+f(2)$

SOLUTION.

1. To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2+3(-1)+4 \\ &= -(1)+(-3)+4 \\ &= 0 \end{aligned}$$

   Similarly, $f(0)=-(0)^2+3(0)+4=4$, and $f(2)=-(2)^2+3(2)+4=-4+6+4=6$.

2. To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2+3(2x)+4 \\ &= -(4x^2)+(6x)+4 \\ &= -4x^2+6x+4 \end{aligned}$$

   The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2+3x+4\right) \\ &= -2x^2+6x+8 \end{aligned}$$

   Note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the **input** by 2; for $2f(x)$, we are multiplying the **output** by 2. As we see, we get entirely different results. Also note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

3. To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -\left(x^2+4x+4\right) + (3x+6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find $f(x)+2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x)+2 &= \left(-x^2+3x+4\right) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

Once again, we see there is a dramatic difference between modifying the input and modifying the output. Finally, in $f(x) + f(2)$ we are adding the value $f(2)$ to the expression $f(x)$. From our work above, we see $f(2) = 6$ so that

$$\begin{aligned} f(x)+f(2) &= \left(-x^2+3x+4\right) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

Notice that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three **different** expressions. Even though function notation uses parentheses, as does multiplication, there is no general 'distributive property' of function notation. $\square$

Suppose we wish to find $r(3)$ for $r(x) = \dfrac{2x}{x^2-9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2-9} = \frac{6}{0},$$

which is undefined. The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation as $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$. When a formula for a function is given, we assume (unless the contrary is explicitly stated) that the domain of the function is the set of all real numbers for which the formula makes arithmetic sense when the number is substituted into the formula. This set of numbers is often called the **implied domain**[1] of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

[1] or, 'implicit domain'

EXAMPLE 1.2.3. Find the domain[2] of the following functions.

1. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x-3}}$
2. $g(x) = \sqrt{4-3x}$
3. $h(x) = \sqrt[5]{4-3x}$
4. $r(x) = \dfrac{4}{6 - \sqrt{x+3}}$
5. $I(x) = \dfrac{3x^2}{x}$

SOLUTION.

1. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x - 3 = 0$ or $x = 3$. For the 'large' denominator

$$\begin{aligned}
1 - \frac{4x}{x-3} &= 0 \\
1 &= \frac{4x}{x-3} \\
(1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\
x - 3 &= 4x \\
-3 &= 3x \\
-1 &= x
\end{aligned}$$

So we get two real numbers which make denominators 0, namely $x = -1$ and $x = 3$. Our domain is all real numbers **except** -1 and 3: $(-\infty, -1) \cup (-1, 3) \cup (3, \infty)$.

2. The potential disaster for g is if the radicand[3] is negative. To avoid this, we set $4 - 3x \geq 0$

$$\begin{aligned}
4 - 3x &\geq 0 \\
4 &\geq 3x \\
\frac{4}{3} &\geq x
\end{aligned}$$

Hence, as long as $x \leq \frac{4}{3}$, the expression $4 - 3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

3. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference – whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula

[2] The word 'implied' is, well, implied.

[3] The 'radicand' is the expression 'inside' the radical.

for $h(x)$ involves an odd indexed root (the fifth root.) Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

4. To find the domain of r, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a square root in that denominator. To satisfy the square root, we set the radicand $x+3 \geq 0$ so $x \geq -3$. Setting the denominator equal to zero gives

$$\begin{aligned} 6-\sqrt{x+3} &= 0 \\ 6 &= \sqrt{x+3} \\ 6^2 &= \left(\sqrt{x+3}\right)^2 \\ 36 &= x+3 \\ 33 &= x \end{aligned}$$

Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $x=33$, $6-\sqrt{x+3}=6-\sqrt{36}=0$, and so $x=33$ will cause problems in the denominator. At last we can find the domain of r: we need $x \geq -3$, but $x \neq 33$. Our final answer is $[-3,33) \cup (33,\infty)$.

5. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined.[4] Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty,0) \cup (0,\infty)$. $\Box$

It is worth reiterating the importance of finding the domain of a function **before** simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

Our next example shows how a function can be used to model real-world phenomena.

Example 1.2.4. The height h in feet of a model rocket above the ground t seconds after lift off is given by

$$h(t) = \begin{cases} -5t^2+100t, & \text{if} \quad 0 \leq t \leq 20 \\ 0, & \text{if} \quad t > 20 \end{cases}$$

Find and interpret $h(10)$ and $h(60)$.

Solution. There are a few qualities of h which may be off-putting. The first is that, unlike previous examples, the independent variable is t, not x. In this context, t is chosen because it represents time. The second is that the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. To find $h(10)$, we first notice that 10 is between 0 and 20 so we use the first formula listed: $h(t) = -5t^2 + 100t$. Hence,

[4] More precisely, the fraction $\frac{0}{0}$ is an 'indeterminant form'. Much time will be spent in Calculus wrestling with such creatures.

$h(10) = -5(10)^2 + 100(10) = 500$. In terms of the model rocket, this means that 10 seconds after lift off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that 60 is greater than 20, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift off, the rocket is 0 feet above the ground; in other words, a minute after lift off, the rocket has already returned to earth. $\square$

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena (e.g. postal rates,[5] income tax formulas[6]) are modeled by such functions. Also note that the domain of h in the above example is restricted to $t \geq 0$. For example, $h(-3)$ is not defined because $t = -3$ doesn't satisfy any of the conditions in any of the function's pieces. There is no inherent arithmetic reason which prevents us from calculating, say, $-5(-3)^2 + 100(-3)$, it's just that in this applied setting, $t = -3$ is meaningless. In this case, we say h has an **applied domain**[7] of $[0, \infty)$

1.2.1 Exercises

1. Suppose f is a function that takes a real number x and performs the following three steps in the order given: (1) square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4. Find an expression for $f(x)$ and find its domain.

2. Suppose g is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) square root; (3) make the quantity the denominator of a fraction with numerator 4. Find an expression for $g(x)$ and find its domain.

3. Suppose h is a function that takes a real number x and performs the following three steps in the order given: (1) square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13. Find an expression for $h(x)$ and find its domain.

4. Suppose k is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) square root; (3) subtract 13. Find an expression for $k(x)$ and find its domain.

5. For $f(x) = x^2 - 3x + 2$, find and simplify the following:

   (a) $f(3)$
   (b) $f(-1)$
   (c) $f\left(\frac{3}{2}\right)$
   (d) $f(4x)$
   (e) $4f(x)$
   (f) $f(-x)$
   (g) $f(x-4)$
   (h) $f(x) - 4$
   (i) $f\left(x^2\right)$

6. Repeat Exercise **??** above for $f(x) = \dfrac{2}{x^3}$

7. Let $f(x) = 3x^2 + 3x - 2$. Find and simplify the following:

[5] See the United States Postal Service website http://www.usps.com/prices/first-class-mail-prices.htm

[6] See the Internal Revenue Service's website http://www.irs.gov/pub/irs-pdf/i1040tt.pdf

[7] or, 'explicit domain'

(a) $f(2)$
(b) $f(-2)$
(c) $f(2a)$
(d) $2f(a)$
(e) $f(a+2)$
(f) $f(a)+f(2)$
(g) $f\left(\frac{2}{a}\right)$
(h) $\frac{f(a)}{2}$
(i) $f(a+h)$

8. Let $f(x) = \begin{cases} x+5, & x \le -3 \\ \sqrt{9-x^2}, & -3 < x \le 3 \\ -x+5, & x > 3 \end{cases}$

(a) $f(-4)$
(b) $f(-3)$
(c) $f(3)$
(d) $f(3.001)$
(e) $f(-3.001)$
(f) $f(2)$

9. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$ Compute the following function values.

(a) $f(4)$
(b) $f(-3)$
(c) $f(1)$
(d) $f(0)$
(e) $f(-1)$
(f) $f(-0.999)$

10. Find the (implied) domain of the function.

(a) $f(x) = x^4 - 13x^3 + 56x^2 - 19$
(b) $f(x) = x^2 + 4$
(c) $f(x) = \dfrac{x+4}{x^2-36}$
(d) $f(x) = \sqrt{6x-2}$
(e) $f(x) = \dfrac{6}{\sqrt{6x-2}}$
(f) $f(x) = \sqrt[3]{6x-2}$
(g) $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$
(h) $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
(i) $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$
(j) $s(t) = \dfrac{t}{t-8}$
(k) $Q(r) = \dfrac{\sqrt{r}}{r-8}$
(l) $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$
(m) $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$
(n) $A(x) = \sqrt{x-7} + \sqrt{9-x}$
(o) $g(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$
(p) $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

11. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \dfrac{150t}{t+15}$, where $t=0$ represents the year 1803. What is the applied domain of P? What range "makes sense" for this function? What does $P(0)$ represent? What does $P(205)$ represent?

12. Recall that the **integers** is the set of numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[8] The **greatest integer of x**, $\lfloor x \rfloor$, is defined to be the largest integer k with $k \leq x$.

    (a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$

    (b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piece-wise defined function.

    HINT: There are infinitely many pieces!

    (c) Is $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if a or b is an integer? Test some values, make a conjecture, and explain your result.

13. We have through our examples tried to convince you that, in general, $f(a + b) \neq f(a) + f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function f for which the following properties are always true.

    (a) $f(0) = f(-1 + 1) = f(-1) + f(1)$

    (b) $f(5) = f(2 + 3) = f(2) + f(3)$

    (c) $f(-6) = f(0 - 6) = f(0) - f(6)$

    (d) $f(a + b) = f(a) + f(b)$ regardless of what two numbers we give you for a and b.

    How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \dfrac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a + b) \neq f(a) + f(b)$.

1.2.2 Answers

1. $f(x) = \dfrac{4}{\sqrt{x} - 13}$
   Domain: $[0, 169) \cup (169, \infty)$

2. $g(x) = \dfrac{4}{\sqrt{x - 13}}$
   Domain: $(13, \infty)$

3. $h(x) = \dfrac{4}{\sqrt{x}} - 13$
   Domain: $(0, \infty)$

4. $k(x) = \sqrt{\dfrac{4}{x} - 13}$
   Domain: $(0, \infty)$

5. (a) 2
   (b) 6
   (c) $-\dfrac{1}{4}$
   (d) $16x^2 - 12x + 2$
   (e) $4x^2 - 12x + 8$
   (f) $x^2 + 3x + 2$
   (g) $x^2 - 11x + 30$
   (h) $x^2 - 3x - 2$
   (i) $x^4 - 3x^2 + 2$

[8] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count.'

6. (a) $\frac{2}{27}$
   (b) -2
   (c) $\frac{16}{27}$
   (d) $\frac{1}{32x^3}$
   (e) $\frac{8}{x^3}$
   (f) $-\frac{2}{x^3}$
   (g) $\frac{2}{(x-4)^3} = \frac{2}{x^3 - 12x^2 + 48x - 64}$
   (h) $\frac{2}{x^3} - 4 = \frac{2 - 4x^3}{x^3}$
   (i) $\frac{2}{x^6}$

7. (a) 16
   (b) 4
   (c) $12a^2 + 6a - 2$
   (d) $6a^2 + 6a - 4$
   (e) $3a^2 + 15a + 14$
   (f) $3a^2 + 3a + 14$
   (g) $\frac{12}{a^2} + \frac{6}{a} - 2$
   (h) $\frac{3a^2}{2} + \frac{3a}{2} - 1$
   (i) $3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

8. (a) $f(-4) = 1$
   (b) $f(-3) = 2$
   (c) $f(3) = 0$
   (d) $f(3.001) = 1.999$
   (e) $f(-3.001) = 1.999$
   (f) $f(2) = \sqrt{5}$

9. (a) $f(4) = 4$
   (b) $f(-3) = 9$
   (c) $f(1) = 0$
   (d) $f(0) = 1$
   (e) $f(-1) = 1$
   (f) $f(-0.999) \approx 0.0447101778$

10. (a) $(-\infty, \infty)$
    (b) $(-\infty, \infty)$
    (c) $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$
    (d) $\left[\frac{1}{3}, \infty\right)$
    (e) $\left(\frac{1}{3}, \infty\right)$
    (f) $(-\infty, \infty)$
    (g) $\left[\frac{1}{3}, 3\right) \cup (3, \infty)$
    (h) $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$
    (i) $(-\infty, \infty)$
    (j) $(-\infty, 8) \cup (8, \infty)$
    (k) $[0, 8) \cup (8, \infty)$
    (l) $(8, \infty)$
    (m) $(-\infty, 8) \cup (8, \infty)$
    (n) $[7, 9]$
    (o) $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
    (p) $[0, 25) \cup (25, \infty)$

11. The applied domain of P is $[0, \infty)$. The range is some subset of the natural numbers because we cannot have fractional Sasquatch. This was a bit of a trick question and we'll address the notion of mathematical modeling more thoroughly in later chapters. $P(0) = 0$ means that there were no Sasquatch in Portage County in 1803. $P(205) \approx 139.77$ would mean there were 139 or 140 Sasquatch in Portage County in 2008.

12. (a) $\lfloor 0.785 \rfloor = 0$, $\lfloor 117 \rfloor = 117$, $\lfloor -2.001 \rfloor = -3$, and $\lfloor \pi + 6 \rfloor = 9$

1.3 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x) + 2$' and '$2f(x)$' for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Function Arithmetic

Suppose f and g are functions and x is an element common to the domains of f and g.

- The **sum** of f and g, denoted $f + g$, is the function defined by the formula:

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of f and g, denoted $f - g$, is the function defined by the formula:

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula:

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

  provided $g(x) \neq 0$.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on.

Example 1.3.1. Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$. Find and simplify expressions for for the following functions. In addition, find the domain of each of these functions.

1. $(f + g)(x)$
2. $(g - f)(x)$
3. $(fg)(x)$
4. $\left(\dfrac{g}{f}\right)(x)$

SOLUTION.

1. $(f+g)(x)$ is defined to be $f(x)+g(x)$. To that end, we get

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
&= \left(6x^2-2x\right)+\left(3-\frac{1}{x}\right)\\
&= 6x^2-2x+3-\frac{1}{x}\\
&= \frac{6x^3}{x}-\frac{2x^2}{x}+\frac{3x}{x}-\frac{1}{x} \qquad \text{get common denominators}\\
&= \frac{6x^3-2x^2+3x-1}{x}
\end{aligned}
$$

To find the domain of $(f+g)$ we do so **before** we simplify, that is, at the step

$$\left(6x^2-2x\right)+\left(3-\frac{1}{x}\right)$$

We see $x \neq 0$, but everything else is fine. Hence, the domain is $(-\infty,0)\cup(0,\infty)$.

2. $(g-f)(x)$ is defined to be $g(x)-f(x)$. To that end, we get

$$
\begin{aligned}
(g-f)(x) &= g(x)-f(x)\\
&= \left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)\\
&= 3-\frac{1}{x}-6x^2+2x\\
&= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x} \qquad \text{get common denominators}\\
&= \frac{-6x^3+2x^2+3x-1}{x}
\end{aligned}
$$

Looking at the expression for $(g-f)$ before we simplified

$$\left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)$$

we see, as before, $x \neq 0$ is the only restriction. The domain is $(-\infty,0)\cup(0,\infty)$.

3. $(fg)(x)$ is defined to be $f(x)g(x)$. Substituting yields

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2-2x\right)\left(3-\frac{1}{x}\right) \\
&= \left(6x^2-2x\right)\left(\frac{3x-1}{x}\right) \\
&= \left(\frac{2x(3x-1)}{1}\right)\left(\frac{3x-1}{x}\right) && \text{factor} \\
&= \left(\frac{2\cancel{x}(3x-1)}{1}\right)\left(\frac{3x-1}{\cancel{x}}\right) && \text{cancel} \\
&= 2(3x-1)^2 \\
&= 2\left(9x^2-6x+1\right) \\
&= 18x^2-12x+2
\end{aligned}$$

To determine the domain, we check the step just after we substituted

$$\left(6x^2-2x\right)\left(3-\frac{1}{x}\right)$$

which gives us, as before, the domain: $(-\infty,0)\cup(0,\infty)$.

4. $\left(\frac{g}{f}\right)(x)$ is defined to be $\frac{g(x)}{f(x)}$. Thus we have

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x} \\
&= \frac{3-\frac{1}{x}}{6x^2-2x}\cdot\frac{x}{x} && \text{simplify complex fractions} \\
&= \frac{\left(3-\frac{1}{x}\right)x}{\left(6x^2-2x\right)x} \\
&= \frac{3x-1}{\left(6x^2-2x\right)x}
\end{aligned}$$

$$\begin{aligned} &= \frac{3x-1}{2x^2(3x-1)} && \text{factor} \\ &= \frac{\cancel{(3x-1)}^{1}}{2x^2\cancel{(3x-1)}} && \text{cancel} \\ &= \frac{1}{2x^2} \end{aligned}$$

To find the domain, we consider the first step after substitution:

$$\frac{3-\dfrac{1}{x}}{6x^2-2x}$$

To avoid division by zero in the 'little' fraction, $\frac{1}{x}$, we need $x \neq 0$. For the 'big' fraction we set $6x^2 - 2x = 0$ and solve: $2x(3x-1) = 0$ and get $x = 0, \frac{1}{3}$. Thus we must exclude $x = \frac{1}{3}$ as well, resulting in a domain of $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$. $\square$

We close this section with concept of the **difference quotient** of a function. It is a critical tool for Calculus and also a great way to practice function notation.[1]

Definition 1.3. Given a function, f, the **difference quotient** of f is the expression:

$$\frac{f(x+h) - f(x)}{h}$$

Example 1.3.2. Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$

Solution.

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

[1] You may need to brush up on your Intermediate Algebra skills, as well.

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\
&= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\
&= \frac{2xh+h^2-h}{h} \\
&= \frac{h\,(2x+h-1)}{h} && \text{factor} \\
&= \frac{\cancel{h}\,(2x+h-1)}{\cancel{h}} && \text{cancel} \\
&= 2x+h-1.
\end{aligned}
$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \dfrac{3}{2x+1}$ with the quantity $(x+h)$

$$
\begin{aligned}
g(x+h) &= \frac{3}{2(x+h)+1} \\
&= \frac{3}{2x+2h+1},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\frac{g(x+h)-g(x)}{h} &= \frac{\dfrac{3}{2x+2h+1}-\dfrac{3}{2x+1}}{h} \\
&= \frac{\dfrac{3}{2x+2h+1}-\dfrac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\
&= \frac{3(2x+1)-3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\
&= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \\
&= \frac{-6}{(2x+2h+1)(2x+1)}.
\end{aligned}
$$

For reasons which will become clear in Calculus, we do not expand the denominator. $\square$

1.3.1 Exercises

1. Let $f(x) = \sqrt{x}$, $g(x) = x + 10$ and $h(x) = \dfrac{1}{x}$.

   (a) Compute the following function values.

   i. $(f+g)(4)$ ii. $(g-h)(7)$ iii. $(fh)(25)$ iv. $\left(\dfrac{h}{g}\right)(3)$

   (b) Find the domain of the following functions then simplify their expressions.

   i. $(f+g)(x)$ ii. $(g-h)(x)$ iii. $(fh)(x)$ iv. $\left(\dfrac{h}{g}\right)(x)$ v. $\left(\dfrac{g}{h}\right)(x)$ vi. $(h-f)(x)$

2. Let $f(x) = 3\sqrt{x} - 1$, $g(x) = 2x^2 - 3x - 2$ and $h(x) = \dfrac{3}{2-x}$.

   (a) Compute the following function values.

   i. $(f+g)(4)$ ii. $(g-h)(1)$ iii. $(fh)(0)$ iv. $\left(\dfrac{h}{g}\right)(-1)$

   (b) Find the domain of the following functions then simplify their expressions.

   i. $(f-g)(x)$ ii. $(gh)(x)$ iii. $\left(\dfrac{f}{g}\right)(x)$ iv. $\left(\dfrac{f}{h}\right)(x)$

3. Let $f(x) = \sqrt{6x-2}$, $g(x) = x^2 - 36$, and $h(x) = \dfrac{1}{x-4}$.

   (a) Compute the following function values.

   i. $(f+g)(3)$ ii. $(g-h)(8)$ iii. $\left(\dfrac{f}{g}\right)(4)$ iv. $(fh)(8)$ v. $(g+h)(-4)$ vi. $\left(\dfrac{h}{g}\right)(-12)$

   (b) Find the domain of the following functions and simplify their expressions.

   i. $(f+g)(x)$ ii. $(g-h)(x)$ iii. $\left(\dfrac{f}{g}\right)(x)$ iv. $(fh)(x)$ v. $(g+h)(x)$ vi. $\left(\dfrac{h}{g}\right)(x)$

4. Find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the following functions.

(a) $f(x) = 2x - 5$
(b) $f(x) = -3x + 5$
(c) $f(x) = 6$
(d) $f(x) = 3x^2 - x$
(e) $f(x) = -x^2 + 2x - 1$
(f) $f(x) = x^3 + 1$
(g) $f(x) = \dfrac{2}{x}$
(h) $f(x) = \dfrac{3}{1-x}$
(i) $f(x) = \dfrac{x}{x-9}$
(j) $f(x) = \sqrt{x}$ [2]
(k) $f(x) = mx + b$ where $m \neq 0$
(l) $f(x) = ax^2 + bx + c$ where $a \neq 0$

1.3.2 Answers

1. (a) i. $(f+g)(4) = 16$ ii. $(g-h)(7) = \dfrac{118}{7}$ iii. $(fh)(25) = \dfrac{1}{5}$ iv. $\left(\dfrac{h}{g}\right)(3) = \dfrac{1}{39}$

   (b) i. $(f+g)(x) = \sqrt{x} + x + 10$
   Domain: $[0, \infty)$
   ii. $(g-h)(x) = x + 10 - \dfrac{1}{x}$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   iii. $(fh)(x) = \dfrac{1}{\sqrt{x}}$
   Domain: $(0, \infty)$
   iv. $\left(\dfrac{h}{g}\right)(x) = \dfrac{1}{x(x+10)}$
   Domain: $(-\infty, -10) \cup (-10, 0) \cup (0, \infty)$
   v. $\left(\dfrac{g}{h}\right)(x) = x(x+10)$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   vi. $(h-f)(x) = \dfrac{1}{x} - \sqrt{x}$
   Domain: $(0, \infty)$

2. (a) i. $(f+g)(4) = 23$ ii. $(g-h)(1) = -6$ iii. $(fh)(0) = -\dfrac{3}{2}$ iv. $\left(\dfrac{h}{g}\right)(-1) = \dfrac{1}{3}$

   (b) i. $(f-g)(x) = -2x^2 + 3x + 3\sqrt{x} + 1$
   Domain: $[0, \infty)$
   ii. $(gh)(x) = -6x - 3$
   Domain: $(-\infty, 2) \cup (2, \infty)$
   iii. $\left(\dfrac{f}{g}\right)(x) = \dfrac{3\sqrt{x} - 1}{2x^2 - 3x - 2}$
   Domain: $[0, 2) \cup (2, \infty)$
   iv. $\left(\dfrac{f}{h}\right)(x) = -x\sqrt{x} + \frac{1}{3}x + 2\sqrt{x} - \frac{2}{3}$
   Domain: $[0, 2) \cup (2, \infty)$

3. (a) i. $(f+g)(3) = -23$
   ii. $(g-h)(8) = \dfrac{111}{4}$
   iii. $\left(\dfrac{f}{g}\right)(4) = -\dfrac{\sqrt{22}}{20}$
   iv. $(fh)(8) = \dfrac{\sqrt{46}}{4}$
   v. $(g+h)(-4) = -\dfrac{161}{8}$
   vi. $\left(\dfrac{h}{g}\right)(-12) = -\dfrac{1}{1728}$

   (b)

[2] Rationalize the numerator. It won't look 'simplified' per se, but work through until you can cancel the 'h'.

i. $(f+g)(x) = x^2 - 36 + \sqrt{6x-2}$
Domain: $\left[\frac{1}{3}, \infty\right)$

ii. $(g-h)(x) = x^2 - 36 - \dfrac{1}{x-4}$
Domain: $(-\infty, 4) \cup (4, \infty)$

iii. $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$
Domain: $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

iv. $(fh)(x) = \dfrac{\sqrt{6x-2}}{x-4}$
Domain: $\left[\frac{1}{3}, 4\right) \cup (4, \infty)$

v. $(g+h)(x) = x^2 - 36 + \dfrac{1}{x-4}$
Domain: $(-\infty, 4) \cup (4, \infty)$

vi. $\left(\dfrac{h}{g}\right)(x) = \dfrac{1}{(x-4)(x^2-36)}$
Domain:
$(-\infty, -6) \cup (-6, 4) \cup (4, 6) \cup (6, \infty)$

4. (a) 2
(b) -3
(c) 0
(d) $6x + 3h - 1$
(e) $-2x - h + 2$
(f) $3x^2 + 3xh + h^2$
(g) $-\dfrac{2}{x(x+h)}$
(h) $\dfrac{3}{(1-x-h)(1-x)}$
(i) $\dfrac{-9}{(x-9)(x+h-9)}$
(j) $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$
(k) m
(l) $2ax + ah + b$

1.4 Graphs of Functions

In Section **??** we defined a function as a special type of relation; one in which each x-coordinate was matched with only one y-coordinate. We spent most of our time in that section looking at functions graphically because they were, after all, just sets of points in the plane. Then in Section **??** we described a function as a process and defined the notation necessary to work with functions algebraically. So now it's time to look at functions graphically again, only this time we'll do so with the notation defined in Section **??**. We start with what should not be a surprising connection.

The Fundamental Graphing Principle for Functions

The graph of a function f is the set of points which satisfy the equation $y = f(x)$. That is, the point (x, y) is on the graph of f if and only if $y = f(x)$.

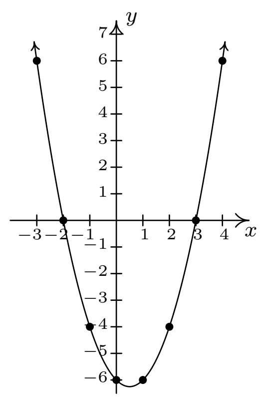
Example 1.4.1. Graph $f(x) = x^2 - x - 6$.

Solution. To graph f, we graph the equation $y = f(x)$. To this end, we use the techniques outlined in Section **??**. Specifically, we check for intercepts, test for symmetry, and plot additional points as needed. To find the x-intercepts, we set $y = 0$. Since $y = f(x)$, this means $f(x) = 0$.

$$\begin{array}{rcll} f(x) & = & x^2 - x - 6 & \\ 0 & = & x^2 - x - 6 & \\ 0 & = & (x-3)(x+2) & \text{factor} \\ x - 3 = 0 & \text{or} & x + 2 = 0 & \\ x & = & -2, 3 & \end{array}$$

So we get $(-2, 0)$ and $(3, 0)$ as x-intercepts. To find the y-intercept, we set $x = 0$. Using function notation, this is the same as finding $f(0)$ and $f(0) = 0^2 - 0 - 6 = -6$. Thus the y-intercept is $(0, -6)$. As far as symmetry is concerned, we can tell from the intercepts that the graph possesses none of the three symmetries discussed thus far. (You should verify this.) We can make a table analogous to the ones we made in Section **??**, plot the points and connect the dots in a somewhat pleasing fashion to get the graph below on the right.

x	$f(x)$	$(x, f(x))$
-3	6	$(-3, 6)$
-2	0	$(-2, 0)$
-1	-4	$(-1, -4)$
0	-6	$(0, -6)$
1	-6	$(1, -6)$
2	-4	$(2, -4)$
3	0	$(3, 0)$
4	6	$(4, 6)$



□

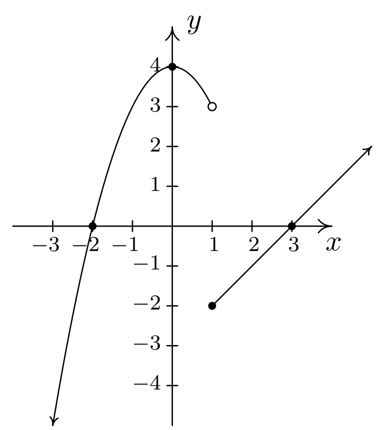
Graphing piecewise-defined functions is a bit more of a challenge.

Example 1.4.2. Graph: $f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x < 1 \\ x - 3, & \text{if} \quad x \geq 1 \end{cases}$

Solution. We proceed as before: finding intercepts, testing for symmetry and then plotting additional points as needed. To find the x-intercepts, as before, we set $f(x) = 0$. The twist is that we have two formulas for $f(x)$. For $x < 1$, we use the formula $f(x) = 4 - x^2$. Setting $f(x) = 0$ gives $0 = 4 - x^2$, so that $x = \pm 2$. However, of these two answers, only $x = -2$ fits in the domain $x < 1$ for this piece. This means the only x-intercept for the $x < 1$ region of the x-axis is $(-2, 0)$. For $x \geq 1$, $f(x) = x - 3$. Setting $f(x) = 0$ gives $0 = x - 3$, or $x = 3$. Since $x = 3$ satisfies the inequality $x \geq 1$, we get $(3, 0)$ as another x-intercept. Next, we seek the y-intercept. Notice that $x = 0$ falls in the domain $x < 1$. Thus $f(0) = 4 - 0^2 = 4$ yields the y-intercept $(0, 4)$. As far as symmetry is concerned, you can check that the equation $y = 4 - x^2$ is symmetric about the y-axis; unfortunately, this equation (and its symmetry) is valid only for $x < 1$. You can also verify

$y = x - 3$ possesses none of the symmetries discussed in the Section **??**. When plotting additional points, it is important to keep in mind the restrictions on x for each piece of the function. The sticking point for this function is $x = 1$, since this is where the equations change. When $x = 1$, we use the formula $f(x) = x - 3$, so the point on the graph $(1, f(1))$ is $(1, -2)$. However, for all values less than 1, we use the formula $f(x) = 4 - x^2$. As we have discussed earlier in Section **??**, there is no real number which immediately precedes $x = 1$ on the number line. Thus for the values $x = 0.9$, $x = 0.99$, $x = 0.999$, and so on, we find the corresponding y values using the formula $f(x) = 4 - x^2$. Making a table as before, we see that as the x values sneak up to $x = 1$ in this fashion, the $f(x)$ values inch closer and closer[1] to $4 - 1^2 = 3$. To indicate this graphically, we use an open circle at the point $(1, 3)$. Putting all of this information together and plotting additional points, we get

x	$f(x)$	$(x, f(x))$
0.9	3.19	$(0.9, 3.19)$
0.99	≈ 3.02	$(0.99, 3.02)$
0.999	≈ 3.002	$(0.999, 3.002)$



□

In the previous two examples, the x-coordinates of the x-intercepts of the graph of $y = f(x)$ were found by solving $f(x) = 0$. For this reason, they are called the **zeros** of f.

DEFINITION 1.4. The **zeros** of a function f are the solutions to the equation $f(x) = 0$. In other words, x is a zero of f if and only if $(x, 0)$ is an x-intercept of the graph of $y = f(x)$.

Of the three symmetries discussed in Section **??**, only two are of significance to functions: symmetry about the y-axis and symmetry about the origin.[2] Recall that we can test whether the graph of an equation is symmetric about the y-axis by replacing x with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for x results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. In a similar fashion, we recall that to test an equation's graph for symmetry about the origin, we replace x and y with $-x$ and $-y$, respectively. Doing

[1] We've just stepped into Calculus here!

[2] Why are we so dismissive about symmetry about the x-axis for graphs of functions?

this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for y gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. These results are summarized below.

Steps for testing if the graph of a function possesses symmetry

The graph of a function f is symmetric:

- About the y-axis if and only if $f(-x) = f(x)$ for all x in the domain of f.
- About the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

For reasons which won't become clear until we study polynomials, we call a function **even** if its graph is symmetric about the y-axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd,[3] functions fall into one of three distinct categories: even, odd, or neither even nor odd.

Example 1.4.3. Analytically determine if the following functions are even, odd, or neither even nor odd. Verify your result with a graphing calculator.

1. $f(x) = \dfrac{5}{2-x^2}$
2. $g(x) = \dfrac{5x}{2-x^2}$
3. $h(x) = \dfrac{5x}{2-x^3}$
4. $i(x) = \dfrac{5x}{2x-x^3}$
5. $j(x) = x^2 - \dfrac{x}{100} - 1$

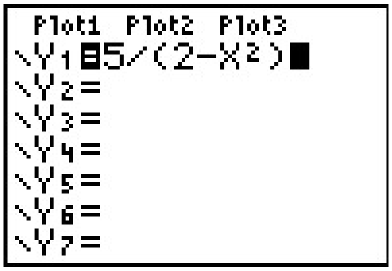
Solution. The first step in all of these problems is to replace x with $-x$ and simplify.

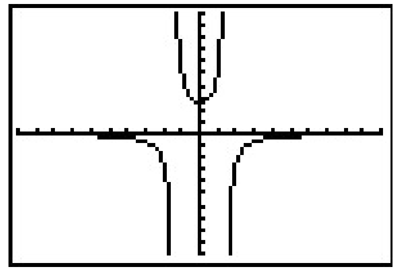
1.

$$\begin{aligned} f(x) &= \frac{5}{2-x^2} \\ f(-x) &= \frac{5}{2-(-x)^2} \\ f(-x) &= \frac{5}{2-x^2} \\ f(-x) &= f(x) \end{aligned}$$

Hence, f is **even**. The graphing calculator furnishes the following:

[3] Any ideas?





This suggests[4] the graph of f is symmetric about the y-axis, as expected.

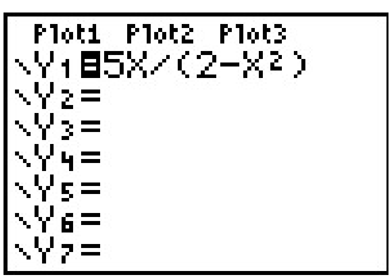
2.

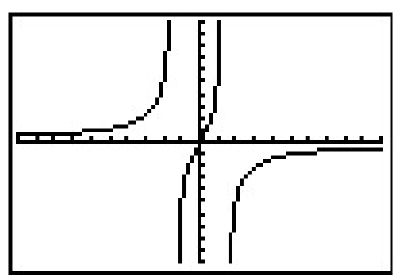
$$\begin{aligned} g(x) &= \frac{5x}{2-x^2} \\ g(-x) &= \frac{5(-x)}{2-(-x)^2} \\ g(-x) &= \frac{-5x}{2-x^2} \end{aligned}$$

It doesn't appear that $g(-x)$ is equivalent to $g(x)$. To prove this, we check with an x value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that g is not even, but it doesn't rule out the possibility that g is odd. (Why not?) To check if g is odd, we compare $g(-x)$ with $-g(x)$

$$\begin{aligned} -g(x) &= -\frac{5x}{2-x^2} \\ &= \frac{-5x}{2-x^2} \\ -g(x) &= g(-x) \end{aligned}$$

Hence, g is odd. Graphically,





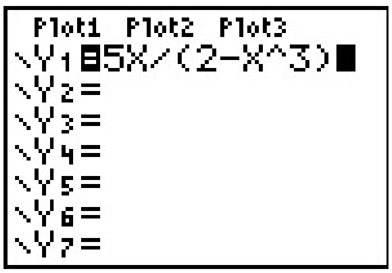
The calculator indicates the graph of g is symmetric about the origin, as expected.

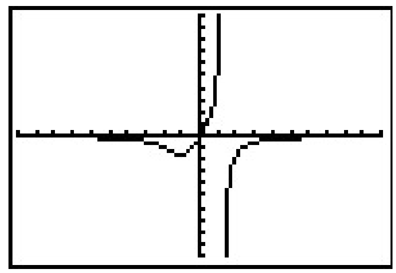
[4] 'Suggests' is about the extent of what a graphing calculator can do.

3.

$$
\begin{aligned}
h(x) &= \frac{5x}{2-x^3} \\
h(-x) &= \frac{5(-x)}{2-(-x)^3} \\
h(-x) &= \frac{-5x}{2+x^3}
\end{aligned}
$$

Once again, $h(-x)$ doesn't appear to be equivalent to $h(x)$. We check with an x value, for example, $h(1)=5$ but $h(-1)=-\frac{5}{3}$. This proves that h is not even and it also shows h is not odd. (Why?) Graphically,





The graph of h appears to be neither symmetric about the y-axis nor the origin.

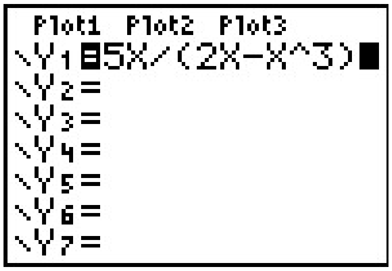
4.

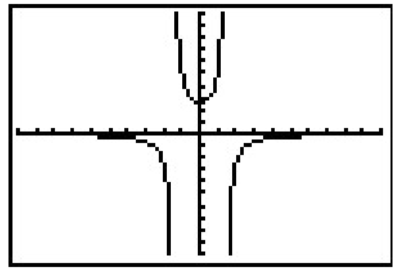
$$
\begin{aligned}
i(x) &= \frac{5x}{2x-x^3} \\
i(-x) &= \frac{5(-x)}{2(-x)-(-x)^3} \\
i(-x) &= \frac{-5x}{-2x+x^3}
\end{aligned}
$$

The expression $i(-x)$ doesn't appear to be equivalent to $i(x)$. However, after checking some x values, for example $x=1$ yields $i(1)=5$ and $i(-1)=5$, it appears that $i(-x)$ does, in fact, equal $i(x)$. However, while this suggests i is even, it doesn't **prove** it. (It does, however, prove i is not odd.) To prove $i(-x)=i(x)$, we need to manipulate our expressions for $i(x)$ and $i(-x)$ and show they are equivalent. A clue as to how to proceed is in the numerators: in the formula for $i(x)$, the numerator is $5x$ and in $i(-x)$ the numerator is $-5x$. To re-write $i(x)$ with a numerator of $-5x$, we need to multiply its numerator by -1. To keep the value of the fraction the same, we need to multiply the denominator by -1 as well. Thus

$$
\begin{aligned}
i(x) &= \frac{5x}{2x-x^3} \\
&= \frac{(-1)5x}{(-1)\left(2x-x^3\right)} \\
&= \frac{-5x}{-2x+x^3}
\end{aligned}
$$

Hence, $i(x) = i(-x)$, so i is even. The calculator supports our conclusion.





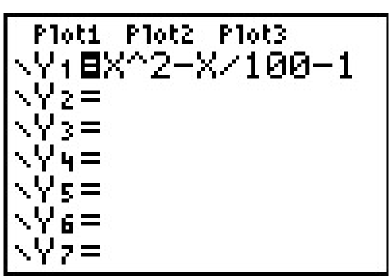
5.

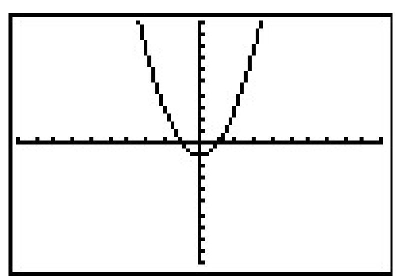
$$
\begin{aligned}
j(x) &= x^2 - \frac{x}{100} - 1 \\
j(-x) &= (-x)^2 - \frac{-x}{100} - 1 \\
j(-x) &= x^2 + \frac{x}{100} - 1
\end{aligned}
$$

The expression for $j(-x)$ doesn't seem to be equivalent to $j(x)$, so we check using $x = 1$ to get $j(1) = -\frac{1}{100}$ and $j(-1) = \frac{1}{100}$. This rules out j being even. However, it doesn't rule out j being odd. Examining $-j(x)$ gives

$$
\begin{aligned}
j(x) &= x^2 - \frac{x}{100} - 1 \\
-j(x) &= -\left(x^2 - \frac{x}{100} - 1\right) \\
-j(x) &= -x^2 + \frac{x}{100} + 1
\end{aligned}
$$

The expression $-j(x)$ doesn't seem to match $j(-x)$ either. Testing $x = 2$ gives $j(2) = \frac{149}{50}$ and $j(-2) = \frac{151}{50}$, so j is not odd, either. The calculator gives:





The calculator suggests that the graph of j is symmetric about the y-axis which would imply that j is even. However, we have proven that is not the case. □

There are two lessons to be learned from the last example. The first is that sampling function values at particular x values is not enough to prove that a function is even or odd – despite the fact that $j(-1) = -j(1)$, j turned out not to be odd. Secondly, while the calculator may **suggest** mathematical truths, it is the algebra which **proves** mathematical truths.[5]

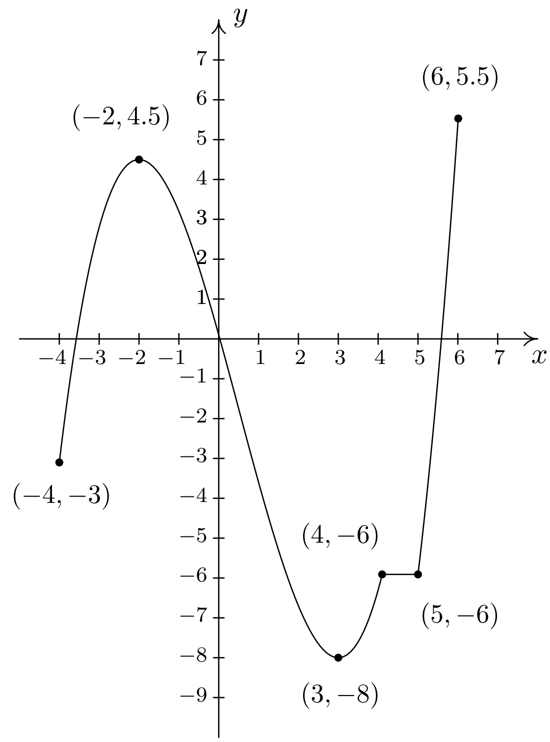
1.4.1 General Function Behavior

The last topic we wish to address in this section is general function behavior. As you shall see in the next several chapters, each family of functions has its own unique attributes and we will study them all in great detail. The purpose of this section's discussion, then, is to lay the foundation for that further study by investigating aspects of function behavior which apply to all functions. To start, we will examine the concepts of **increasing**, **decreasing**, and **constant**. Before defining the concepts algebraically, it is instructive to first look at them graphically. Consider the graph of the function f given on the next page.

Reading from left to right, the graph 'starts' at the point $(-4, -3)$ and 'ends' at the point $(6, 5.5)$. If we imagine walking from left to right on the graph, between $(-4, -3)$ and $(-2, 4.5)$, we are walking 'uphill'; then between $(-2, 4.5)$ and $(3, -8)$, we are walking 'downhill'; and between $(3, -8)$ and $(4, -6)$, we are walking 'uphill' once more. From $(4, -6)$ to $(5, -6)$, we 'level off', and then resume walking 'uphill' from $(5, -6)$ to $(6, 5.5)$. In other words, for the x values between -4 and -2 (inclusive), the y-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y = f(x)$, the y values on the graph are the function values, and we say that the function f is **increasing** on the interval $[-4, -2]$. Analogously, we say that f is **decreasing** on the interval $[-2, 3]$ increasing once more on the interval $[3, 4]$, **constant** on $[4, 5]$, and finally increasing once again on $[5, 6]$. It is extremely important to notice that the behavior (increasing, decreasing or constant) occurs on an interval on the x-axis. When we say that the function f is increasing on $[-4, -2]$ we do not mention the actual y values that f attains along the way. Thus, we report **where** the behavior occurs, not to what extent the behavior occurs.[6] Also notice that we do not say that a function is increasing, decreasing or constant at a single x value. In fact, we would run into serious trouble in our previous example if we tried to do so because $x = -2$ is contained in an interval on which f was increasing and one on which it is decreasing. (There's more on this issue and many others in the exercises.)

[5] Or, in other words, don't rely too heavily on the machine!

[6] The notions of how quickly or how slowly a function increases or decreases are explored in Calculus.



The graph of $y = f(x)$

We're now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

Definition 1.5. Suppose f is a function defined on an interval I. We say f is:

- **increasing** on I if and only if $f(a) < f(b)$ for all real numbers a, b in I with $a < b$.
- **decreasing** on I if and only if $f(a) > f(b)$ for all real numbers a, b in I with $a < b$.
- **constant** on I if and only if $f(a) = f(b)$ for all real numbers a, b in I.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing, and constant as stated in Definition **??** agree with our graphical descriptions given earlier. You should look back through the examples and exercise sets in previous sections where graphs were given to see if you can determine the intervals on which the functions are increasing, decreasing or constant. Can you find an example of a function for which none of the concepts in Definition **??** apply?

Now let's turn our attention to a few of the points on the graph. Clearly the point $(-2, 4.5)$ does not have the largest y value of all of the points on the graph of f – indeed that honor goes to $(6, 5.5)$ − but $(-2, 4.5)$ should get some sort of consolation prize for being 'the top of the hill' between $x = -4$ and $x = 3$. We say that the function f has a **local maximum**[7] at the point $(-2, 4.5)$, because the y-coordinate 4.5 is the largest y-value (hence, function value) on the curve 'near'[8] $x = -2$. Similarly, we say that the function f has a **local minimum**[9] at the point $(3, -8)$, since the y-coordinate -8 is the smallest function value near $x = 3$. Although it is tempting to say that local extrema[10] occur when the function changes from increasing to decreasing or vice versa, it is not a precise enough way to define the concepts for the needs of Calculus. At the risk of being pedantic, we will present the traditional definitions and thoroughly vet the pathologies they induce in the exercises. We have one last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph, we see the largest y value (hence the largest function value) is 5.5 at $x = 6$. In this case, we say the **maximum**[11] of f is 5.5; similarly, the **minimum**[12] of f is -8. We formalize these concepts in the following definitions.

Definition 1.6. Suppose f is a function with $f(a) = b$.

- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I different than a. The value $f(a) = b$ is called 'a local maximum value of f' in this case.
- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I different than a. The value $f(a) = b$ is called 'a local minimum value of f' in this case.
- The value b is called the **maximum** of f if $b \geq f(x)$ for all x in the domain of f.
- The value b is called the **minimum** of f if $b \leq f(x)$ for all x in the domain of f.

It's important to note that not every function will have all of these features. Indeed, it is possible to have a function with no local or absolute extrema at all! (Any ideas of what such a function's graph would have to look like?) We shall see in the exercises examples of functions which have one or two, but not all, of these features, some that have instances of each type of extremum and some functions that seem to defy common sense. In all cases, though, we shall adhere to the algebraic definitions above as we explore the wonderful diversity of graphs that functions provide to us.

[7] Also called 'relative maximum'.

[8] We will make this more precise in a moment.

[9] Also called a 'relative minimum'.

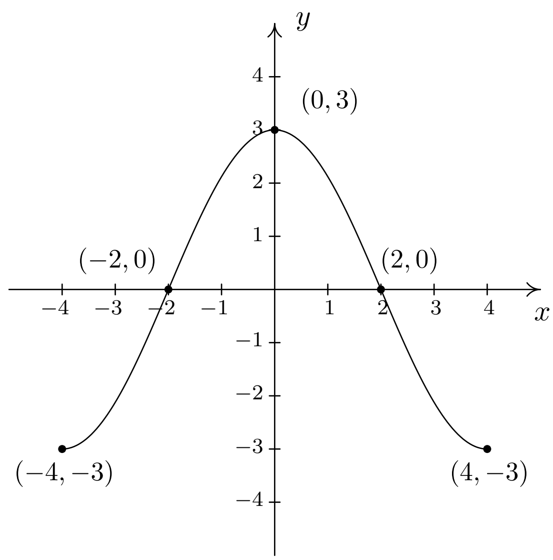
[10] 'Maxima' is the plural of 'maximum' and 'mimima' is the plural of 'minimum'. 'Extrema' is the plural of 'extremum' which combines maximum and minimum.

[11] Sometimes called the 'absolute' or 'global' maximum.

[12] Again, 'absolute' or 'global' minimum can be used.

Here is the 'tame' example which was promised earlier. It summarizes all of the concepts presented in this section as well as some from previous sections so you should spend some time thinking deeply about it before proceeding to the exercises.

EXAMPLE 1.4.4. Given the graph of $y = f(x)$ below, answer all of the following questions.



1. Find the domain of f.
2. Find the range of f.
3. Determine $f(2)$.
4. List the x-intercepts, if any exist.
5. List the y-intercepts, if any exist.
6. Find the zeros of f.
7. Solve $f(x) < 0$.
8. Determine the number of solutions to the equation $f(x) = 1$.
9. List the intervals on which f is increasing.
10. List the intervals on which f is decreasing.
11. List the local maximums, if any exist.
12. List the local minimums, if any exist.
13. Find the maximum, if it exists.
14. Find the minimum, if it exists.
15. Does f appear to be even, odd, or neither?

SOLUTION.

1. To find the domain of f, we proceed as in Section **??**. By projecting the graph to the x-axis, we see the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

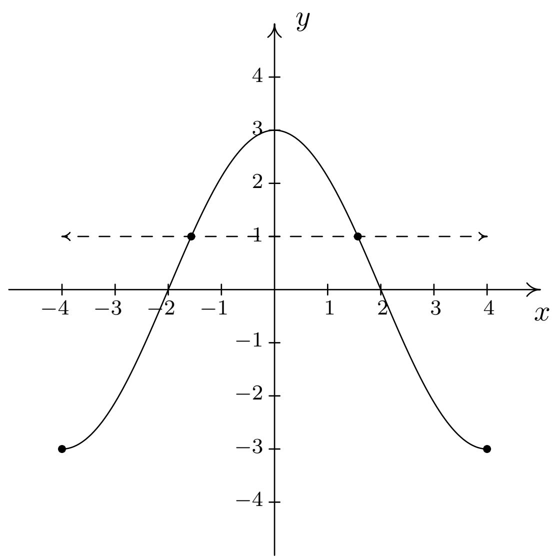
4. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

5. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

6. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

7. To solve $f(x) < 0$, we look for the x values of the points on the graph where the y-coordinate is less than 0. Graphically, we are looking where the graph is **below** the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

8. To find where $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. Even though these points aren't specified, we see that the curve has two points with a y value of 1, as seen in the graph below. That means there are two solutions to $f(x) = 1$.



9. As we move from left to right, the graph rises from $(-4, -3)$ to $(0, 3)$. This means f is increasing on the interval $[-4, 0]$. (Remember, the answer here is an interval on the x-axis.)

10. As we move from left to right, the graph falls from $(0, 3)$ to $(4, -3)$. This means f is decreasing on the interval $[0, 4]$. (Remember, the answer here is an interval on the x-axis.)

11. The function has its only local maximum at $(0, 3)$.

12. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval I which contains $x = -4$ such that $f(-4) < f(x)$ for all x in I different

from -4. But if we put an open interval around $x = -4$ a portion of that interval will lie outside of the domain of f. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason – no open interval around $x = 4$ stays within the domain of f.

13. The maximum value of f is the largest y-coordinate which is 3.

14. The minimum value of f is the smallest y-coordinate which is -3.

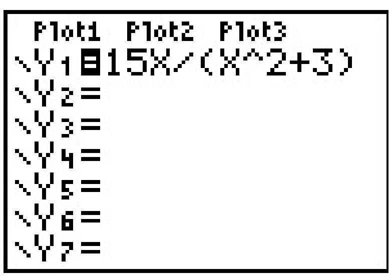
15. The graph appears to be symmetric about the y-axis. This suggests[13] that f is even.

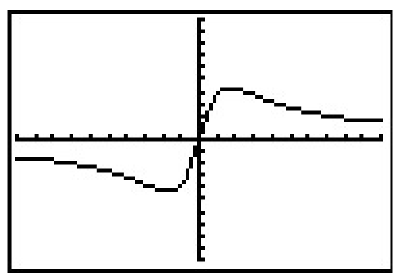
$\square$.

With few exceptions, we will not develop techniques in College Algebra which allow us to determine the intervals on which a function is increasing, decreasing or constant or to find the local maximums and local minimums analytically; this is the business of Calculus.[14] When we have need to find such beasts, we will resort to the calculator. Most graphing calculators have 'Minimum' and 'Maximum' features which can be used to approximate these values, as demonstrated below.

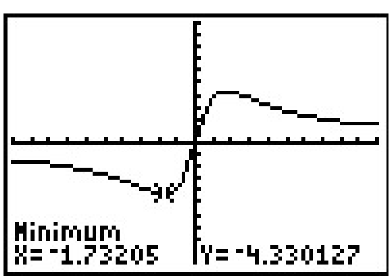
Example 1.4.5. Let $f(x) = \dfrac{15x}{x^2+3}$. Use a graphing calculator to approximate the intervals on which f is increasing and those on which it is decreasing. Approximate all extrema.

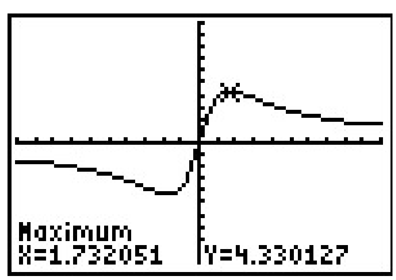
Solution. Entering this function into the calculator gives





Using the Minimum and Maximum features, we get





[13]but does not prove

[14]Although, truth be told, there is only one step of Calculus involved, followed by several pages of algebra.

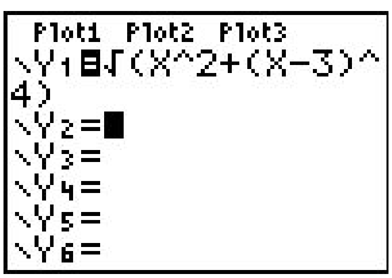
To two decimal places, f appears to have its only local minimum at $(-1.73, -4.33)$ and its only local maximum at $(1, 73, 4.33)$. Given the symmetry about the origin suggested by the graph, the relation between these points shouldn't be too surprising. The function appears to be increasing on $[-1.73, 1.73]$ and decreasing on $(-\infty, -1.73] \cup [1.73, \infty)$. This makes -4.33 the (absolute) minimum and 4.33 the (absolute) maximum. $\square$

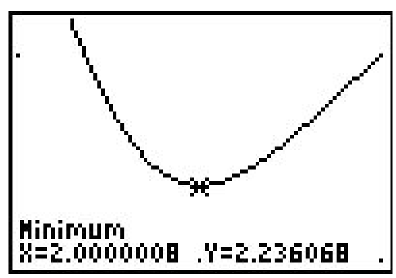
Example 1.4.6. Find the points on the graph of $y = (x-3)^2$ which are closest to the origin. Round your answers to two decimal places.

Solution. Suppose a point (x, y) is on the graph of $y = (x-3)^2$. Its distance to the origin, $(0, 0)$, is given by

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2} \\ &= \sqrt{x^2 + [(x-3)^2]^2} && \text{Since } y = (x-3)^2 \\ &= \sqrt{x^2 + (x-3)^4} \end{aligned}$$

Given a value for x, the formula $d = \sqrt{x^2 + (x-3)^4}$ is the distance from $(0, 0)$ to the point (x, y) on the curve $y = (x-3)^2$. What we have defined, then, is a function $d(x)$ which we wish to minimize over all values of x. To accomplish this task analytically would require Calculus so as we've mentioned before, we can use a graphing calculator to find an approximate solution. Using the calculator, we enter the function $d(x)$ as shown below and graph.





Using the Minimum feature, we see above on the right that the (absolute) minimum occurs near $x = 2$. Rounding to two decimal places, we get that the minimum distance occurs when $x = 2.00$. To find the y value on the parabola associated with $x = 2.00$, we substitute 2.00 into the equation to get $y = (x-3)^2 = (2.00-3)^2 = 1.00$. So, our final answer is $(2.00, 1.00)$.[15] (What does the y value listed on the calculator screen mean in this problem?) $\square$

[15] It seems silly to list a final answer as $(2.00, 1.00)$. Indeed, Calculus confirms that the **exact** answer to this problem is, in fact, $(2, 1)$. As you are well aware by now, the author is a pedant, and as such, uses the decimal places to remind the reader that **any** result garnered from a calculator in this fashion is an approximation, and should be treated as such.

1.4.2 Exercises

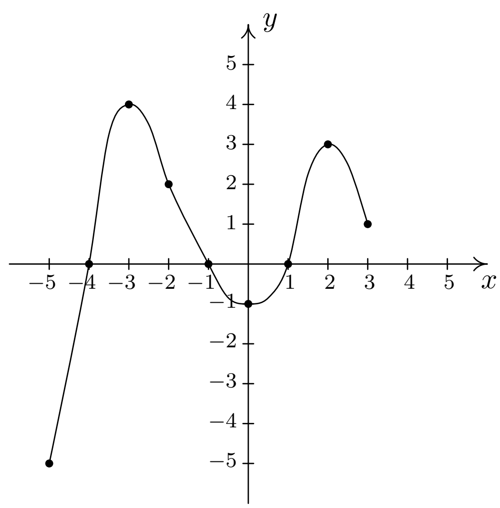
1. Sketch the graphs of the following functions. State the domain of the function, identify any intercepts and test for symmetry.

   (a) $f(x) = \dfrac{x-2}{3}$ (b) $f(x) = \sqrt{5-x}$ (c) $f(x) = \sqrt[3]{x}$ (d) $f(x) = \dfrac{1}{x^2+1}$

2. Analytically determine if the following functions are even, odd or neither.

   (a) $f(x) = 7x$
   (b) $f(x) = 7x + 2$
   (c) $f(x) = \dfrac{1}{x^3}$
   (d) $f(x) = 4$
   (e) $f(x) = 0$
   (f) $f(x) = x^6 - x^4 + x^2 + 9$
   (g) $f(x) = -x^5 - x^3 + x$
   (h) $f(x) = x^4+x^3+x^2+x+1$
   (i) $f(x) = \sqrt{5-x}$
   (j) $f(x) = x^2 - x - 6$

3. Given the graph of $y = f(x)$ below, answer all of the following questions.



   (a) Find the domain of f.
   (b) Find the range of f.
   (c) Determine $f(-2)$.
   (d) List the x-intercepts, if any exist.
   (e) List the y-intercepts, if any exist.
   (f) Find the zeros of f.
   (g) Solve $f(x) \geq 0$.
   (h) Determine the number of solutions to the equation $f(x) = 2$.
   (i) List the intervals where f is increasing.
   (j) List the intervals where f is decreasing.
   (k) List the local maximums, if any exist.
   (l) List the local minimums, if any exist.
   (m) Find the maximum, if it exists.
   (n) Find the minimum, if it exists.
   (o) Is f even, odd, or neither?

4. Use your graphing calculator to approximate the local and absolute extrema of the following functions. Approximate the intervals on which the function is increasing and those on which it is decreasing. Round your answers to two decimal places.

   (a) $f(x) = x^4 - 3x^3 - 24x^2 + 28x + 48$

   (b) $f(x) = x^{2/3}(x - 4)$

   (c) $f(x) = \sqrt{9 - x^2}$

   (d) $f(x) = x\sqrt{9 - x^2}$

5. Sketch the graphs of the following piecewise-defined functions.

   (a) $f(x) = \begin{cases} -2x - 4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \geq 0 \end{cases}$

   (b) $f(x) = \begin{cases} \sqrt{x + 4} & \text{if} \quad -4 \leq x < 5 \\ \sqrt{x - 1} & \text{if} \quad x \geq 5 \end{cases}$

   (c) $f(x) = \begin{cases} x^2 & \text{if} \quad x \leq -2 \\ 3 - x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \geq 2 \end{cases}$

   (d) $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

6. Let $f(x) = \lfloor x \rfloor$, the greatest integer function defined in Exercise **??** in Section **??**.

   (a) Graph $y = f(x)$. Be careful to correctly describe the behavior of the graph near the integers.

   (b) Is f even, odd, or neither? Explain.

   (c) Discuss with your classmates which points on the graph are local minimums, local maximums or both. Is f ever increasing? Decreasing? Constant?

7. Use your graphing calculator to show that the following functions do not have any extrema, neither local nor absolute.

   (a) $f(x) = x^3 + x - 12$

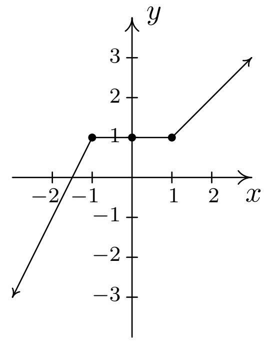
   (b) $f(x) = -5x + 2$

8. In Exercise **??** in Section **??**, we saw that the population of Sasquatch in Portage County could be modeled by the function $P(t) = \dfrac{150t}{t + 15}$, where $t = 0$ represents the year 1803. Use your graphing calculator to analyze the general function behavior of P. Will there ever be a time when 200 Sasquatch roam Portage County?

9. One of the most important aspects of the Cartesian Coordinate Plane is its ability to put Algebra into geometric terms and Geometry into algebraic terms. We've spent most of this chapter looking at this very phenomenon and now you should spend some time with your classmates reviewing what we've done. What major results do we have that tie Algebra and Geometry together? What concepts from Geometry have we not yet described algebraically? What topics from Intermediate Algebra have we not yet discussed geometrically?

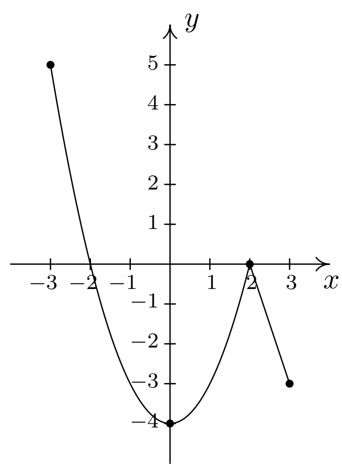
10. It's now time to "thoroughly vet the pathologies induced" by the precise definitions of local maximum and local minimum. We'll do this by providing you and your classmates a series of exercises to discuss. You will need to refer back to Definition **??** (Increasing, Decreasing and Constant) and Definition **??** (Maximum and Minimum) during the discussion.

    (a) Consider the graph of the function f given below.

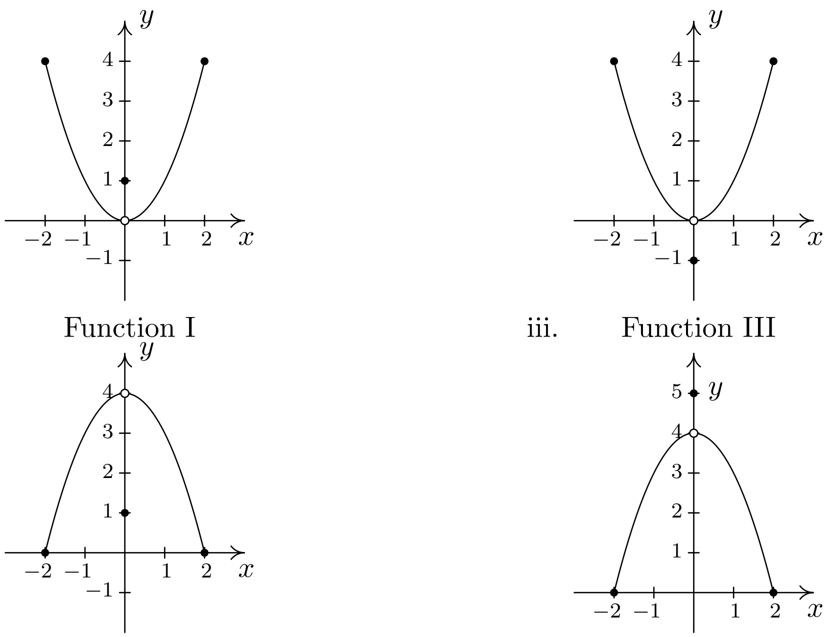
    

    i. Show that f has a local maximum but not a local minimum at the point $(-1, 1)$.
    ii. Show that f has a local minimum but not a local maximum at the point $(1, 1)$.
    iii. Show that f has a local maximum AND a local minimum at the point $(0, 1)$.
    iv. Show that f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

    (b) Using Example **??** as a guide, show that the function g whose graph is given below does not have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute. What's unique about the point $(0, -4)$ on this graph? Also find the intervals on which g is increasing and those on which g is decreasing.

    (c) We said earlier in the section that it is not good enough to say local extrema exist where a function changes from increasing to decreasing or vice versa. As a previous exercise showed, we could have local extrema when a function is constant so now we need to examine some functions whose graphs do indeed change direction. Consider the functions graphed below. Notice that all four of them change direction at an open circle on the graph. Examine each for local extrema. What is the effect of placing the “dot” on the y-axis above or below the open circle? What could you say if no function value was assigned to $x = 0$?

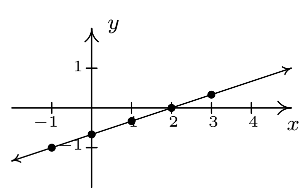


i. Function I   iii. Function III

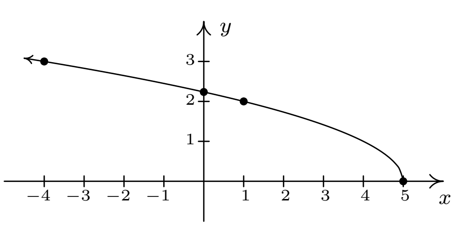
ii. Function II   iv. Function IV

1.4.3 Answers

1. (a) $f(x) = \dfrac{x-2}{3}$
   Domain: $(-\infty, \infty)$
   x-intercept: $(2, 0)$
   y-intercept: $\left(0, -\frac{2}{3}\right)$
   No symmetry

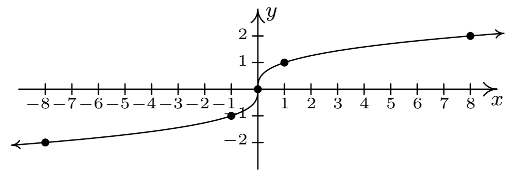


   (b) $f(x) = \sqrt{5-x}$
   Domain: $(-\infty, 5]$
   x-intercept: $(5, 0)$
   y-intercept: $(0, \sqrt{5})$
   No symmetry

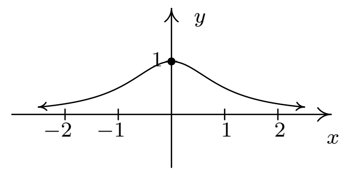


   (c)

$f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Symmetry about the origin



(d) $f(x) = \dfrac{1}{x^2+1}$
Domain: $(-\infty, \infty)$
No x-intercepts
y-intercept: $(0, 1)$
Symmetry about the y-axis



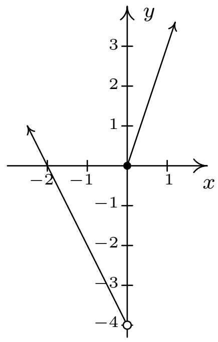
2. (a) $f(x) = 7x$ is odd
   (b) $f(x) = 7x + 2$ is neither
   (c) $f(x) = \dfrac{1}{x^3}$ is odd
   (d) $f(x) = 4$ is even
   (e) $f(x) = 0$ is even **and** odd
   (f) $f(x) = x^6 - x^4 + x^2 + 9$ is even
   (g) $f(x) = -x^5 - x^3 + x$ is odd
   (h) $f(x) = x^4 + x^3 + x^2 + x + 1$ is neither
   (i) $f(x) = \sqrt{5-x}$ is neither
   (j) $f(x) = x^2 - x - 6$ is neither

3. (a) $[-5, 3]$
   (b) $[-5, 4]$
   (c) $f(-2) = 2$
   (d) $(-4, 0)$, $(-1, 0)$, $(1, 0)$
   (e) $(0, -1)$
   (f) -4, -1, 1
   (g) $[-4, -1]$, $[1, 3]$
   (h) 4
   (i) $[-5, -3]$, $[0, 2]$
   (j) $[-3, 0]$, $[2, 3]$
   (k) $(-3, 4)$, $(2, 3)$
   (l) $(0, -1)$
   (m) 4
   (n) -5
   (o) Neither

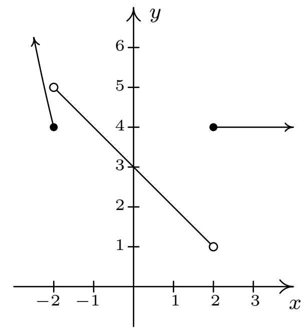
4. (a) No absolute maximum
   Absolute minimum $f(4.55) \approx -175.46$
   Local minimum at $(-2.84, -91.32)$
   Local maximum at $(0.54, 55.73)$
   Local minimum at $(4.55, -175.46)$
   Increasing on $[-2.84, 0.54], [4.55, \infty)$
   Decreasing on $(-\infty, -2.84], [0.54, 4.55]$
   (b) No absolute maximum
   No absolute minimum
   Local maximum at $(0, 0)$
   Local minimum at $(1.60, -3.28)$
   Increasing on $(-\infty, 0], [1.60, \infty)$
   Decreasing on $[0, 1.60]$
   (c) Absolute maximum $f(0) = 3$
   Absolute minimum $f(\pm 3) = 0$
   Local maximum at $(0, 3)$
   No local minimum
   Increasing on $[-3, 0]$
   Decreasing on $[0, 3]$
   (d) Absolute maximum $f(2.12) \approx 4.50$
   Absolute minimum $f(-2.12) \approx -4.50$
   Local maximum $(2.12, 4.50)$
   Local minimum $(-2.12, -4.50)$
   Increasing on $[-2.12, 2.12]$
   Decreasing on $[-3, -2.12], [2.12, 3]$

5.

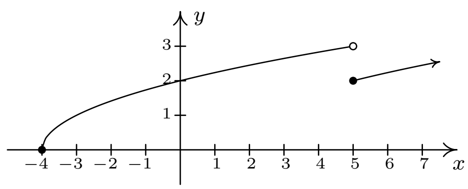
(a)



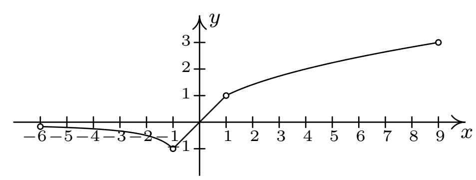
(c)



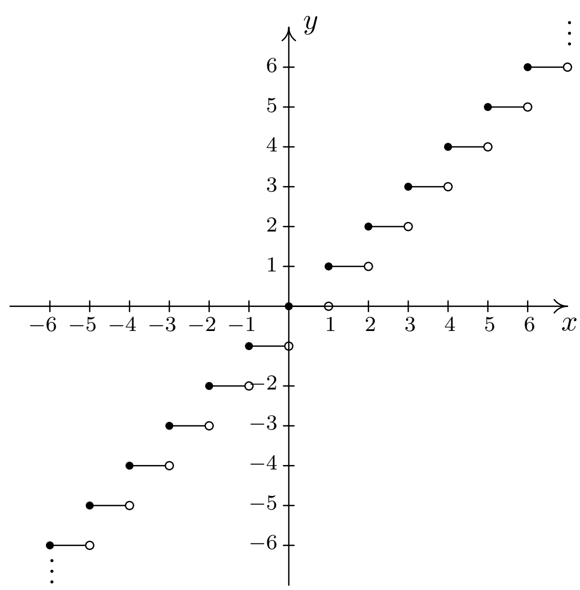
(b)



(d)



6. (a)

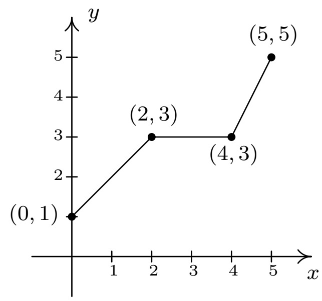


The graph of $f(x) = \lfloor x \rfloor$.

(b) Note that $f(1.1) = 1$, but $f(-1.1) = -2$, and so f is neither even nor odd.

1.5 Transformations

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections, and scalings, and we will present them in that order. Suppose the graph below is the complete graph of f.

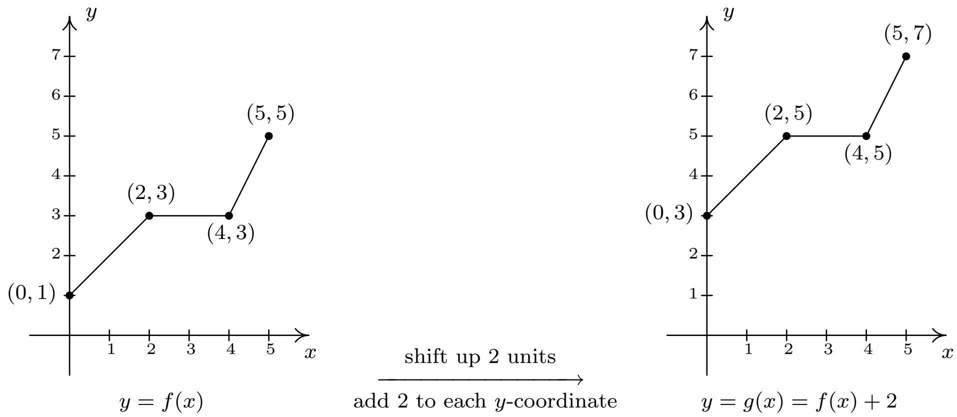


$y = f(x)$

The Fundamental Graphing Principle for Functions says that for a point (a, b) to be on the graph, $f(a) = b$. In particular, we know $f(0) = 1$, $f(2) = 3$, $f(4) = 3$ and $f(5) = 5$. Suppose we wanted to graph the function defined by the formula $g(x) = f(x) + 2$. Let's take a minute to remind ourselves of what g is doing. We start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. In order to graph g, we need to graph the points $(x, g(x))$. How are we to find the values for $g(x)$ without a formula for $f(x)$? The answer is that we don't need a *formula* for $f(x)$, we just need the *values* of $f(x)$. The values of $f(x)$ are the y values on the graph of $y = f(x)$. For example, using the points indicated on the graph of f, we can make the following table.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	3	$(0, 3)$
2	$(2, 3)$	3	5	$(2, 5)$
4	$(4, 3)$	3	5	$(4, 5)$
5	$(5, 5)$	5	7	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f. Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units **above** its previous location. Adding 2 to every y-coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before. We have the results side-by-side below.



You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. What we have discussed is generalized in the following theorem.

THEOREM 1.2. **Vertical Shifts.** Suppose f is a function and k is a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ **up** k units by **adding** k to the **y-coordinates** of the points on the graph of f.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ **down** k units by **subtracting** k from the **y-coordinates** of the points on the graph of f.

The key to understanding Theorem **??** and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If (a, b) is on the graph of f, then $f(a) = b$. Substituting $x = a$ into the equation $y = f(x) + k$ gives $y = f(a) + k = b + k$. Hence, $(a, b+k)$ is on the graph of $y = f(x)+k$, and we have the result. In the language of 'inputs' and 'outputs', Theorem **??** can be paraphrased as "Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively". So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function.[1] Let's

[1] We have spent a lot of time in this text showing you that $f(x + 2)$ and $f(x) + 2$ are, in general, wildly different algebraic animals. We will see momentarily that the geometry is also dramatically different.

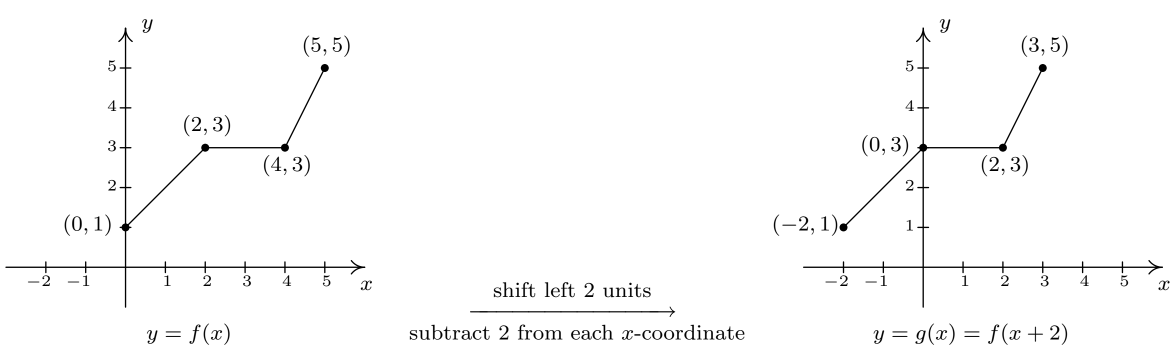
try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0,1)$	1	$f(0+2) = f(2) = 3$	$(0,3)$
2	$(2,3)$	3	$f(2+2) = f(4) = 3$	$(2,3)$
4	$(4,3)$	3	$f(4+2) = f(6) = ?$	
5	$(5,5)$	5	$f(5+2) = f(7) = ?$	

When we substitute $x = 4$ into the formula $g(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$ which doesn't exist because the domain of f is only $[0,5]$. The same thing happens when we attempt to find $g(5)$. What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2)+2) = f(0) = 1$ so $(-2,1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we get

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(0) = 1$	$(-2,1)$
0	2	$g(0) = f(2) = 3$	$(0,3)$
2	4	$g(2) = f(4) = 3$	$(2,3)$
3	5	$g(3) = f(5) = 5$	$(3,5)$

In summary, the points $(0,1)$, $(2,3)$, $(4,3)$ and $(5,5)$ on the graph of $y = f(x)$ give rise to the points $(-2,1)$, $(0,3)$, $(2,3)$ and $(3,5)$ on the graph of $y = g(x)$, respectively. In general, if (a,b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2,b)$ is on the graph of $y = g(x)$. The point $(a-2,b)$ is exactly 2 units to the **left** of the point (a,b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.



Note that while the ranges of f and g are the same, the domain of g is $[-2,3]$ whereas the domain of f is $[0,5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the **right** 2 units. Generalizing, we have the following result.

THEOREM 1.3. **Horizontal Shifts.** Suppose f is a function and h is a positive number.

- To graph $y = f(x+h)$, shift the graph of $y = f(x)$ **left** h units by **subtracting** h from the ***x*-coordinates** of the points on the graph of f.
- To graph $y = f(x-h)$, shift the graph of $y = f(x)$ **right** h units by **adding** h to the ***x*-coordinates** of the points on the graph of f.

In other words, Theorem **??** says adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems **??** and **??** present a theme which will run common throughout the section: changes to the **outputs** from a function affect the ***y*-coordinates** of the graph, resulting in some kind of vertical change; changes to the **inputs** to a function affect the ***x*-coordinates** of the graph, resulting in some kind of horizontal change.

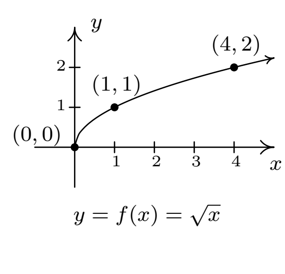
EXAMPLE 1.5.1.

1. Graph $f(x) = \sqrt{x}$. Plot at least three points.
2. Use your graph in 1 to graph $g(x) = \sqrt{x} - 1$.
3. Use your graph in 1 to graph $j(x) = \sqrt{x-1}$.
4. Use your graph in 1 to graph $m(x) = \sqrt{x+3} - 2$.

SOLUTION.

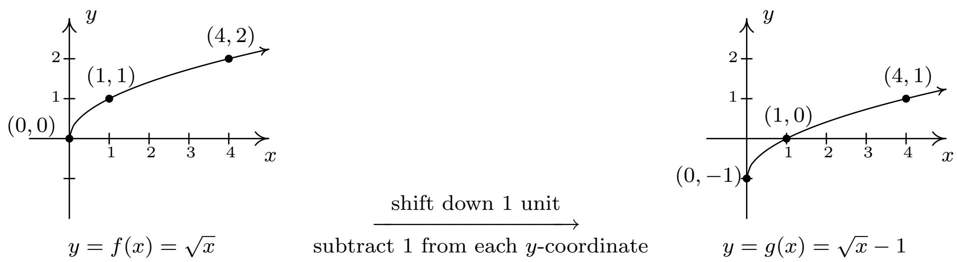
1. Owing to the square root, the domain of f is $x \geq 0$, or $[0, \infty)$. We choose perfect squares to build our table and graph below. From the graph we verify the domain of f is $[0, \infty)$ and the range of f is also $[0, \infty)$.

x	$f(x)$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

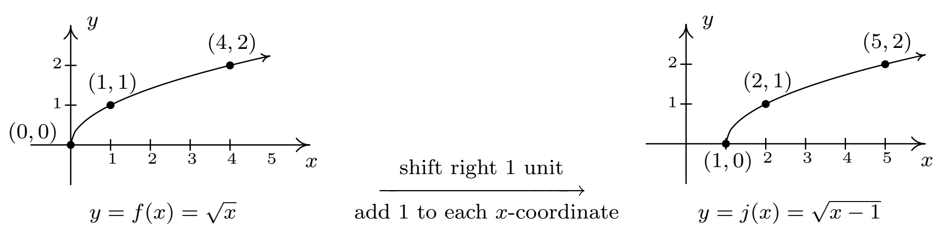


$y = f(x) = \sqrt{x}$

2. The domain of g is the same as the domain of f, since the only condition on both functions is that $x \geq 0$. If we compare the formula for $g(x)$ with $f(x)$, we see that $g(x) = f(x) - 1$. In other words, we have subtracted 1 from the output of the function f. By Theorem **??**, we know that in order to graph g, we shift the graph of f **down** one unit by **subtracting** 1 from each of the y-coordinates of the points on the graph of f. Applying this to the three points we have specified on the graph, we move $(0, 0)$ to $(0, -1)$, $(1, 1)$ to $(1, 0)$, and $(4, 2)$ to $(4, 1)$. The rest of the points follow suit, and we connect them with the same basic shape as before. We confirm the domain of g is $[0, \infty)$ and find the range of g to be $[-1, \infty)$.



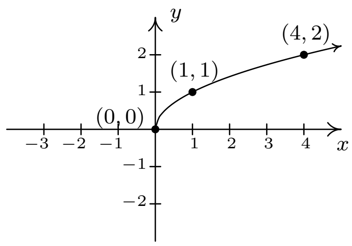
3. Solving $x - 1 \geq 0$ gives $x \geq 1$, so the domain of j is $[1, \infty)$. To graph j, we note that $j(x) = f(x - 1)$. In other words, we are subtracting 1 from the *input* of f. According to Theorem **??**, this induces a shift to the **right** of the graph of f. We **add** 1 to the x-coordinates of the points on the graph of f and get the result below. The graph reaffirms the domain of j is $[1, \infty)$ and tells us that the range is $[0, \infty)$.



4. To find the domain of m, we solve $x + 3 \geq 0$ and get $[-3, \infty)$. Comparing the formulas of $f(x)$ and $m(x)$, we have $m(x) = f(x + 3) - 2$. We have 3 being added to an input, indicating a horizontal shift, and 2 being subtracted from an output, indicating a vertical shift. We leave it to the reader to verify that, in this particular case, the order in which we perform these transformations is immaterial; we will arrive at the same graph regardless as to which transformation we apply first.[2] We follow the convention 'inputs first',[3] and to that end we first tackle the horizontal shift. Letting $m_1(x) = f(x + 3)$ denote this intermediate step, Theorem **??** tells us that the graph of $y = m_1(x)$ is the graph of f shifted to the **left** 3 units. Hence, we **subtract** 3 from each of the x-coordinates of the points on the graph of f.

[2] We shall see in the next example that order is generally important when applying more than one transformation to a graph.

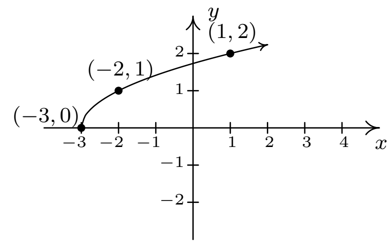
[3] We could equally have chosen the convention 'outputs first'.



$y = f(x) = \sqrt{x}$

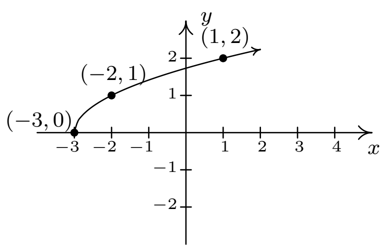
shift left 3 units

subtract 3 from each x-coordinate



$y = m_1(x) = f(x+3) = \sqrt{x+3}$

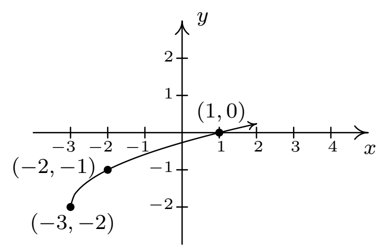
Since $m(x) = f(x+3) - 2$ and $f(x+3) = m_1(x)$, we have $m(x) = m_1(x) - 2$. We can apply Theorem **??** and obtain the graph of m by **subtracting** 2 from the y-coordinates of each of the points on the graph of $m_1(x)$. The graph verifies that the domain of m is $[-3, \infty)$ and we find the range of m is $[-2, \infty)$.



$y = m_1(x) = f(x+3) = \sqrt{x+3}$

shift down 2 units

subtract 2 from each y-coordinate



$y = m(x) = m_1(x) - 2 = \sqrt{x+3} - 2$

Keep in mind that we can check our answer to any of these kinds of problems by showing that any of the points we've moved lie on the graph of our final answer. For example, we can check that $(-3, -2)$ is on the graph of m, by computing $m(-3) = \sqrt{(-3)+3} - 2 = \sqrt{0} - 2 = -2$ ✓ □

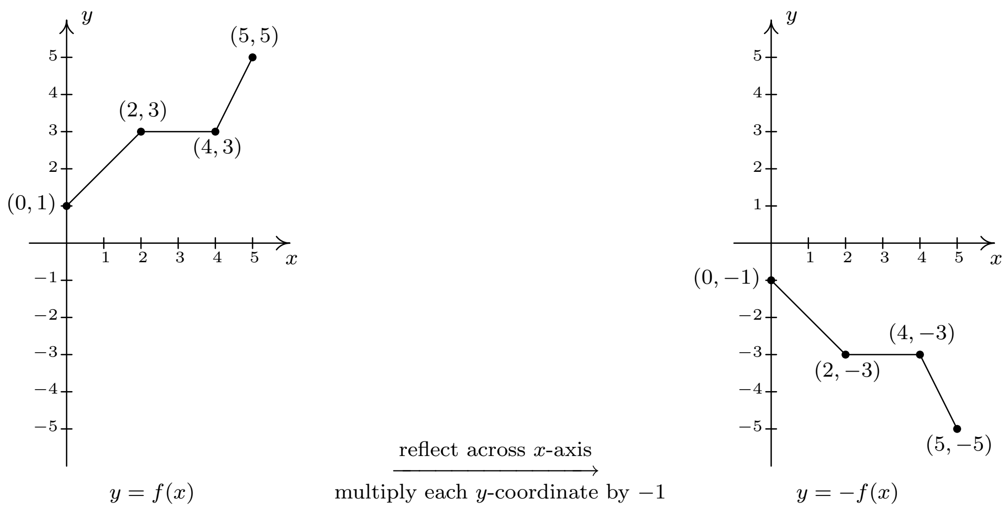
We now turn our attention to reflections. We know from Section **??** that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of f reflected across the y-axis. Returning to inputs and outputs, multiplying the output from a function by -1 reflects its graph across the x-axis, while multiplying the input to a function by -1 reflects the graph across the y-axis.[4]

[4]The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section **??** to test if a function was even, odd, or neither. The interested reader is invited to explore the role of reflections and symmetry of functions. What happens if you reflect an even function across the y-axis? What happens if you reflect an odd function across the y-axis? What about the x-axis?

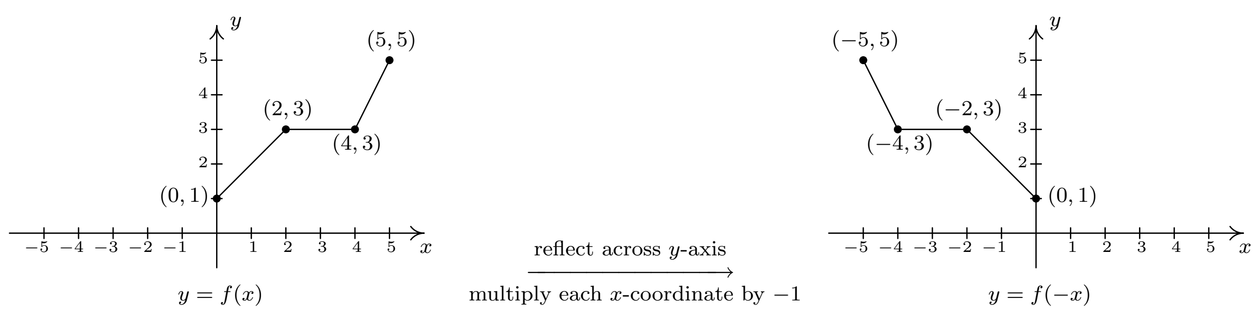
THEOREM 1.4. **Reflections.** Suppose f is a function.

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the ***x*-axis** by multiplying the ***y*-coordinates** of the points on the graph of f by -1.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the ***y*-axis** by multiplying the ***x*-coordinates** of the points on the graph of f by -1.

Applying Theroem **??** to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis



By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.



With the addition of reflections, it is now more important than ever to consider the **order** of transformations, as the next example illustrates.

EXAMPLE 1.5.2. Let $f(x) = \sqrt{x}$. Use the graph of f from Example **??** to graph the following functions below. Also, state their domains and ranges.

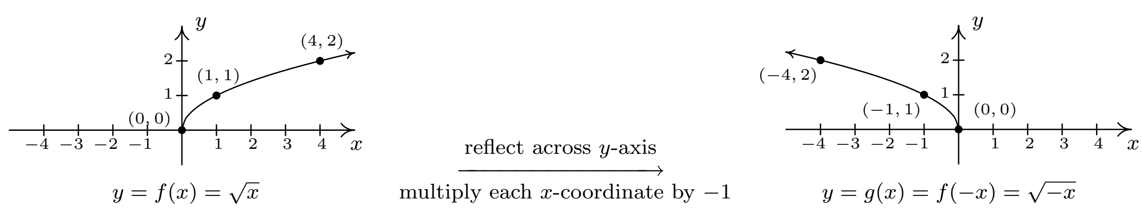
1. $g(x) = \sqrt{-x}$

2. $j(x) = \sqrt{3-x}$

3. $m(x) = 3 - \sqrt{x}$

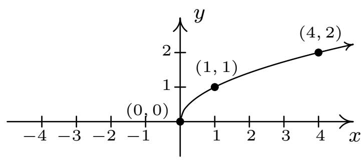
Solution.

1. The mere sight of $\sqrt{-x}$ usually causes alarm, if not panic. When we discussed domains in Section **??**, we clearly banished negatives from the radicals of even roots. However, we must remember that x is a variable, and as such, the quantity $-x$ isn't always negative. For example, if $x=-4$, $-x=4$, thus $\sqrt{-x} = \sqrt{-(-4)} = 2$ is perfectly well-defined. To find the domain analytically, we set $-x \geq 0$ which gives $x \leq 0$, so that the domain of g is $(-\infty, 0]$. Since $g(x) = f(-x)$, Theorem **??** tells us the graph of g is the reflection of the graph of f across the y-axis. We can accomplish this by multiplying each x-coordinate on the graph of f by -1, so that the points $(0,0)$, $(1,1)$, and $(4,2)$ move to $(0,0)$, $(-1,1)$, and $(-4,2)$, respectively. Graphically, we see that the domain of g is $(-\infty, 0]$ and the range of g is the same as the range of f, namely $[0,\infty)$.



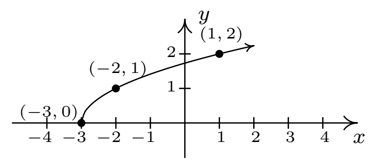
2. To determine the domain of $j(x) = \sqrt{3-x}$, we solve $3 - x \geq 0$ and get $x \leq 3$, or $(-\infty, 3]$. To determine which transformations we need to apply to the graph of f to obtain the graph of j, we rewrite $j(x) = \sqrt{-x+3} = f(-x+3)$. Comparing this formula with $f(x) = \sqrt{x}$, we see that not only are we multiplying the input x by -1, which results in a reflection across the y-axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point $(4,2)$ on the graph of f. We refer to the discussion leading up to Theorem **??**. We know $f(4) = 2$ and wish to find the point on $y = j(x) = f(-x+3)$ which corresponds to $(4,2)$. We set $-x+3 = 4$ and solve. Our first step is to subtract 3 from both sides to get $-x = 1$. Subtracting 3 from the x-coordinate 4 is shifting the point $(4,2)$ to the left. From $-x = 1$, we then multiply[5] both sides by -1 to get $x = -1$. Multiplying the x-coordinate by -1 corresponds to reflecting the point about the y-axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the y-axis. Starting with $f(x) = \sqrt{x}$, we let $j_1(x)$ be the intermediate function which shifts the graph of f 3 units to the left, $j_1(x) = f(x+3)$.

[5] Or divide - it amounts to the same thing.



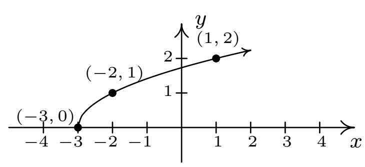
$y = f(x) = \sqrt{x}$

$\xrightarrow[\text{subtract 3 from each } x\text{-coordinate}]{\text{shift left 3 units}}$



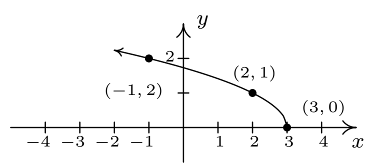
$y = j_1(x) = f(x+3) = \sqrt{x+3}$

To obtain the function j, we reflect the graph of j_1 about y-axis. Theorem **??** tells us we have $j(x) = j_1(-x)$. Putting it all together, we have $j(x) = j_1(-x) = f(-x+3) = \sqrt{-x+3}$, which is what we want.[6] From the graph, we confirm the domain of j is $(-\infty, 3]$ and we get the range is $[0, \infty)$.



$y = j_1(x) = \sqrt{x+3}$

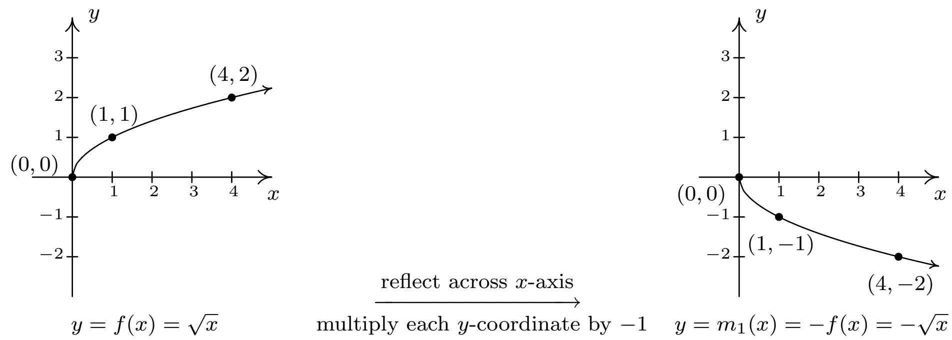
$\xrightarrow[\text{multiply each } x\text{-coordinate by } -1]{\text{reflect across } y\text{-axis}}$



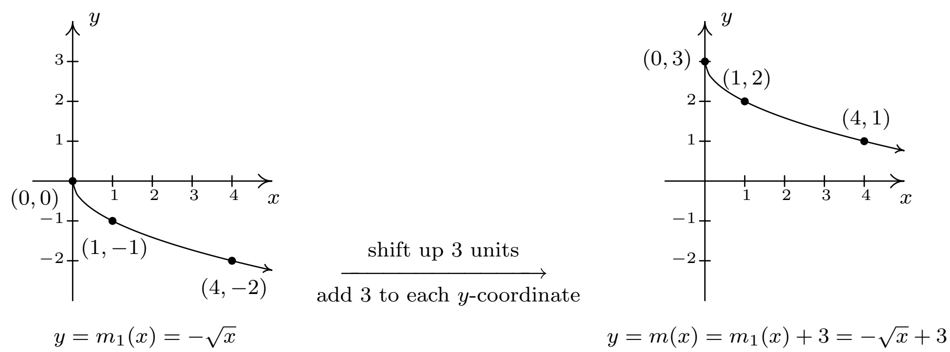
$y = j(x) = j_1(-x) = \sqrt{-x+3}$

3. The domain of m works out to be the domain of f, $[0, \infty)$. Rewriting $m(x) = -\sqrt{x} + 3$, we see $m(x) = -f(x) + 3$. Since we are multiplying the output of f by -1 and then adding 3, we once again have two transformations to deal with: a reflection across the x-axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of m which corresponds to $(4, 2)$ on the graph of f. Since in the formula for $m(x)$, the input to f is just x, we substitute to find $m(4) = -f(4) + 3 = -2 + 3 = 1$. Hence, $(4, 1)$ is the corresponding point on the graph of m. If we closely examine the arithmetic, we see that we first multiply $f(4)$ by -1, which corresponds to the reflection across the x-axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function $m_1(x) = -f(x)$ to take care of the reflection, we get

[6] If we had done the reflection first, then $j_1(x) = f(-x)$. Following this by a shift left would give us $j(x) = j_1(x+3) = f(-(x+3)) = f(-x-3) = \sqrt{-x-3}$ which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function $j(x)$. We leave it to the reader to verify the details.

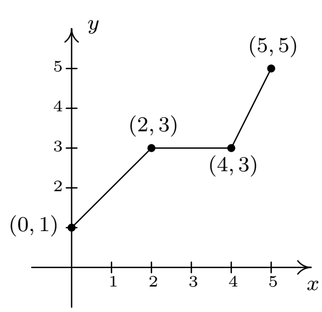


To shift the graph of m_1 up 3 units, we set $m(x) = m_1(x) + 3$. Since $m_1(x) = -f(x)$, when we put it all together, we get $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$. We see from the graph that the range of m is $(-\infty, 3]$.



□

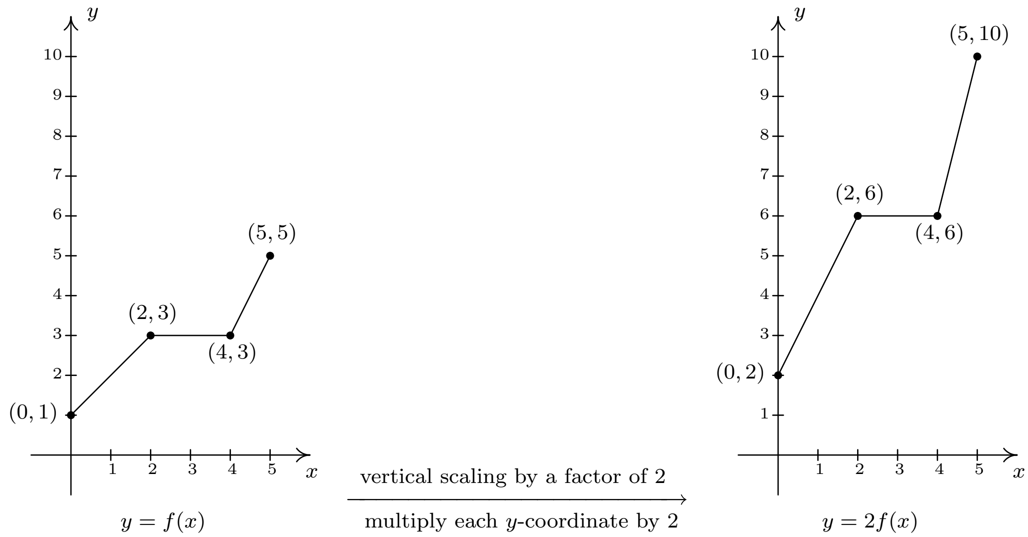
We now turn our attention to our last class of transformations, scalings. Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.



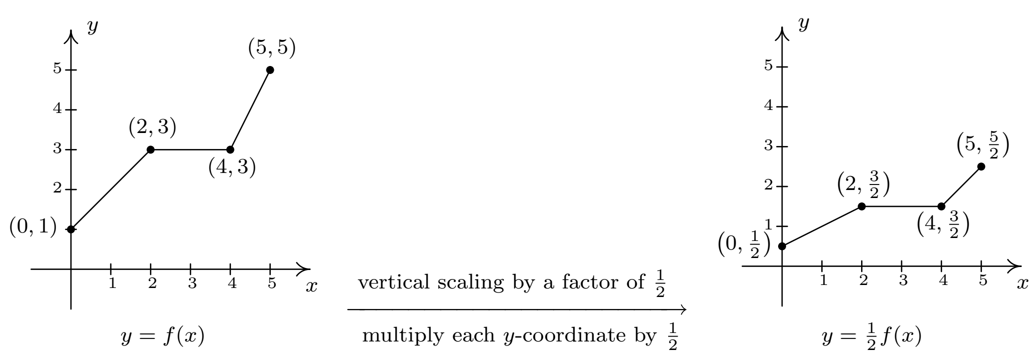
$y = f(x)$

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[7] by a factor of 2', and the results are given below.



If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[8] by a factor of $\frac{1}{2}$' as seen below.



These results are generalized in the following theorem.

[7] Also called a 'vertical stretch', 'vertical expansion' or 'vertical dilation' by a factor of 2.
[8] Also called 'vertical shrink,''vertical compression' or 'vertical contraction' by a factor of 2.

Theorem 1.5. **Vertical Scalings.** Suppose f is a function and $a > 0$. To graph $y = af(x)$, **multiply** all of the ***y*-coordinates** of the points on the graph of f by a. We say the graph of f has been **vertically scaled** by a factor of a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch (expansion, dilation) by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about Theorem **??** are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$.' This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem. Second, in terms of inputs and outputs, Theorem **??** says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

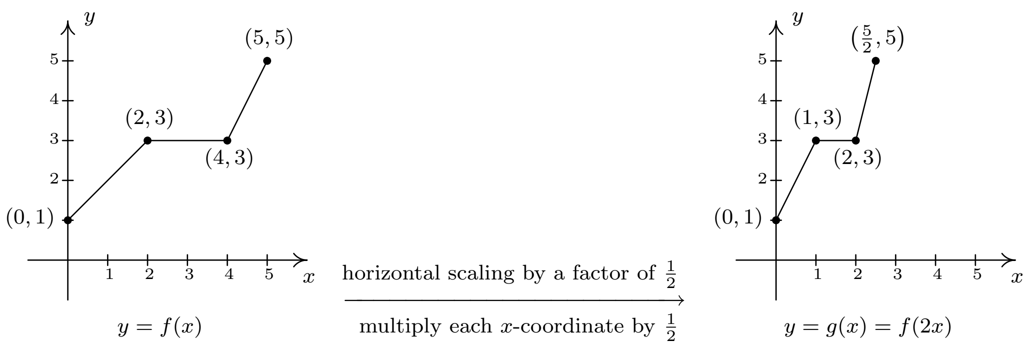
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem **??**, as seen in the table on the left below. We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

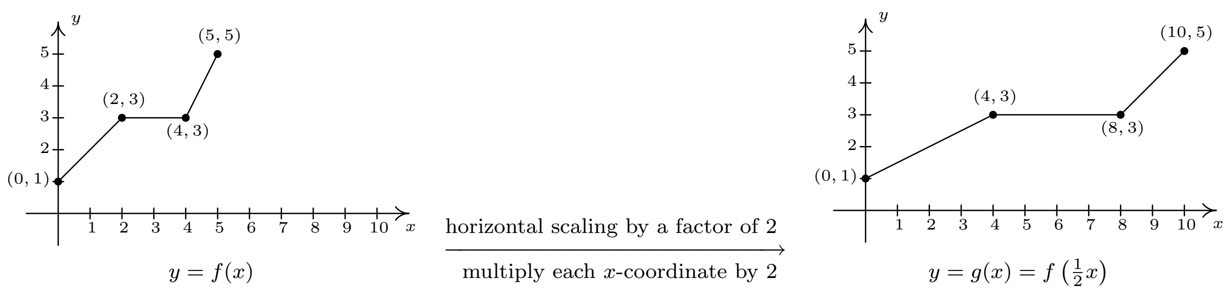
x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[9] by a factor of $\frac{1}{2}$.

[9] Also called 'horizontal shrink,''horizontal compression' or 'horizontal contraction' by a factor of 2.



If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[10] by a factor of 2, as demonstrated below.



We have the following theorem.

THEOREM 1.6. **Horizontal Scalings.** Suppose f is a function and $b > 0$. To graph $y = f(bx)$, **divide** all of the ***x*-coordinates** of the points on the graph of f by b. We say the graph of f has been **horizontally scaled** by a factor of $\frac{1}{b}$.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch (expansion, dilation) by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink (compression, contraction) by a factor of b.

Theorem **??** tells us that if we multiply the **input** to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$ since the x-values are divided by b to produce corresponding points on the graph of $f(bx)$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

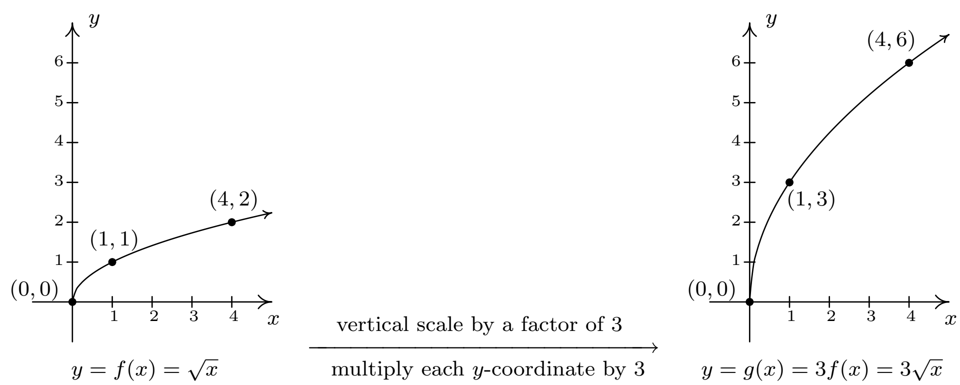
[10] Also called 'horizontal stretch,''horizontal expansion' or 'horizontal dilation' by a factor of 2.

Example 1.5.3. Let $f(x) = \sqrt{x}$. Use the graph of f from Example **??** to graph the following functions below. Also, state their domains and ranges.

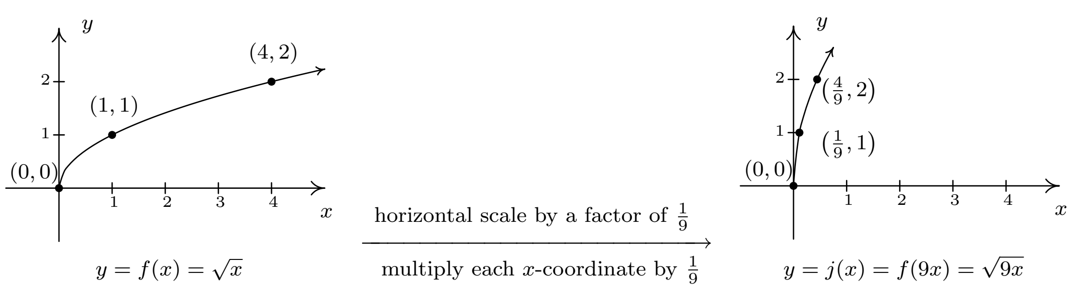
1. $g(x) = 3\sqrt{x}$
2. $j(x) = \sqrt{9x}$
3. $m(x) = 1 - \sqrt{\frac{x+3}{2}}$

Solution.

1. First we note that the domain of g is $[0, \infty)$ for the usual reason. Next, we have $g(x) = 3f(x)$ so by Theorem **??**, we obtain the graph of g by multiplying all of the y-coordinates of the points on the graph of f by 3. The result is a vertical scaling of the graph of f by a factor of 3. We find the range of g is also $[0, \infty)$.

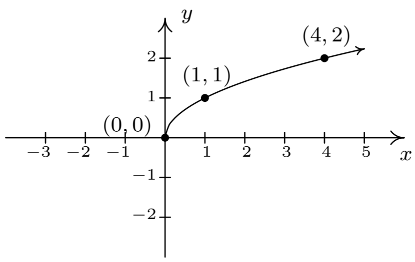


2. To determine the domain of j, we solve $9x \geq 0$ to find $x \geq 0$. Our domain is once again $[0, \infty)$. We recognize $j(x) = f(9x)$ and by Theorem **??**, we obtain the graph of j by dividing the x-coordinates of the points on the graph of f by 9. From the graph, we see the range of j is also $[0, \infty)$.

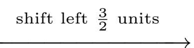


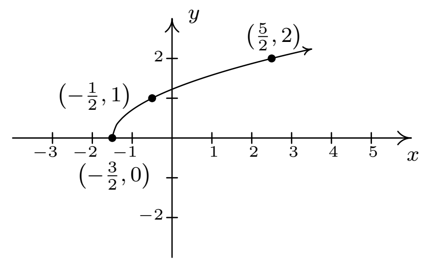
3. Solving $\frac{x+3}{2} \geq 0$ gives $x \geq -3$, so the domain of m is $[-3, \infty)$. To take advantage of what we know of transformations, we rewrite $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$, or $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$.

Focusing on the inputs first, we note that the input to f in the formula for $m(x)$ is $\frac{1}{2}x + \frac{3}{2}$. Multiplying the x by $\frac{1}{2}$ corresponds to a horizontal stretch by a factor of 2, and adding the $\frac{3}{2}$ corresponds to a shift to the left by $\frac{3}{2}$. As before, we resolve which to perform first by thinking about how we would find the point on m corresponding to a point on f, in this case, $(4, 2)$. To use $f(4) = 2$, we solve $\frac{1}{2}x + \frac{3}{2} = 4$. Our first step is to subtract the $\frac{3}{2}$ (the horizontal shift) to obtain $\frac{1}{2}x = \frac{5}{2}$. Next, we multiply by 2 (the horizontal stretch) and obtain $x = 5$. We define two intermediate functions to handle first the shift, then the stretch. In accordance with Theorem **??**, $m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$ will shift the graph of f to the left $\frac{3}{2}$ units.



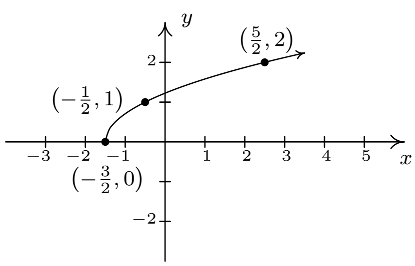
$y = f(x) = \sqrt{x}$



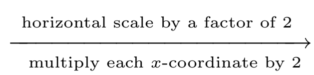


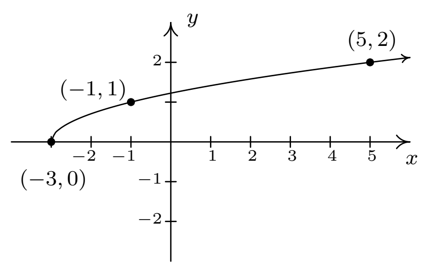
$y = m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$

Next, $m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ will, according to Theorem **??**, horizontally stretch the graph of m_1 by a factor of 2.



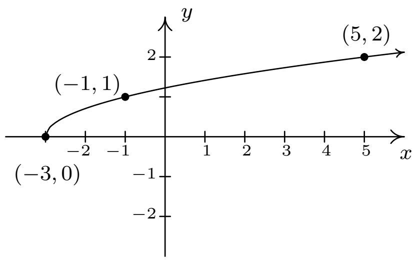
$y = m_1(x) = \sqrt{x + \frac{3}{2}}$



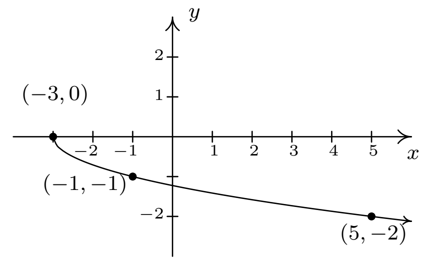


$y = m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

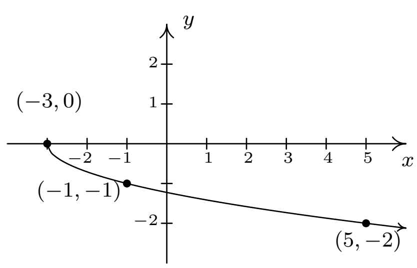
We now examine what's happening to the outputs. From $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$, we see the output from f is being multiplied by -1 (a reflection about the x-axis) and then a 1 is added (a vertical shift up 1). As before, we can determine the correct order by looking at how the point $(4, 2)$ is moved. We have already determined that to make use of the equation $f(4) = 2$, we need to substitute $x = 5$. We get $m(5) = -f\left(\frac{1}{2}(5) + \frac{3}{2}\right) + 1 = -f(4) + 1 = -2 + 1 = -1$. We see that $f(4)$ (the output from f) is first multiplied by -1 then the 1 is added meaning we first reflect the graph about the x-axis then shift up 1. Theorem **??** tells us $m_3(x) = -m_2(x)$ will handle the reflection.



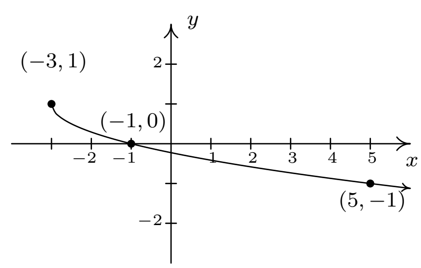
$y = m_2(x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ $\xrightarrow[\text{multiply each } y\text{-coordinate by } -1]{\text{reflect across } x\text{-axis}}$ $y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$



Finally, to handle the vertical shift, Theorem **??** gives $m(x) = m_3(x) + 1$, and we see that the range of m is $(-\infty, 1]$.



$y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$ $\xrightarrow[\text{add 1 to each } y\text{-coordinate}]{\text{shift up 1 unit}}$ $y = m(x) = m_3(x) + 1 = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$



□

Some comments about Example **??** are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions g and j are the same, since j and g have the same domains and $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$. (We invite the reader to verify that the all of the points we plotted on the graph of g lie on the graph of j and vice-versa.) Hence, for $f(x) = \sqrt{x}$, a vertical stretch by a factor of 3 and a horizontal shrink by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function m, we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

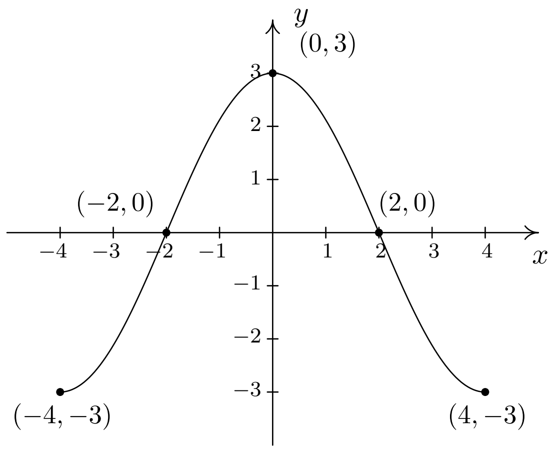
THEOREM 1.7. **Transformations.** Suppose f is a function. To graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract H from each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by B. This results in a horizontal scaling, but may also include a reflection about the y-axis if $B < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by A. This results in a vertical scaling, but may also include a reflection about the x-axis if $A < 0$.
4. Add K to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Theorem **??** can be established by generalizing the techniques developed in this section. Suppose (a, b) is on the graph of f. Then $f(a) = b$, and to make good use of this fact, we set $Bx + H = a$ and solve. We first subtract the H (causing the horizontal shift) and then divide by B. If B is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{B}$. If $B < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{a-H}{B}$ as the input to g which corresponds to the input $x = a$ to f. We now evaluate $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$. We notice that the output from f is first multiplied by A. As with the constant B, if $A > 0$, this induces only a vertical scaling. If $A < 0$, then the -1 induces a reflection across the x-axis. Finally, we add K to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem **??** is to think of the quantity $Bx + H$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation. What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations agreement: we first multiply by A then add K to find the corresponding y-coordinates on the graph of g.

EXAMPLE 1.5.4. Below is the complete graph of $y = f(x)$. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.



Solution. We use Theorem **??** to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations. We first rewrite $g(x)$ in the form presented in Theorem **??**, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the x-coordinates of the key points and solve. For example, solving $-2x+1=-4$, we first subtract 1 to get $-2x=-5$ then divide by -2 to get $x=\frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in the table below.

$(a, f(a))$	a	$-2x+1=a$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

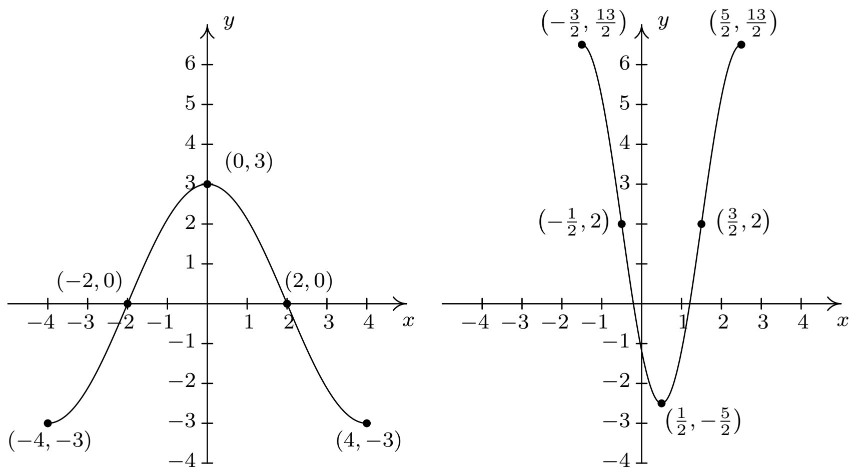
Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x=\frac{5}{2}$, and using the fact that $f(-4)=-3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we see we have a vertical stretch by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below.

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2}, \frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2}, 2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2}, -\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2}, 2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2}, \frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives



The reader is strongly encouraged[11] to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g. We have outlined the sequence of transformations in the above exposition; all that remains is to plot all five intermediate stages. $\square$

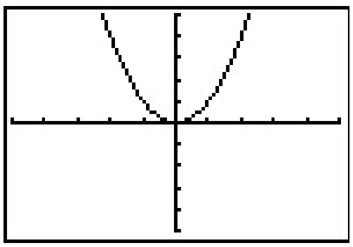
Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

Example 1.5.5. Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing the following sequence of transformations. Check your answer using a graphing calculator.

1. Vertical shift up 2 units
2. Reflection across the x-axis
3. Horizontal shift right 1 unit
4. Horizontal stretch by a factor of 2

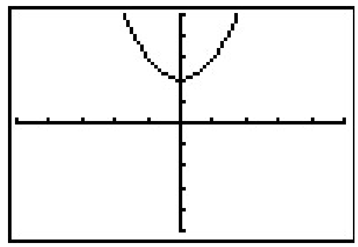
[11] You really should do this once in your life.

Solution. We build up to a formula for $g(x)$ using intermediate functions as we've seen in previous examples. We let g_1 take care of our first step. Theorem **??** tells us $g_1(x) = f(x) + 2 = x^2 + 2$. Next, we reflect the graph of g_1 about the x-axis using Theorem **??**: $g_2(x) = -g_1(x) = -\left(x^2+2\right) = -x^2 - 2$. We shift the graph to the right 1 unit, according to Theorem **??**, by setting $g_3(x) = g_2(x-1) = -(x-1)^2 - 2 = -x^2 + 2x - 3$. Finally, we induce a horizontal stretch by a factor of 2 using Theorem **??** to get $g(x) = g_3\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right) - 3$ which yields $g(x) = -\frac{1}{4}x^2 + x - 3$. We use the calculator to graph the stages below to confirm our result.

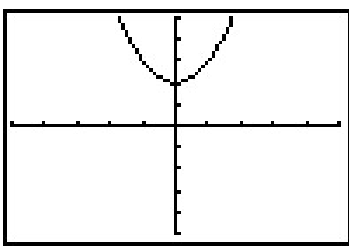


$y = f(x) = x^2$

$$\xrightarrow[\text{add 2 to each } y\text{-coordinate}]{\text{shift up 2 units}}$$

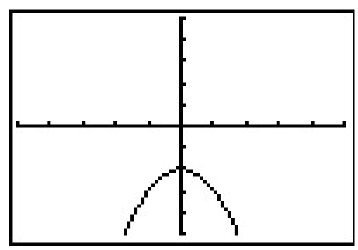


$y = g_1(x) = f(x) + 2 = x^2 + 2$

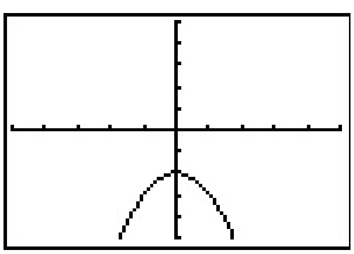


$y = g_1(x) = x^2 + 2$

$$\xrightarrow[\text{multiply each } y\text{-coordinate by } -1]{\text{reflect across } x\text{-axis}}$$

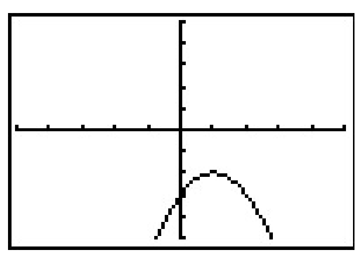


$y = g_2(x) = -g_1(x) = -x^2 - 2$

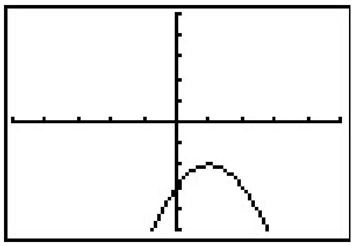


$y = g_2(x) = -x^2 - 2$

$$\xrightarrow[\text{add 1 to each } x\text{-coordinate}]{\text{shift right 1 unit}}$$

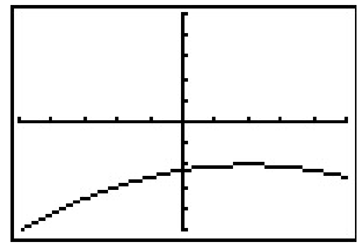


$y = g_3(x) = g_2(x-1) = -x^2 + 2x - 3$



$y = g_3(x) = -x^2 + 2x - 3$

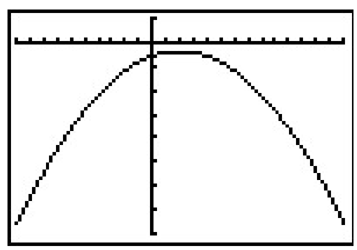
$$\xrightarrow[\text{multiply each } x\text{-coordinate by } 2]{\text{horizontal stretch by a factor of } 2}$$



$y = g(x) = g_3\left(\frac{1}{2}x\right) = -\frac{1}{4}x^2 + x - 3$

□

We have kept the viewing window the same in all of the graphs above. This had the undesirable consequence of making the last graph look 'incomplete' in that we cannot see the original shape of $f(x) = x^2$. Altering the viewing window results in a more complete graph of the transformed function as seen below.

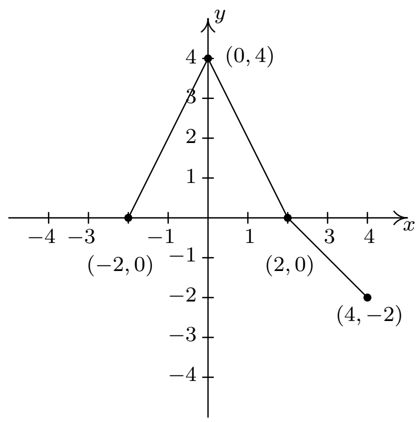


$y = g(x)$

This example brings our first chapter to a close. In the chapters which lie ahead, be on the lookout for the concepts developed here to resurface as we study different families of functions.

1.5.1 Exercises

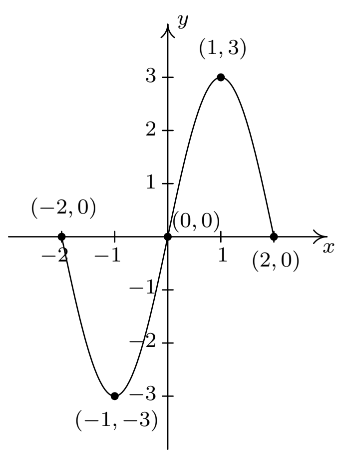
1. The complete graph of $y = f(x)$ is given below. Use it to graph the following functions.



The graph of $y = f(x)$

(a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = \frac{1}{2}f(x)$
(d) $y = f(2x)$
(e) $y = -f(x)$
(f) $y = f(-x)$
(g) $y = f(x + 1) - 1$
(h) $y = 1 - f(x)$
(i) $y = \frac{1}{2}f(x + 1) - 1$

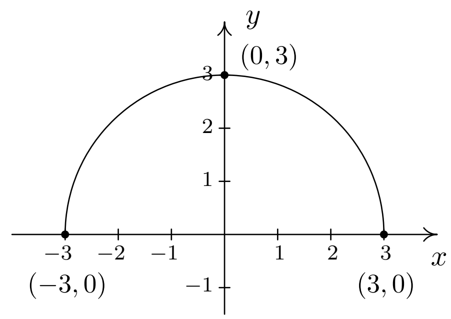
2. The complete graph of $y = S(x)$ is given below. Use it to graph the following functions.



The graph of $y = S(x)$

(a) $y = S(x + 1)$
(b) $y = S(-x + 1)$
(c) $y = \frac{1}{2}S(-x + 1)$
(d) $y = \frac{1}{2}S(-x + 1) + 1$

3. The complete graph of $y = f(x)$ is given below. Use it to graph the following functions.



(a) $g(x) = f(x) + 3$

(b) $h(x) = f(x) - \frac{1}{2}$

(c) $j(x) = f\left(x - \frac{2}{3}\right)$

(d) $a(x) = f(x + 4)$

(e) $b(x) = f(x + 1) - 1$

(f) $c(x) = \frac{3}{5}f(x)$

(g) $d(x) = -2f(x)$

(h) $k(x) = f\left(\frac{2}{3}x\right)$

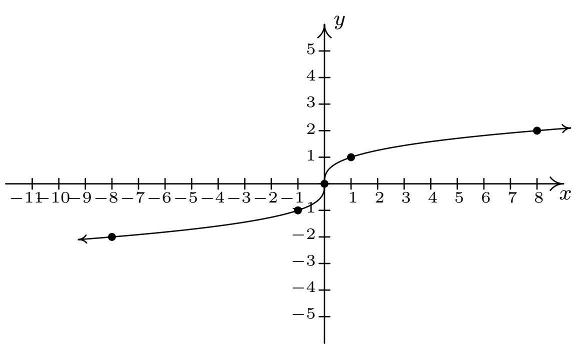
(i) $m(x) = -\frac{1}{4}f(3x)$

(j) $n(x) = 4f(x - 3) - 6$

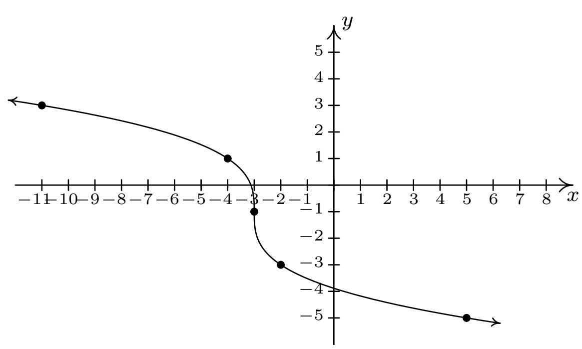
(k) $p(x) = 4 + f(1 - 2x)$

(l) $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

4. The graph of $y = f(x) = \sqrt[3]{x}$ is given below on the left and the graph of $y = g(x)$ is given on the right. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.



$y = \sqrt[3]{x}$



$y = g(x)$

5. For many common functions, the properties of algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x} = 3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$. What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

6. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

7. What happens if you reflect an even function across the y-axis?

8. What happens if you reflect an odd function across the y-axis?

9. What happens if you reflect an even function across the x-axis?

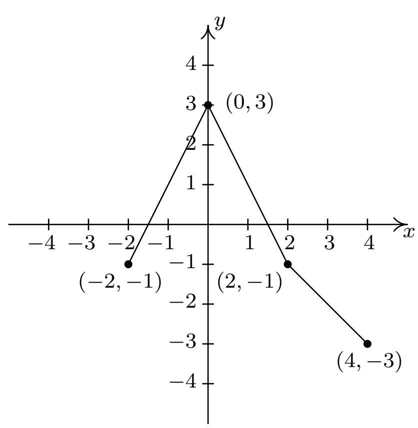
10. What happens if you reflect an odd function across the x-axis?

11. How would you describe symmetry about the origin in terms of reflections?

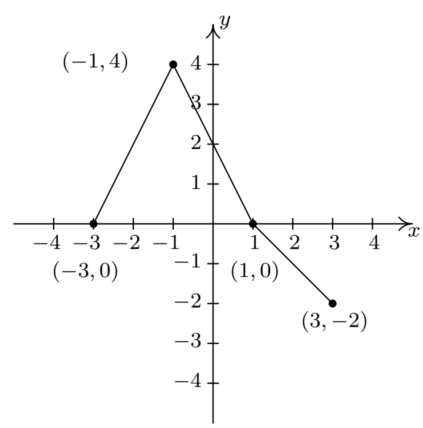
12. As we saw in Example **??**, the viewing window on the graphing calculator affects how we see the transformations done to a graph. Using two different calculators, find viewing windows so that $f(x) = x^2$ on the one calculator looks like $g(x) = 3x^2$ on the other.

1.5.2 Answers

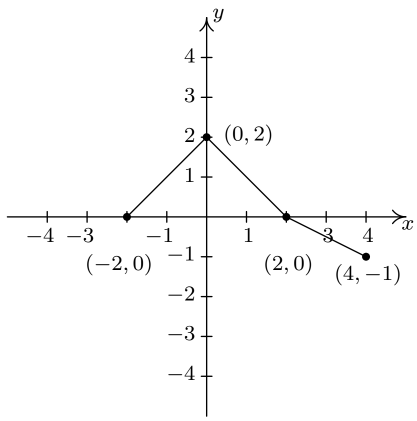
1. (a) $y = f(x) - 1$



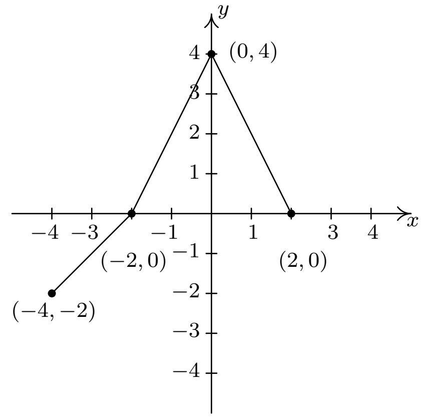
(b) $y = f(x + 1)$



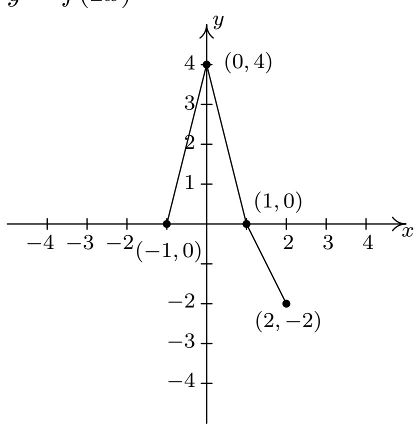
(c) $y = \frac{1}{2}f(x)$



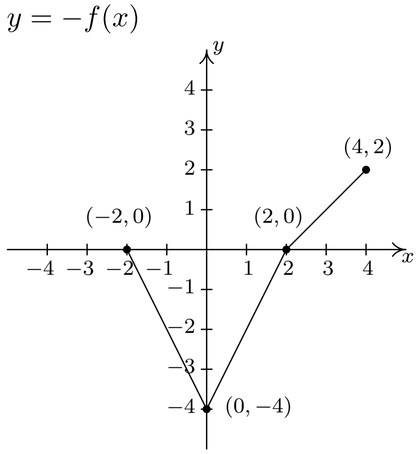
(f) $y = f(-x)$



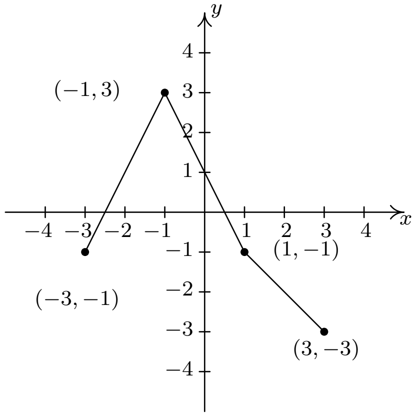
(d) $y = f(2x)$



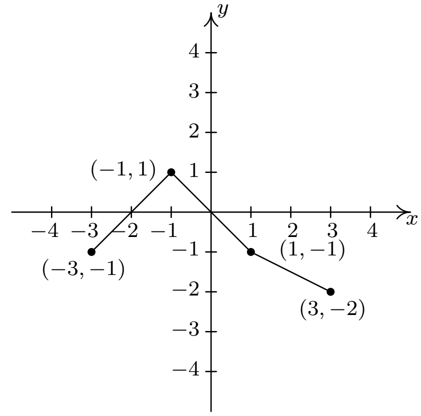
(e) $y = -f(x)$



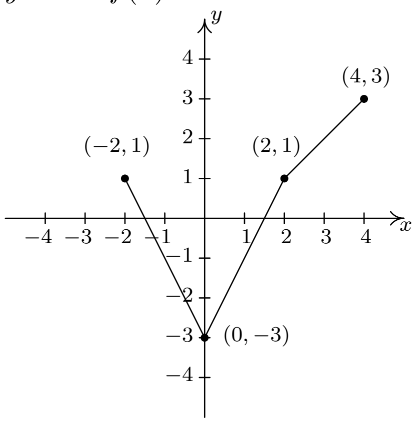
(g) $y = f(x+1) - 1$



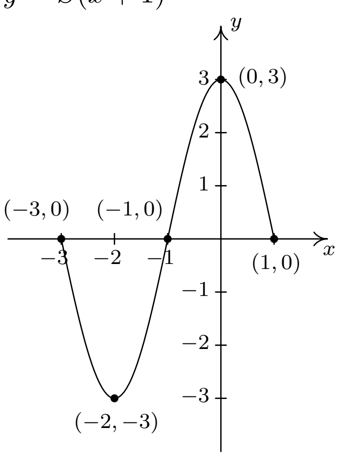
(i) $y = \frac{1}{2}f(x+1) - 1$



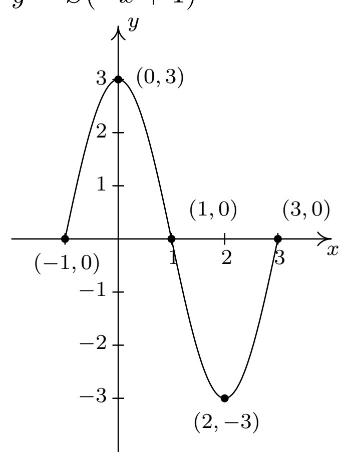
(h) $y = 1 - f(x)$



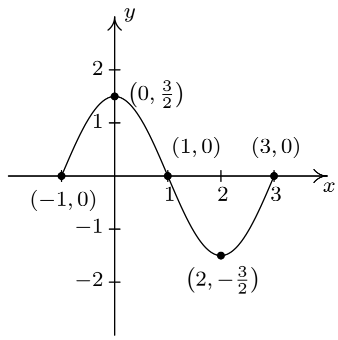
2. (a) $y = S(x+1)$



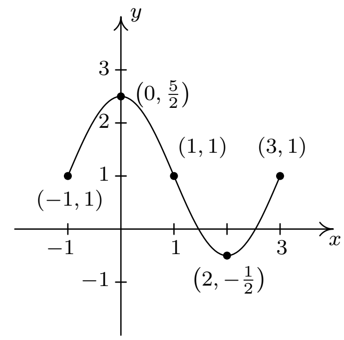
(b) $y = S(-x+1)$



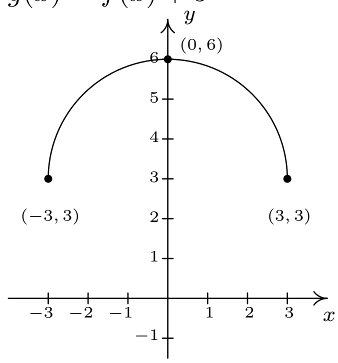
(c) $y = \frac{1}{2}S(-x+1)$



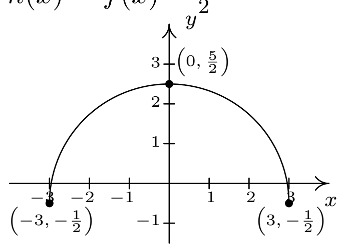
(d) $y = \frac{1}{2}S(-x+1) + 1$



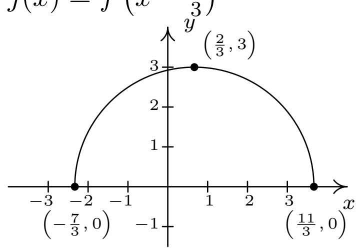
3\. (a) $g(x) = f(x) + 3$



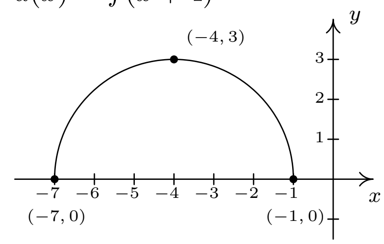
(b) $h(x) = f(x) - \frac{1}{2}$



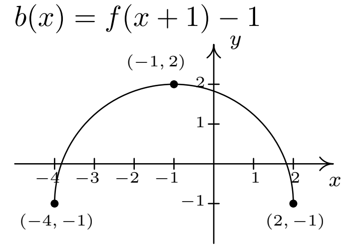
(c) $j(x) = f\left(x - \frac{2}{3}\right)$



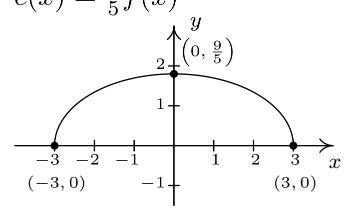
(d) $a(x) = f(x+4)$



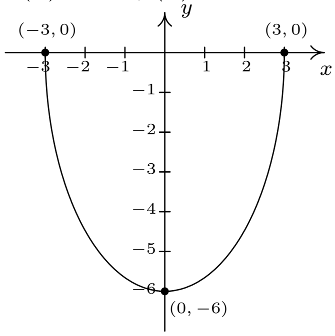
(e) $b(x) = f(x+1) - 1$



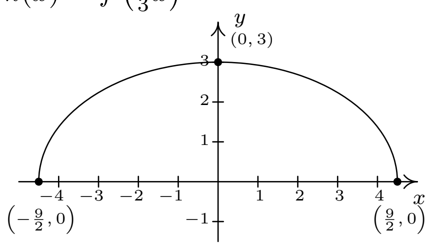
(f) $c(x) = \frac{3}{5}f(x)$



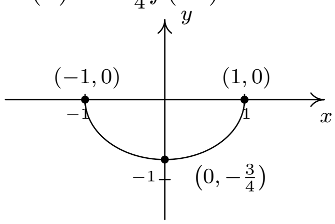
(g) $d(x) = -2f(x)$



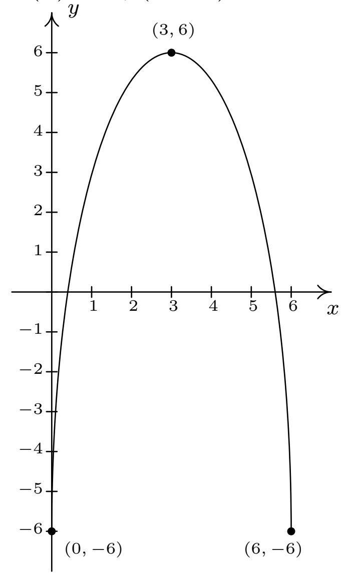
(h) $k(x) = f\left(\frac{2}{3}x\right)$



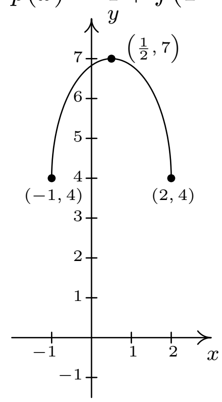
(i) $m(x) = -\frac{1}{4}f(3x)$



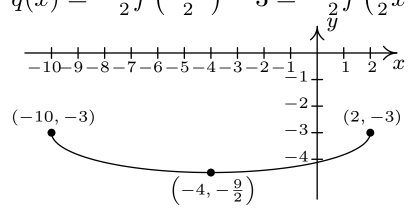
(j) $n(x) = 4f(x-3) - 6$



(k) $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$



(l) $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x+2\right) - 3$



4. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

1.6 Function Composition

Before we embark upon any further adventures with functions, we need to take some time to gather our thoughts and gain some perspective. Chapter **??** first introduced us to functions in Section **??**. At that time, functions were specific kinds of relations - sets of points in the plane which passed the Vertical Line Test, Theorem **??**. In Section **??**, we developed the idea that functions are processes - rules which match inputs to outputs - and this gave rise to the concepts of domain and range. We spoke about how functions could be combined in Section **??** using the four basic arithmetic operations, took a more detailed look at their graphs in Section **??** and studied how their graphs behaved under certain classes of transformations in Section **??**. In Chapter **??**, we took a closer look at three families of functions: linear functions (Section **??**), absolute value functions[1] (Section **??**), and quadratic functions (Section **??**). Linear and quadratic functions were special cases of polynomial functions, which we studied in generality in Chapter **??**. Chapter **??** culminated with the Real Factorization Theorem, Theorem **??**, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field[2] of study to rational functions in Chapter **??**. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Chapters **??**, **??**, and **??** can be thought of as an exhaustive study of linear and quadratic[3] functions and their arithmetic combinations as described in Section **??**. We now wish to study other algebraic functions, such as $f(x) = \sqrt{x}$ and $g(x) = x^{2/3}$, and the purpose of the first two sections of this chapter is to see how these kinds of functions arise from polynomial and rational functions. To that end, we first study a new way to combine functions as defined below.

> Definition 1.7. Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is an element of the domain of f and $f(x)$ is an element of the domain of g.

The quantity $g \circ f$ is also read 'g composed with f' or, more simply 'g of f.' At its most basic level, Definition **??** tells us to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$. If we take a step back and look at this from a procedural, 'inputs and outputs' perspective, Defintion **??** tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have

[1]These were introduced, as you may recall, as piecewise-defined linear functions.

[2]This is a really bad math pun.

[3]If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem **??**, tells us every function we have studied thus far is a combination of linear functions.

In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. There are two ways to go about evaluating composite functions - 'inside out' and 'outside in' - depending on which function we replace with its formula first. Both ways are demonstrated in the following example.

EXAMPLE 1.6.1. Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$. Find and simplify the indicated composite functions. State the domain of each.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $(h \circ h)(x)$
6. $(h \circ (g \circ f))(x)$
7. $((h \circ g) \circ f)(x)$

SOLUTION.

1. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate the two ways to evaluate this.

   - *inside out*: We insert the expression $f(x)$ into g first to get

   $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

   Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

   $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

   We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. We accomplish this by following the rule set forth in Section **??**, that is, we find the domain **before** we simplify. To that end, we examine $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$. To keep the square root happy, we solve the inequality $x^2 - 4x + 3 \geq 0$ by creating a sign diagram. If we let $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$. We obtain

Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

2. To find $(f \circ g)(x)$, we find $f(g(x))$.

   - *inside out*: We insert the expression $g(x)$ into f first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(2 - \sqrt{x+3}\right) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= 4 - 4\sqrt{x+3} + \left(\sqrt{x+3}\right)^2 - 8 + 4\sqrt{x+3} \\
&= 4 + x + 3 - 8 \\
&= x - 1
\end{aligned}$$

   - *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= (g(x))^2 - 4\,(g(x)) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= x - 1 && \text{same algebra as before}
\end{aligned}$$

   Thus we get $(f \circ g)(x) = x - 1$. To find the domain of $(f \circ g)$, we look to the step before we did any simplification and find $(f \circ g)(x) = \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right)$. To keep the square root happy, we set $x + 3 \geq 0$ and find our domain to be $[-3, \infty)$.

3. To find $(g \circ h)(x)$, we compute $g(h(x))$.

   - *inside out*: We insert the expression $h(x)$ into g first to get

$$(g \circ h)(x) = g(h(x))$$

$$
\begin{aligned}
&= g\left(\frac{2x}{x+1}\right) \\
&= 2-\sqrt{\left(\frac{2x}{x+1}\right)+3} \\
&= 2-\sqrt{\frac{2x}{x+1}+\frac{3(x+1)}{x+1}} \qquad \text{get common denominators} \\
&= 2-\sqrt{\frac{5x+3}{x+1}}
\end{aligned}
$$

- *outside in*: We use the formula for $g(x)$ first to get

$$
\begin{aligned}
(g \circ h)(x) &= g(h(x)) \\
&= 2-\sqrt{h(x)+3} \\
&= 2-\sqrt{\left(\frac{2x}{x+1}\right)+3} \\
&= 2-\sqrt{\frac{5x+3}{x+1}} \qquad \text{get common denominators as before}
\end{aligned}
$$

Hence, $(g \circ h)(x) = 2-\sqrt{\frac{5x+3}{x+1}}$. To find the domain, we look to the step before we began to simplify: $(g \circ h)(x) = 2-\sqrt{\left(\frac{2x}{x+1}\right)+3}$. To avoid division by zero, we need $x \neq -1$. To keep the radical happy, we need to solve $\frac{2x}{x+1}+3 \geq 0$. Getting common denominators as before, this reduces to $\frac{5x+3}{x+1} \geq 0$. Defining $r(x) = \frac{5x+3}{x+1}$, we have that r is undefined at $x=-1$ and $r(x)=0$ at $x=-\frac{3}{5}$. We get

#3

Our domain is $(-\infty,-1) \cup \left[-\frac{3}{5},\infty\right)$.

4. We find $(h \circ g)(x)$ by finding $h(g(x))$.

- *inside out*: We insert the expression $g(x)$ into h first to get

$$
\begin{aligned}
(h \circ g)(x) &= h(g(x)) \\
&= h\left(2-\sqrt{x+3}\right) \\
&= \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}
\end{aligned}
$$

$$= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned}
(h \circ g)(x) &= h(g(x)) \\
&= \frac{2\,(g(x))}{(g(x)) + 1} \\
&= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}
\end{aligned}$$

Hence, $(h \circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. To find the domain of $h \circ g$, we look to the step before any simplification: $(h \circ g)(x) = \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}$. To keep the square root happy, we require $x + 3 \geq 0$ or $x \geq -3$. Setting the denominator equal to zero gives $\left(2 - \sqrt{x+3}\right) + 1 = 0$ or $\sqrt{x+3} = 3$. Squaring both sides gives us $x + 3 = 9$, or $x = 6$. Since $x = 6$ checks in the original equation, $\left(2 - \sqrt{x+3}\right) + 1 = 0$, we know $x = 6$ is the only zero of the denominator. Hence, the domain of $h \circ g$ is $[-3, 6) \cup (6, \infty)$.

5. To find $(h \circ h)(x)$, we substitute the function h into itself, $h(h(x))$.

- *inside out*: We insert the expression $h(x)$ into h to get

$$\begin{aligned}
(h \circ h)(x) &= h(h(x)) \\
&= h\left(\frac{2x}{x+1}\right) \\
&= \frac{2\left(\dfrac{2x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right) + 1} \\
&= \frac{\dfrac{4x}{x+1}}{\dfrac{2x}{x+1} + 1} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{\dfrac{4x}{x+1} \cdot (x+1)}{\left(\dfrac{2x}{x+1}\right) \cdot (x+1) + 1 \cdot (x+1)}
\end{aligned}$$

$$= \frac{\frac{4x}{\cancel{(x+1)}_1} \cdot \cancel{(x+1)}}{\frac{2x}{\cancel{(x+1)}_1} \cdot \cancel{(x+1)} + x + 1}$$
$$= \frac{4x}{3x+1}$$

- *outside in*: This approach yields

$$\begin{aligned}(h \circ h)(x) &= h(h(x)) \\ &= \frac{2(h(x))}{h(x)+1} \\ &= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1} \\ &= \frac{4x}{3x+1} \qquad \text{same algebra as before}\end{aligned}$$

To find the domain of $h \circ h$, we analyze $(h \circ h)(x) = \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1}$. To keep the denominator $x+1$ happy, we need $x \neq -1$. Setting the denominator $\frac{2x}{x+1} + 1 = 0$ gives $x = -\frac{1}{3}$. Our domain is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

6. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 1 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$. We now proceed as usual.

- *inside out*: We insert the expression $(g \circ f)(x)$ into h first to get

$$\begin{aligned}(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\ &= h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned}
(h\circ(g\circ f))(x) &= h((g\circ f)(x))\\
&= \frac{2\left((g\circ f)(x)\right)}{((g\circ f)(x))+1}\\
&= \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1}\\
&= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}
\end{aligned}$$

So we get $(h\circ(g\circ f))(x) = \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}$. To find the domain, we look at the step before we began to simplify, $(h\circ(g\circ f))(x) = \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1}$. For the square root, we need $x^2-4x+3 \geq 0$, which we determined in number 1 to be $(-\infty,1]\cup[3,\infty)$. Next, we set the denominator to zero and solve: $\left(2-\sqrt{x^2-4x+3}\right)+1=0$. We get $\sqrt{x^2-4x+3}=3$, and, after squaring both sides, we have $x^2-4x+3=9$. To solve $x^2-4x-6=0$, we use the quadratic formula and get $x=2\pm\sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $\left(2-\sqrt{x^2-4x+3}\right)+1=0$. Hence we must exclude these numbers from the domain of $h\circ(g\circ f)$. Our final domain for $h\circ(f\circ g)$ is $(-\infty,2-\sqrt{10})\cup(2-\sqrt{10},1]\cup\left[3,2+\sqrt{10}\right)\cup\left(2+\sqrt{10},\infty\right)$.

7. The expression $((h\circ g)\circ f)(x)$ indicates that we first find the composite $h\circ g$ and then compose that with f. From number 4, we gave $(h\circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. We now proceed as before.

- *inside out*: We insert the expression $f(x)$ into $h\circ g$ first to get

$$\begin{aligned}
((h\circ g)\circ f)(x) &= (h\circ g)(f(x))\\
&= (h\circ g)\left(x^2-4x\right)\\
&= \frac{4-2\sqrt{(x^2-4x)+3}}{3-\sqrt{(x^2-4x)+3}}\\
&= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}
\end{aligned}$$

- *outside in*: We use the formula for $(h\circ g)(x)$ first to get

$$\begin{aligned}
((h\circ g)\circ f)(x) &= (h\circ g)(f(x))\\
&= \frac{4-2\sqrt{(f(x))+3}}{3-\sqrt{f(x))+3}}\\
&= \frac{4-2\sqrt{(x^2-4x)+3}}{3-\sqrt{(x^2-4x)+3}}
\end{aligned}$$

$$= \quad \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}$$

We note that the formula for $((h \circ g) \circ f)(x)$ before simplification is identical to that of $(h \circ (g \circ f))(x)$ before we simplified it. Hence, the two functions have the same domain, $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

□

It should be clear from Example **??** that, in general, when you compose two functions, such as f and g above, the order matters.[4] We found that the functions $f \circ g$ and $g \circ f$ were different as were $g \circ h$ and $h \circ g$. Thinking of functions as processes, this isn't all that surprising. If we think of one process as putting on our socks, and the other as putting on our shoes, the order in which we do these two tasks does matter.[5] Also note the importance of finding the domain of the composite function **before** simplifying. For instance, the domain of $f \circ g$ is much different than its simplified formula would indicate. Composing a function with itself, as in the case of $h \circ h$, may seem odd. Looking at this from a procedural perspective, however, this merely indicates performing a task h and then doing it again - like setting the washing machine to do a 'double rinse'. Composing a function with itself is called 'iterating' the function, and we could easily spend an entire course on just that. The last two problems in Example **??** serve to demonstrate the **associative** property of functions. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. This property as well as another important property are listed in the theorem below.

THEOREM 1.8. **Properties of Function Composition:** Suppose f, g, and h are functions.

- $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- If I is defined as $I(x) = x$ for all real numbers x, then $I \circ f = f \circ I = f$.

By repeated applications of Definition **??**, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition **??**. Recall that

[4]This shows us function composition isn't **commutative**. An example of an operation we perform on two functions which is commutative is function addition, which we defined in Section **??**. In other words, the functions $f + g$ and $g + f$ are always equal. Which of the remaining operations on functions we have discussed are commutative?

[5]A more mathematical example in which the order of two processes matters can be found in Section **??**. In fact, all of the transformations in that section can be viewed in terms of composing functions with linear functions.

the function $I(x) = x$ is called the *identity function* and was introduced in Exercise **??** in Section **??**. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(x)$. This establishes that we have an identity for function composition much in the same way the real number 1 is an identity for real number multiplication. That is, just as for any real number x, $1 \cdot x = x \cdot 1 = x$, we have for any function f, $I \circ f = f \circ I = f$. We shall see the concept of an identity take on great significance in the next section. Out in the wild, function composition is often used to relate two quantities which may not be directly related, but have a variable in common, as illustrated in our next example.

Example 1.6.2. The surface area S of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'middle man' r. □

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

Example 1.6.3. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x) = |3x - 1|$
2. $G(x) = \dfrac{2}{x^2 + 1}$
3. $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition **??**, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$, so we are done.

2. We attack deconstructing G from an operational approach. Given an input x, the first step is to square x, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of three functions: f, g and h. Our first function, f, is the function that squares its input, $f(x) = x^2$. The next function is the function that adds 1 to its input, $g(x) = x + 1$. Our last function takes its input and divides it into 2, $h(x) = \frac{2}{x}$. The claim is that $G = h \circ g \circ f$. We find $(h \circ g \circ f)(x) = h(g(f(x))) = h(g\left(x^2\right)) = h\left(x^2 + 1\right) = \frac{2}{x^2+1} = G(x)$.

3. If we look $H(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{x}$ is a simple piece of the larger function. If we define $f(x) = \sqrt{x}$, we have $H(x) = \frac{f(x)+1}{f(x)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula from $g(x)$ by looking at what is being done to $f(x)$. We find $g(x) = \frac{x+1}{x-1}$. We check $(g \circ f)(x) = g(f(x)) = \frac{f(x)+1}{f(x)-1} = \frac{\sqrt{x}+1}{\sqrt{x}-1} = H(x)$, as required. $\square$

1.6.1 Exercises

1. Let $f(x) = 3x - 6$, $g(x) = |x|$, $h(x) = \sqrt{x}$ and $k(x) = \dfrac{1}{x}$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ k)(x)$
   (h) $(k \circ f)(x)$
   (i) $(h \circ k)(x)$
   (j) $(k \circ h)(x)$
   (k) $(f \circ g \circ h)(x)$
   (l) $(h \circ g \circ k)(x)$
   (m) $(k \circ h \circ f)(x)$
   (n) $(h \circ k \circ g \circ f)(x)$

2. Let $f(x) = 2x + 1$, $g(x) = x^2 - x - 6$ and $h(x) = \dfrac{x+6}{x-6}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(g \circ f)(x)$
   (b) $(h \circ f)(x)$
   (c) $(h \circ g)(x)$
   (d) $(h \circ h)(x)$

3. Let $f(x) = \sqrt{x-3}$, $g(x) = 4x + 3$ and $h(x) = \dfrac{x-2}{x+3}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ f)(x)$
   (h) $(g \circ g)(x)$
   (i) $(h \circ h)(x)$

4. Let $f(x) = \sqrt{9 - x^2}$ and $g(x) = x^2 - 9$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ f)(x)$
   (b) $(g \circ g)(x)$
   (c) $(g \circ f)(x)$
   (d) $(f \circ g)(x)$

5. Let f be the function defined by $f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$ and let g be the function defined $g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$.

   Find the each of the following values if it exists.

   (a) $(f \circ g)(3)$
   (b) $f(g(-1))$
   (c) $(f \circ f)(0)$
   (d) $(f \circ g)(-3)$
   (e) $(g \circ f)(3)$
   (f) $g(f(-3))$
   (g) $(g \circ g)(-2)$
   (h) $(g \circ f)(-2)$
   (i) $g(f(g(0)))$
   (j) $f(f(f(-1)))$
   (k) $f(f(f(f(f(1)))))$
   (l) $(\overbrace{g \circ g \circ \cdots \circ g}^{n \text{ times}})(0)$

6. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

7. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

8. Write the following as a composition of two or more non-identity functions.

   (a) $h(x) = \sqrt{2x-1}$
   (b) $r(x) = \dfrac{2}{5x+1}$
   (c) $F(x) = \left(x^2-1\right)^3$
   (d) $R(x) = \dfrac{2x^3+1}{x^3-1}$

9. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

10. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

11. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.
    (b) Find and simplify $(R \circ x)(t)$. What does this represent?
    (c) What is the revenue per hour at noon?

12. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

1.6.2 Answers

1. (a) $(f \circ g)(x) = 3|x| - 6$
   Domain: $(-\infty, \infty)$
   (b) $(g \circ f)(x) = |3x - 6|$
   Domain: $(-\infty, \infty)$
   (c) $(f \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$
   (d) $(h \circ f)(x) = \sqrt{3x - 6}$
   Domain: $[2, \infty)$
   (e) $(g \circ h)(x) = \sqrt{x}$
   Domain: $[0, \infty)$
   (f) $(h \circ g)(x) = \sqrt{|x|}$
   Domain: $(-\infty, \infty)$
   (g) $(f \circ k)(x) = \dfrac{3}{x} - 6$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   (h) $(k \circ f)(x) = \dfrac{1}{3x - 6}$
   Domain: $(-\infty, 2) \cup (2, \infty)$
   (i) $(h \circ k)(x) = \sqrt{\dfrac{1}{x}}$
   Domain: $(0, \infty)$
   (j) $(k \circ h)(x) = \dfrac{1}{\sqrt{x}}$
   Domain: $(0, \infty)$
   (k) $(f \circ g \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$
   (l) $(h \circ g \circ k)(x) = \sqrt{\left|\dfrac{1}{x}\right|}$
   Domain: $(-\infty, 0) \cup (0, \infty)$
   (m) $(k \circ h \circ f)(x) = \dfrac{1}{\sqrt{3x - 6}}$
   Domain: $(2, \infty)$
   (n) $(h \circ k \circ g \circ f)(x) = \sqrt{\dfrac{1}{|3x - 6|}}$
   Domain: $(-\infty, 2) \cup (2, \infty)$

2. (a) $(g \circ f)(x) = 4x^2 + 2x - 6$
   Domain: $(-\infty, \infty)$
   (b) $(h \circ f)(x) = \dfrac{2x + 7}{2x - 5}$
   Domain: $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$
   (c) $(h \circ g)(x) = \dfrac{x^2 - x}{x^2 - x - 12}$
   Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$
   (d) $(h \circ h)(x) = -\dfrac{7x - 30}{5x - 42}$
   Domain: $(-\infty, 6) \cup \left(6, \frac{42}{5}\right) \cup \left(\frac{42}{5}, \infty\right)$

3. (a) $(f \circ g)(x) = 2\sqrt{x}$
   Domain: $[0, \infty)$
   (b) $(g \circ f)(x) = 4\sqrt{x - 3} + 3$
   Domain: $[3, \infty)$
   (c) $(f \circ h)(x) = \sqrt{\dfrac{-2x - 11}{x + 3}}$
   Domain: $\left[-\frac{11}{2}, -3\right)$
   (d) $(h \circ f)(x) = \dfrac{\sqrt{x - 3} - 2}{\sqrt{x - 3} + 3}$
   Domain: $[3, \infty)$
   (e) $(g \circ h)(x) = \dfrac{7x + 1}{x + 3}$
   Domain: $(-\infty, -3) \cup (-3, \infty)$
   (f) $(h \circ g)(x) = \dfrac{4x + 1}{4x + 6}$
   Domain: $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$
   (g) $(f \circ f)(x) = \sqrt{\sqrt{x - 3} - 3}$
   Domain: $[12, \infty)$
   (h) $(g \circ g)(x) = 16x + 15$
   Domain: $(-\infty, \infty)$
   (i) $(h \circ h)(x) = \dfrac{-x - 8}{4x + 7}$
   Domain:
   $(-\infty, -3) \cup \left(-3, -\frac{7}{4}\right) \cup \left(-\frac{7}{4}, \infty\right)$

4. (a) $(f \circ f)(x) = |x|$
   Domain: $[-3, 3]$
   (b) $(g \circ g)(x) = x^4 - 18x^2 + 72$
   Domain: $(-\infty, \infty)$
   (c) $(g \circ f)(x) = -x^2$
   Domain: $[-3, 3]$
   (d) $(f \circ g)(x) = \sqrt{-x^4 + 18x^2 - 72}$
   Domain: $[-\sqrt{12}, -\sqrt{6}] \cup [\sqrt{6}, \sqrt{12}]$[6]

5. (a) $(f \circ g)(3) = f(g(3)) = f(2) = 4$
   (b) $f(g(-1)) = f(-4)$ which is undefined
   (c) $(f \circ f)(0) = f(f(0)) = f(1) = 3$
   (d) $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$
   (e) $(g \circ f)(3) = g(f(3)) = g(-1) = -4$
   (f) $g(f(-3)) = g(4)$ which is undefined
   (g) $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$
   (h) $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$
   (i) $g(f(g(0))) = g(f(0)) = g(1) = -3$
   (j) $f(f(f(-1))) = f(f(0)) = f(1) = 3$
   (k) $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$
   (l) $\overbrace{(g \circ g \circ \cdots \circ g)}^{n \text{ times}}(0) = 0$

6. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

7. One possible solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

8. (a) $h(x) = (g \circ f)(x)$ where $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$.
   (b) $r(x) = (g \circ f)(x)$ where $f(x) = 5x + 1$ and $g(x) = \dfrac{2}{x}$.
   (c) $F(x) = (g \circ f)(x)$ where $f(x) = x^2 - 1$ and $g(x) = x^3$.
   (d) $R(x) = (g \circ f)(x)$ where $f(x) = x^3$ and $g(x) = \dfrac{2x+1}{x-1}$.

9. $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \dfrac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

10. $V(x) = x^3$ so $V(x(t)) = (t+1)^3$

11. (a) $R(x) = 2x$
   (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.
   (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

[6]The quantity $-x^4 + 18x^2 - 72$ is a 'quadratic in disguise' which factors nicely. See Example **??** is Section **??**.

1.7 Inverse Functions

Thinking of a function as a process like we did in Section **??**, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, as we saw in Section **??**

1. multiply by 3

2. add 4

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, the function g should

1. *subtract* 4

2. *divide* by 3

Following this procedure, we get $g(x) = \frac{x-4}{3}$. Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and put that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section **??**, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively. Abstractly, we can visualize the relationship between f and g in the diagram below.

The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

DEFINITION 1.8. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f **and**
2. $(f \circ g)(x) = x$ for all x in the domain of g.

Then f and g are said to be **inverses** of each other. The functions f and g are said to be **invertible**.

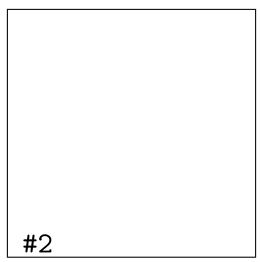
Our first result of the section formalizes the concepts that inverse functions exchange inputs and outputs and is a consequence of Definition **??** and the Fundamental Graphing Principle for Functions.

THEOREM 1.9. **Properties of Inverse Functions:** Suppose f and g are inverse functions.

- The range[a] of f is the domain of g and the domain of f is the range of g
- $f(a) = b$ if and only if $g(b) = a$
- (a, b) is on the graph of f if and only if (b, a) is on the graph of g

[a] Recall this is the set of all outputs of a function.

The third property in Theorem **??** tells us that the graphs of inverse functions are reflections about the line $y = x$. For a proof of this, we refer the reader to Example **??** in Section **??**. A plot of the inverse functions $f(x) = 3x + 4$ and $g(x) = \frac{x-4}{3}$ confirms this to be the case.



If we abstract one step further, we can express the sentiment in Definition **??** by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[1] to the domain of f and I_2 is the identity function restricted to the domain of g. In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem **??**, we can show that function inverses are unique.[2] Suppose g and h are both inverses of a function f. By Theorem **??**, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.[3] We summarize the discussion of the last two paragraphs in the following theorem.[4]

[1] The identity function I, which was introduced in Section **??** and mentioned in Theorem **??**, has a domain of all real numbers. In general, the domains of f and g are not all real numbers, which necessitates the restrictions listed here.

[2] In other words, invertible functions have exactly one inverse.

[3] It is an excellent exercise to explain each step in this string of equalities.

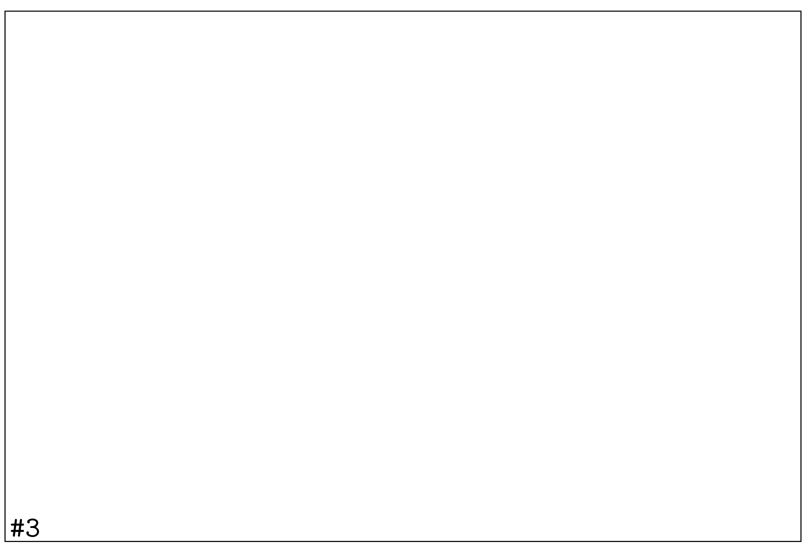
[4] In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.

Theorem 1.10. **Uniqueness of Inverse Functions and Their Graphs :** Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read f-inverse)
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely **not** the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$. Why does this confusing notation persist? As we mentioned in Section **??**, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Let's turn our attention to the function $f(x) = x^2$. Is f invertible? A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. Checking the composition yields $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x for all x in the domain $(-\infty, \infty)$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$, which means g failed to return the input -2 from its output 4. What g did, however, is match the output 4 to a **different** input, namely 2, which satisfies $f(2) = 4$. This issue is presented schematically in the picture below.



We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a **function** which takes 4 back to **both** $x = 2$ and $x = -2$. (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if $y = f^{-1}(x)$ exists, its graph can be obtained by reflecting $y = x^2$ about the line $y = x$, in accordance with Theorem **??**. Doing so produces

#4

$y = f(x) = x^2$

$$\xrightarrow[\text{switch } x \text{ and } y \text{ coordinates}]{\text{reflect across } y = x}$$

#5

$y = f^{-1}(x)$?

We see that the line $x = 4$ intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, Theorem **??**, and as such, does not represent y as a function of x. The vertical line $x = 4$ on the graph on the right corresponds to the *horizontal line* $y = 4$ on the graph of $y = f(x)$. The fact that the horizontal line $y = 4$ intersects the graph of f twice means two **different** inputs, namely $x = -2$ and $x = 2$, are matched with the **same** output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, **different** inputs must go to **different** outputs, or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

Definition 1.9. A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

Theorem 1.11. **The Horizontal Line Test:** A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if f is invertible, then f must be one-to-one, otherwise the graph given by reflecting the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function, then f is invertible and we have found f^{-1}. This is precisely what the

Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y. We summarize these results below.

THEOREM 1.12. **Equivalent Conditions for Invertibility:** Suppose f is a function. The following statements are equivalent.

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

We put this result to work in the next example.

EXAMPLE 1.7.1. Determine if the following functions are one-to-one in two ways: (a) analytically using Definition **??** and (b) graphically using the Horizontal Line Test.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$
3. $h(x) = x^2 - 2x + 4$
4. $F = \{(-1,1), (0,2), (2,1)\}$

SOLUTION.

1. (a) To determine if f is one-to-one analytically, we assume $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

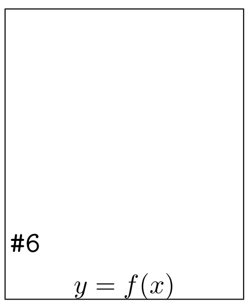
(b) To check if f is one-to-one graphically, we look to see if the graph of $y = f(x)$ passes the Horizontal Line Test. We have that f is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of f passes the Horizontal Line Test as seen below.

2. (a) We begin with the assumption that $g(c) = g(d)$ and try to show $c = d$.

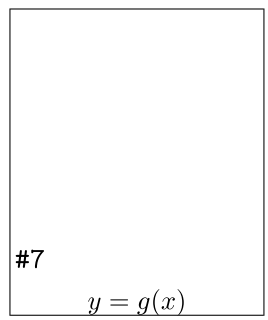
$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c - 2cd &= 2d - 2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that g is one-to-one.

(b) We can graph g using the six step procedure outlined in Section **??**. We get the sole intercept at $(0,0)$, a vertical asymptote $x = 1$ and a horizontal asymptote (which the graph never crosses) $y = -2$. We see from that the graph of g passes the Horizontal Line Test.



$y = f(x)$



$y = g(x)$

3. (a) We begin with $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

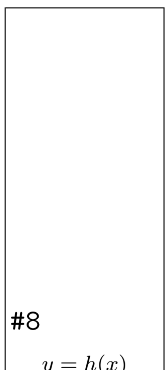
$$\begin{aligned} h(c) &= h(d) \\ c^2 - 2c + 4 &= d^2 - 2d + 4 \\ c^2 - 2c &= d^2 - 2d \\ c^2 - d^2 - 2c + 2d &= 0 \\ (c+d)(c-d) - 2(c-d) &= 0 \\ (c-d)((c+d)-2) &= 0 && \text{factor by grouping} \\ c - d = 0 \quad &\text{or} \quad c + d - 2 = 0 \\ c = d \quad &\text{or} \quad c = 2 - d \end{aligned}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that f may not be one-to-one. Taking $d = 0$, we get $c = 0$ or $c = 2$. With $f(0) = 4$ and $f(2) = 4$, we have produced two different inputs with the same output meaning f is not one-to-one.

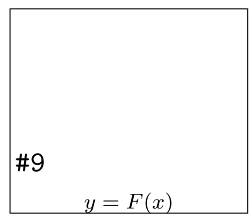
(b) We note that h is a quadratic function and we graph $y = h(x)$ using the techniques presented in Section **??**. The vertex is $(1, 3)$ and the parabola opens upwards. We see immediately from the graph that h is not one-to-one, since there are several horizontal lines which cross the graph more than once.

4. (a) The function F is given to us as a set of ordered pairs. The condition $F(c) = F(d)$ means the outputs from the function (the y-coordinates of the ordered pairs) are the same. We see that the points $(-1, 1)$ and $(2, 1)$ are both elements of F with $F(-1) = 1$ and $F(2) = 1$. Since $-1 \neq 2$, we have established that F is **not** one-to-one.

   (b) Graphically, we see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.



$y = h(x)$



$y = F(x)$

□

We have shown that the functions f and g in Example **??** are one-to-one. This means they are invertible, so it is natural to wonder what $f^{-1}(x)$ and $g^{-1}(x)$ would be. For $f(x) = \frac{1-2x}{5}$, we can think our way through the inverse since there is only one occurrence of x. We can track step-by-step what is done to x and reverse those steps as we did at the beginning of the chapter. The function $g(x) = \frac{2x}{1-x}$ is a bit trickier since x occurs in two places. When one evaluates $g(x)$ for a specific value of x, which is first, the $2x$ or the $1-x$? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem **??** tells us equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

Steps for finding the Inverse of a One-to-one Function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and outputs.

EXAMPLE 1.7.2. Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1. $f(x) = \dfrac{1-2x}{5}$

2. $g(x) = \dfrac{2x}{1-x}$

SOLUTION.

1. As we mentioned earlier, it is possible to think our way through the inverse of f by recording the steps we apply to x and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write $y = f(x)$ and proceed to switch x and y

$$\begin{array}{rcll} y & = & f(x) & \\ y & = & \dfrac{1-2x}{5} & \\ x & = & \dfrac{1-2y}{5} & \text{switch } x \text{ and } y \\ 5x & = & 1-2y & \\ 5x-1 & = & -2y & \\ \dfrac{5x-1}{-2} & = & y & \\ y & = & -\frac{5}{2}x+\frac{1}{2} & \end{array}$$

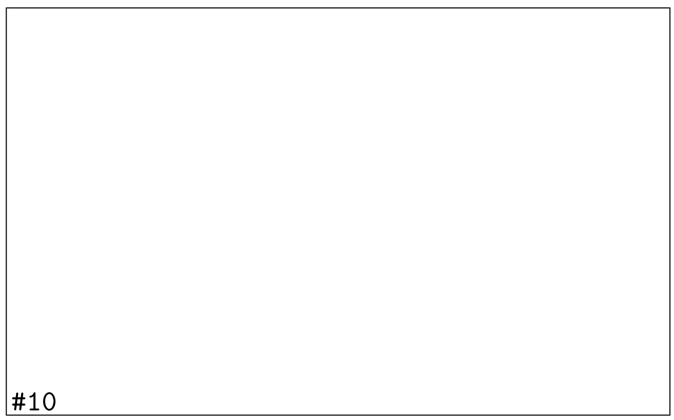
We have $f^{-1}(x) = -\frac{5}{2}x+\frac{1}{2}$. To check this answer analytically, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{array}{rcl} \left(f^{-1} \circ f\right)(x) & = & f^{-1}(f(x)) \\ & = & -\frac{5}{2}f(x)+\frac{1}{2} \\ & = & -\frac{5}{2}\left(\dfrac{1-2x}{5}\right)+\frac{1}{2} \\ & = & -\frac{1}{2}(1-2x)+\frac{1}{2} \\ & = & -\frac{1}{2}+x+\frac{1}{2} \\ & = & x \checkmark \end{array}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the range of f which is also all real numbers. (Recall that the domain of f^{-1}) is the range of f.)

$$
\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f(f^{-1}(x)) \\
&= \frac{1 - 2f^{-1}(x)}{5} \\
&= \frac{1 - 2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} \\
&= \frac{1 + 5x - 1}{5} \\
&= \frac{5x}{5} \\
&= x \checkmark
\end{aligned}
$$

To check our answer graphically, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.[5] They appear to be reflections across the line $y = x$.



2. To find $g^{-1}(x)$, we start with $y = g(x)$. We note that the domain of g is $(-\infty, 1) \cup (1, \infty)$.

$$
\begin{aligned}
y &= g(x) \\
y &= \frac{2x}{1-x} \\
x &= \frac{2y}{1-y} && \text{switch } x \text{ and } y \\
x(1-y) &= 2y \\
x - xy &= 2y \\
x &= xy + 2y \\
x &= y(x+2) && \text{factor} \\
y &= \frac{x}{x+2}
\end{aligned}
$$

[5] Note that if you perform your check on a calculator for more sophisticated functions, you'll need to take advantage of the 'ZoomSquare' feature to get the correct geometric perspective.

We obtain $g^{-1}(x) = \frac{x}{x+2}$. To check this analytically, we first check $\left(g^{-1} \circ g\right)(x) = x$ for all x in the domain of g, that is, for all $x \neq 1$.

$$\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) \\
&= g^{-1}\left(\frac{2x}{1-x}\right) \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right)+2} \cdot \frac{(1-x)}{(1-x)} \quad \text{clear denominators} \\
&= \frac{2x}{2x+2(1-x)} \\
&= \frac{2x}{2x+2-2x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}$$

Next, we check $g\left(g^{-1}(x)\right) = x$ for all x in the range of g. From the graph of g in Example **??**, we have that the range of g is $(-\infty, -2) \cup (-2, \infty)$. This matches the domain we get from the formula $g^{-1}(x) = \frac{x}{x+2}$, as it should.

$$
\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} \quad \text{clear denominators} \\
&= \frac{2x}{(x+2)-x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}
$$

Graphing $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for $y = g^{-1}(x)$. Note that the vertical asymptote $x = 1$ of the graph of g corresponds to the horizontal asymptote $y = 1$ of the graph of g^{-1}, as it should since x and y are switched. Similarly, the horizontal asymptote $y = -2$ of the graph of g corresponds to the vertical asymptote $x = -2$ of the graph of g^{-1}.

#11

$y = g(x)$ and $\boldsymbol{y = g^{-1}(x)}$

$\square$

We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible. However, if we restrict the domain of f, we can produce a new function g which is one-to-one. If we define $g(x) = x^2$, $x \geq 0$, then we have

#12

$y = f(x) = x^2$

$\xrightarrow{\text{restrict domain to } x \geq 0}$

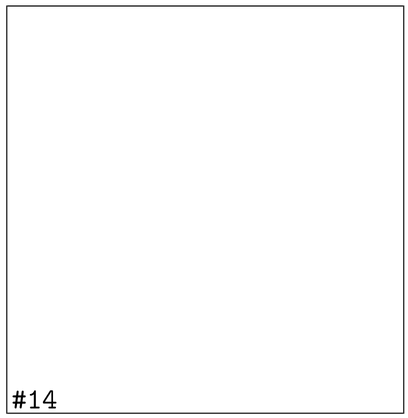
#13

$y = g(x) = x^2$, $x \geq 0$

The graph of g passes the Horizontal Line Test. To find an inverse of g, we proceed as usual

$$\begin{aligned} y &= g(x) \\ y &= x^2, \ \ x \geq 0 \\ x &= y^2, \ \ y \geq 0 \quad \text{switch } x \text{ and } y \\ y &= \pm\sqrt{x} \\ y &= \sqrt{x} \qquad\qquad \text{since } y \geq 0 \end{aligned}$$

We get $g^{-1}(x) = \sqrt{x}$. At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on g saves the day. We get $\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) = g^{-1}\left(x^2\right) = \sqrt{x^2} = |x| = x$, since $x \geq 0$. Checking $\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$. Graphing[6] g and g^{-1} on the same set of axes shows that they are reflections about the line $y = x$.



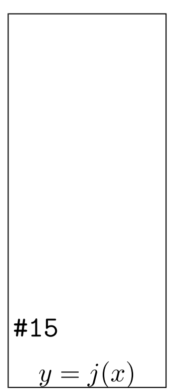
Our next example continues the theme of domain restriction.

Example 1.7.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(x) = \sqrt{x+2} - 1$

Solution.

1. The function j is a restriction of the function h from Example **??**. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. We see that the graph of j passes the Horizontal Line Test and thus j is invertible.



[6]We graphed $y = \sqrt{x}$ in Section **??**.

We now use our algorithm to find $j^{-1}(x)$.

$$\begin{array}{rcll}
y & = & j(x) & \\
y & = & x^2 - 2x + 4, \;\; x \leq 1 & \\
x & = & y^2 - 2y + 4, \;\; y \leq 1 & \text{switch } x \text{ and } y \\
0 & = & y^2 - 2y + 4 - x & \\
y & = & \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c = 4 - x \\
y & = & \dfrac{2 \pm \sqrt{4x - 12}}{2} & \\
y & = & \dfrac{2 \pm \sqrt{4(x-3)}}{2} & \\
y & = & \dfrac{2 \pm 2\sqrt{x-3}}{2} & \\
y & = & \dfrac{2\left(1 \pm \sqrt{x-3}\right)}{2} & \\
y & = & 1 \pm \sqrt{x-3} & \\
y & = & 1 - \sqrt{x-3} & \text{since } y \leq 1.
\end{array}$$

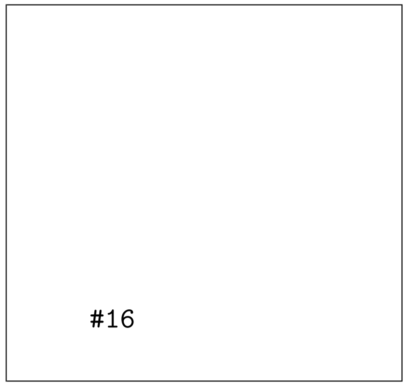
We have $j^{-1}(x) = 1 - \sqrt{x-3}$. When we simplify $\left(j^{-1} \circ j\right)(x)$, we need to remember that the domain of j is $x \leq 1$.

$$\begin{array}{rcll}
\left(j^{-1} \circ j\right)(x) & = & j^{-1}(j(x)) & \\
& = & j^{-1}\left(x^2 - 2x + 4\right), \;\; x \leq 1 & \\
& = & 1 - \sqrt{(x^2 - 2x + 4) - 3} & \\
& = & 1 - \sqrt{x^2 - 2x + 1} & \\
& = & 1 - \sqrt{(x-1)^2} & \\
& = & 1 - |x - 1| & \\
& = & 1 - (-(x-1)) & \text{since } x \leq 1 \\
& = & x \; \checkmark &
\end{array}$$

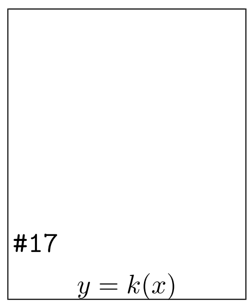
Checking $j \circ j^{-1}$, we get

$$\begin{array}{rcl}
\left(j \circ j^{-1}\right)(x) & = & j\left(j^{-1}(x)\right) \\
& = & j\left(1 - \sqrt{x-3}\right) \\
& = & \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
& = & 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 + 2\sqrt{x-3} + 4 \\
& = & 3 + x - 3 \\
& = & x \; \checkmark
\end{array}$$

We can use what we know from Section **??** to graph $y = j^{-1}(x)$ on the same axes as $y = j(x)$ to get



2. We graph $y = k(x) = \sqrt{x+2} - 1$ using what we learned in Section **??** and see k is one-to-one.



We now try to find k^{-1}.

$$\begin{aligned}
y &= k(x) \\
y &= \sqrt{x+2} - 1 \\
x &= \sqrt{y+2} - 1 \quad \text{switch } x \text{ and } y \\
x + 1 &= \sqrt{y+2} \\
(x+1)^2 &= \left(\sqrt{y+2}\right)^2 \\
x^2 + 2x + 1 &= y + 2 \\
y &= x^2 + 2x - 1
\end{aligned}$$

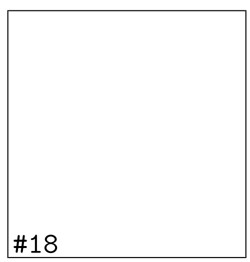
We have $k^{-1}(x) = x^2 + 2x - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem **??** tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. We now check that this works in our compositions.

$$\begin{aligned}
\left(k^{-1} \circ k\right)(x) &= k^{-1}(k(x)) \\
&= k^{-1}\left(\sqrt{x+2}-1\right), \quad x \geq -2 \\
&= \left(\sqrt{x+2}-1\right)^2 + 2\left(\sqrt{x+2}-1\right) - 1 \\
&= \left(\sqrt{x+2}\right)^2 - 2\sqrt{x+2} + 1 + 2\sqrt{x+2} - 2 - 1 \\
&= x + 2 - 2 \\
&= x \checkmark
\end{aligned}$$

and

$$\begin{aligned}
\left(k \circ k^{-1}\right)(x) &= k\left(x^2 + 2x - 1\right) \quad x \geq -1 \\
&= \sqrt{(x^2 + 2x - 1) + 2} - 1 \\
&= \sqrt{x^2 + 2x + 1} - 1 \\
&= \sqrt{(x+1)^2} - 1 \\
&= |x+1| - 1 \\
&= x + 1 - 1 && \text{since } x \geq -1 \\
&= x \checkmark
\end{aligned}$$

Graphically, everything checks out as well, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



$\square$

Our last example of the section gives an application of inverse functions.

Example 1.7.4. Recall from Section **??** that the price-demand equation for the PortaBoy game system is $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$, where x represents the number of systems sold weekly and p is the price per system in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section **??** that the weekly profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example **??**.

Solution.

1. We leave to the reader to show the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the weekly sales and furnished the price per system as the output, p^{-1} takes the price per system and returns the weekly sales as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in a week if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[7] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$. To understand what this means, recall that the original profit function P gave us the weekly profit as a function of the weekly sales. The function p^{-1} gives us the weekly sales as a function of the price. Hence, $P \circ p^{-1}$ takes as its input a price. The function p^{-1} returns the weekly sales, which in turn is fed into P to return the weekly profit. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the weekly profit (in dollars) as a function of the price per system, x, using the weekly sales $p^{-1}(x)$ as the 'middle man'.

4. We know from Section **??** that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation **??**, $x = -\frac{b}{2a} = 165$. Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example **??**, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

[7] It is good review to actually do this!

1.7.1 Exercises

1. Show that the following functions are one-to-one and find the inverse. Check your answers algebraically and graphically. Verify the range of f is the domain of f^{-1} and vice-versa.

   (a) $f(x) = 6x - 2$
   (b) $f(x) = 5x - 3$
   (c) $f(x) = 1 - \dfrac{4+3x}{5}$
   (d) $f(x) = -\sqrt{x-5} + 2$
   (e) $f(x) = \sqrt{3x-1} + 5$
   (f) $f(x) = \sqrt[5]{3x-1}$
   (g) $f(x) = x^2 - 10x,\ x \geq 5$
   (h) $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
   (i) $f(x) = x^2 - 6x + 5,\ x \leq 3$
   (j) $f(x) = 4x^2 + 4x + 1,\ x < -1$
   (k) $f(x) = \dfrac{3}{4-x}$
   (l) $f(x) = \dfrac{x}{1-3x}$
   (m) $f(x) = \dfrac{2x-1}{3x+4}$
   (n) $f(x) = \dfrac{4x+2}{3x-6}$
   (o) $f(x) = \dfrac{-3x-2}{x+3}$

2. Show that the Fahrenheit to Celsius conversion function found in Exercise **??** in Section **??** is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

3. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem **??** to help you compute $f^{-1}(1),\ f^{-1}(5),\ \text{and}\ f^{-1}(-3)$.

4. With the help of your classmates, find a formula for the inverse of the following.

   (a) $f(x) = ax + b,\ a \neq 0$
   (b) $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$
   (c) $f(x) = \frac{ax+b}{cx+d},\ a \neq 0,\ b \neq 0,\ c \neq 0,\ d \neq 0$
   (d) $f(x) = ax^2 + bx + c$ where $a \neq 0,\ x \geq -\frac{b}{2a}$.

5. Let $f(x) = \frac{2x}{x^2-1}$. Using the techniques in Section **??**, graph $y = f(x)$. Verify f is one-to-one on the interval $(-1, 1)$. Use the procedure outlined on Page **??** and your graphing calculator to find the formula for $f^{-1}(x)$. Note that since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

6. Suppose f is an invertible function. Prove that if graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at all, they do so on the line $y = x$.

7. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

8. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

9. What graphical feature must a function f possess for it to be its own inverse?

1.7.2 Answers

1. (a) $f^{-1}(x) = \dfrac{x+2}{6}$
   (b) $f^{-1}(x) = \dfrac{x+3}{5}$
   (c) $f^{-1}(x) = -\frac{5}{3}x + \frac{1}{3}$
   (d) $f^{-1}(x) = (x-2)^2 + 5,\ x \leq 2$
   (e) $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3},\ x \geq 5$
   (f) $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
   (g) $f^{-1}(x) = 5 + \sqrt{x+25}$
   (h) $f^{-1}(x) = -\sqrt{\frac{x+5}{3}} - 4$
   (i) $f^{-1}(x) = 3 - \sqrt{x+4}$
   (j) $f^{-1}(x) = -\frac{\sqrt{x}+1}{2},\ x > 1$
   (k) $f^{-1}(x) = \dfrac{4x-3}{x}$
   (l) $f^{-1}(x) = \dfrac{x}{3x+1}$
   (m) $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
   (n) $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
   (o) $f^{-1}(x) = \dfrac{-3x-2}{x+3}$

3. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$