Chapter 1

Basic Algebra

1.1 The Laws of Algebra

Terminology and Notation. In this section we review the notations used in algebra. Some are peculiar to these notes. For example the notation $A := B$ indicates that the equality holds *by definition* of the notations involved. (See for example Paragraph **??** which follows.) Two other notations which will become important when we solve equations are $\implies$ and $\iff$. The notation $P \implies Q$ means that P *implies* Q i.e. *"If P, then Q"*. For example, $x = 2 \implies x^2 = 4$. (Note however that the converse statement $x^2 = 4 \implies x = 2$ is not always true since it might be that $x = -2$.) The notation $P \iff Q$ means $P \implies Q$ and $Q \implies P$, i.e. *"P if and only if Q"*. For example $3x - 6 = 0 \iff x = 2$. The notations $\implies$ and $\iff$ are explained more carefully in Paragraphs **??** and **??** below.

Implicit Multiplication. In mathematics the absence of an operation symbol usually indicates multiplication: ab mean $a \times b$. Sometimes a dot is used to indicate multiplication and in computer languages an asterisk is often used.

$$ab := a \cdot b := a * b := a \times b$$

Order of operations. Parentheses are used to indicate the order of doing the operations: in evaluating an expression with parentheses the innermost matching pairs are evaluated first as in

$$((1+2)^2+5)^2 = (3^2+5)^2 = (9+5)^2 = 14^2 = 196.$$

There are conventions which allow us not to write the parentheses. For example, multiplication is done before addition

$$ab + c \qquad \text{means } (ab) + c \text{ and not } a(b+c),$$

and powers are done before multiplication:

$$ab^2c \qquad \text{means } a(b^2)c \text{ and not } (ab)^2c.$$

In the absence of other rules and parentheses, the left most operations are done first.

$$a - b - c \qquad \text{means } (a-b) - c \text{ and not } a - (b-c).$$

The long fraction line indicates that the division is done last:

$$\frac{a+b}{c} \qquad \text{means } (a+b)/c \text{ and not } a + (b/c).$$

In writing fractions the length of the fraction line indicates which fraction is evaluated first:

$$\frac{a}{\frac{b}{c}} \qquad \text{means } a/(b/c) \text{ and not } (a/b)/c,$$

$$\frac{\frac{a}{b}}{c} \qquad \text{means } (a/b)/c \text{ and not } a/(b/c).$$

The length of the horizontal line in the radical sign indicates the order of evaluation:

$$\sqrt{a+b} \qquad \text{means } \sqrt{(a+b)} \text{ and not } (\sqrt{a})+b.$$

$$\sqrt{a}+b \qquad \text{means } (\sqrt{a})+b \text{ and not } \sqrt{(a+b)}.$$

The Laws of Algebra. There are four fundamental operations which can be performed on numbers.

1. Addition. The **sum** of a and b is denoted $a+b$.
2. Multiplication. The **product** of a and b is denoted ab.
3. Reversing the sign. The **negative** of a is denoted $-a$.
4. Inverting. The **reciprocal** of a (for $a \neq 0$) is denoted by a^{-1} or by $\frac{1}{a}$.

These operations satisfy the following laws.

Associative	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Commutative	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Identity	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Inverse	$a+(-a)=(-a)+a=0$	$a\cdot a^{-1}=a^{-1}\cdot a=1$
Distributive	$a(b+c)=ab+ac$	$(a+b)c=ac+bc$

The operations of **subtraction** and **division** are then defined by

$$a-b := a+(-b) \qquad\qquad a\div b := \frac{a}{b} := a\cdot b^{-1} = a\cdot\frac{1}{b}.$$

All the rules of calculation that you learned in elementary school follow from the above fundamental laws. In particular, the Commutative and Associative Laws say that you can add a bunch of numbers in any order and similarly you can multiply a bunch of numbers in any order. For example,

$$(A+B)+(C+D)=(A+C)+(B+D), \qquad (A\cdot B)\cdot(C\cdot D)=(A\cdot C)\cdot(B\cdot D).$$

Because both addition and multiplication satisfy the commutative, associative, identity, and inverse laws, there are other analogies:

(i)	$-(-a) = a$	$(a^{-1})^{-1} = a$
(ii)	$-(a+b) = -a-b$	$(ab)^{-1} = a^{-1}b^{-1}$
(iii)	$-(a-b) = b-a$	$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$
(iv)	$(a-b)+(c-d) = (a+c)-(b+d)$	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
(v)	$a-b = (a+c)-(b+c)$	$\frac{a}{b} = \frac{ac}{bc}$
(vi)	$(a-b)-(c-d) = (a-b)+(d-c)$	$\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$

These identities[1] are proved in the Guided Exercises. Here are some further identities which are proved using the distributive law.

(i) $a \cdot 0 = 0$	**(ii)** $-a = (-1)a$
(iii) $a(-b) = -ab$	**(iv)** $(-a)(-b) = ab$
(v) $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$	**(vi)** $(a+b)(c+d) = ab+ad+bc+bd$
(vii) $(a+b)^2 = a^2+2ab+b^2$	**(viii)** $(a+b)(a-b) = a^2-b^2$

An important consequence of the fact that $a \cdot 0 = 0 \cdot a = 0$ is the following **Zero-Product Property**. It is used to solve equations.

[1] An **identity** is an equation which is true for all values of the variables which appear in it.

Zero-Product Property

$$pq = 0 \iff p = 0 \text{ or } q = 0 \text{ (or both).}$$

Proof: If $p = 0$ (or $q = 0$) then $pq = 0$. Conversely, if $p \neq 0$, then $q = p^{-1}pq = p^{-1}0 = 0$. $\square$

DEFINITION 1.1. For a natural number n and any number a the nth **power** of a is

$$a^n := \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

The zeroth power is

$$a^0 := 1$$

and negative powers are defined by

$$a^{-n} := \frac{1}{a^n}.$$

The Laws of Exponents. The following laws are easy to understand when m and n are integers. In Theorem **??** below we will learn that these laws also hold whenever a and b are positive real numbers and m and n are *any* real numbers, not just integers.

(i)	$a^m a^n = a^{m+n}$	e.g.	$a^2a^3 = (aa)(aaa) = a^5$
(ii)	$(a^m)^n = a^{mn}$	e.g.	$(a^2)^3 = (aa)(aa)(aa) = a^6$
(iii)	$\dfrac{a^m}{a^n} = a^{m-n}$	e.g.	$\dfrac{a^2}{a^5} = a^{-3} = \dfrac{1}{a^3}$
(iv)	$(ab)^m = a^m b^m$	e.g.	$(ab)^2 = (ab)(ab) = (aa)(bb) = a^2b^2$
(v)	$\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	e.g.	$\left(\dfrac{a}{b}\right)^2 = \dfrac{a}{b} \cdot \dfrac{a}{b} = \dfrac{aa}{bb} = \dfrac{a^2}{b^2}$

EXAMPLE 1.1.1. Simplify $\left(\dfrac{1}{a} + \dfrac{1}{b}\right)^{-1}$

SOLUTION.

$$\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = \frac{1}{\dfrac{1}{a} + \dfrac{1}{b}} = \frac{ab}{\left(\dfrac{1}{a} + \dfrac{1}{b}\right)ab} = \frac{ab}{\dfrac{ab}{a} + \dfrac{ab}{b}} = \frac{ab}{b+a}.$$

Three Column Calculations. An algebraic calculation often involves substituting expressions for letters in general laws. To avoid making mistakes it is advisable to arrange the computation neatly and use equal signs between quantities which you assert are equal. When you check your work, ask yourself at each step what general principle you used and how you substituted into that general expression. The following **three column calculation** illustrates this technique.[2]

step	by	with
$\frac{1}{3}+\frac{2}{x} = \frac{x}{3x}+\frac{2}{x}$	$\frac{A}{B}=\frac{AC}{CB}$	$A=1,\ B=3,\ C=x$
$= \frac{x}{3x}+\frac{6}{3x}$	$\frac{A}{B}=\frac{CA}{CB}$	$A=2,\ B=x,\ C=3$
$= \frac{x+6}{3x}$	$\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C}$	$A=x,\ B=6,\ C=3x$

When in Doubt. If you are in doubt as to whether some general equation is true you can plug in numbers: if the two side of the equation are not equal the general equation is false.[3] Thus, in general,

$$(A+B)^2 \neq A^2+B^2$$

because when $A=2$ and $B=3$ we have $(A+B)^2=(2+3)^2=5^2=25$ but $A^2+B^2=2^2+3^2=4+9=13$ and $25\neq 13$. The correct general law is

$$(A+B)^2 = A^2+2AB+B^2$$

(when $A=2$ and $B=3$ $A^2+2AB+B^2=4+12+9=25=5^2=(2+3)^2$) and this shows that $(A+B)^2=A^2+B^2 \implies 2AB=0$ so that $(A+B)^2=A^2+B^2$ only when $A=0$ or $B=0$ (or both).

1.1.1 Exercises

1. True or false?

   (i) $\dfrac{a+b}{c}=\dfrac{a}{c}+\dfrac{b}{c}$?

   (ii) $\dfrac{c}{a+b}=\dfrac{c}{a}+\dfrac{c}{b}$?

   (iii) $(a+b)/c=(a/c)+(b/c)$?

   (iv) $c/(a+b)=(c/a)+(c/b)$?

   (v) $\dfrac{a+b}{c+d}=\dfrac{a}{c}+\dfrac{b}{d}$?

   (vi) $\dfrac{a\cdot b}{c\cdot d}=\dfrac{a}{c}\cdot\dfrac{b}{d}$?

2. Factor:

   (a) $2x^3-12x^2+6x=$

   (b) $18a^4b^2-30a^3b^3=$

   (c) $x^2-a^2=$

   (d) $4x^2-\dfrac{1}{9}=$

[2] We *don't* expect this level of detail on exams; we *do* expect you to do calculations like this without making mistakes.

[3] But maybe not conversely!

3. Simplify:

(a) $\dfrac{x^3-9x}{x^2+6x+9}$ (b) $\dfrac{x^2+2x-3}{x^2+4x+4}\cdot\dfrac{x^2-4}{x^2+4x-5}$

(c) $\dfrac{\left(\dfrac{1}{a}-2\right)}{\left(\frac{1}{a^2}-4\right)}$ (d) $\dfrac{\left(\dfrac{1}{ab}+\dfrac{2}{ab^2}\right)}{\left(\frac{3}{a^3b}-\frac{4}{ab}\right)}$

1.1.2 Answers

1. **(i)** True. **(ii)** False.: $\dfrac{1}{2+3}\neq\dfrac{1}{2}+\dfrac{1}{3}$

   (iii) True: $1/A=\dfrac{1}{A}$. **(iv)** False.

   (v) False: $\dfrac{2+3}{4+5}=\dfrac{5}{9}, \quad \dfrac{2}{3}+\dfrac{4}{5}=\dfrac{22}{15}$. **(vi)** True.

2. (a) $2x^3-12x^2+6x=l2x(x^2-6x+3)$ (b) $18a^4b^2-30a^3b^3 = 6a^3b^2(3a-5b)$

   (c) $x^2-a^2=(x-a)(x+a)$ (d) $4x^2-\dfrac{1}{9}=\left(2x-\dfrac{1}{3}\right)\left(2x+\dfrac{1}{3}\right)$

3. (a) $\dfrac{x^3-9x}{x^2+6x+9}=\dfrac{x(x-3)}{x+3}$ (b) $\dfrac{x^2+2x-3}{x^2+4x+4}\cdot\dfrac{x^2-4}{x^2+4x-5}=\dfrac{(x+3)(x-2)}{(x+2)(x+5)}$

   (c) $\dfrac{\left(\dfrac{1}{a}-2\right)}{\left(\dfrac{1}{a^2}-4\right)}=\dfrac{a}{1+2a}$ (d) $\dfrac{\left(\dfrac{1}{ab}+\dfrac{2}{ab^2}\right)}{\left(\dfrac{3}{a^3b}-\dfrac{4}{ab}\right)}=\dfrac{a^2(b+2)}{b(3-4a^2)}$

1.2 Kinds of Numbers

We distinguish the following different kinds of numbers.

- The **natural numbers** are $1, 2, 3\ldots$.
- The **integers** are $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$.
- The **rational numbers** are ratios of integers like $3/2$, $14/99$, $-1/2$.
- The **real numbers** are numbers which have an infinite decimal expansion like

$$\frac{3}{2}=1.5000\ldots,\qquad \frac{14}{99}=0.141414\ldots,\qquad \sqrt{2}=1.4142135623730951\ldots.$$

- The **complex numbers** are those numbers of form $z = x+iy$ where x and y are real numbers and i is a special new number called the **imaginary unit** which has the property that
$$i^2 = -1;$$

Every integer is a rational number (because $n = n/1$), every rational number is a real number (see Remark **??** below), and every real number is a complex number (because $x = x + 0i$). A real number which is not rational is called **irrational**.

New Numbers - New Solutions. Each kind of number enables us to solve equations that the previous kind couldn't solve:

- The solution of the equation $x + 5 = 3$ is $x = -2$ which is an integer but not a natural number.
- The solution of the equation $5x = 3$ is $x = \frac{3}{5}$ which is a rational number but not an integer.
- The equation $x^2 = 2$ has two solutions $x = \sqrt{2}$. The number $\sqrt{2}$ is a real number but not a rational number.
- The equation $x^2 = 4$ has two real solutions $x = \pm 2$ but the equation $z^2 = -4$ has no real solutions because the square of a nonzero real number is always positive. However it does have two complex solutions, namely $z = \pm 2i$.

We will not use complex numbers until Section **??** but may refer to them implicitly as in

The equation $x^2 = -4$ has no (real) solution.

Rational Numbers - Repeating Decimals. It will be proved in Theorem **??** that a real number is rational if and only if its decimal expansion eventually repeats periodically forever as in the following examples:

$$\frac{1}{3} = 0.3333\ldots, \qquad \frac{17}{6} = 2.83333\ldots,$$
$$\frac{7}{4} = 1.250000\ldots, \qquad \frac{22}{7} = 3.142857\,142857\,142857\ldots.$$

Unless the decimal expansion of a real number is eventually zero, as in $\frac{1}{2} = 0.5000\ldots$, any finite part of the decimal expansion is close to, but not exactly equal to, the real number. For example 1.414 is close to the square root of two but not exactly equal:

$$(1.414)^2 = 1.999396 \neq 2, \qquad (\sqrt{2})^2 = 2.$$

If we compute the square root to more decimal places we get a better approximation, but it still isn't exactly correct:

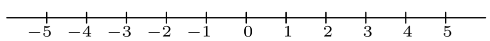
$$(1.4142135623730951)^2 = 2.0000000000000001448106923536 4401.$$

The square root of 2 is Irrational. Here is a proof that $\sqrt{2}$ is irrational. If it were rational there would be integers m and n with

$$\left(\frac{m}{n}\right)^2 = 2.$$

By canceling common factors we may assume that m and n have no common factors and hence that they are not both even. Now $m^2 = 2n^2$ so m^2 is even so m is even, say $m = 2p$. Then $4p^2 = (2p)^2 = m^2 = 2n^2$ so $2p^2 = n^2$ so n^2 is even so n is even. This contradicts the fact m and n are not both even.

The Number Line. The choice of two points (representing 0 and 1) on a line determines a correspondence between the points of the line and the real numbers as indicated in the following picture.



The correspondence is called a **coordinate system** on the line. The line is called a **number line**. When the point A corresponds to the number a we say that the number a is the **coordinate** of the point A. The **positive numbers** are the real numbers on the same side of 0 as 1 and the **negative numbers** are on the other side. We usually draw the number line as above so that it is horizontal and 1 is to the right of 0. We write say a is **less than** b and write $a < b$ b is to the right of a, i.e. when $b - a$ is positive. it is equivalent to say that b is **greater than** a or a to the left of b and to write $b > a$. The notation $a \leq b$ means that a is less than or equal to b i.e. either $a < b$ or else $a = b$. Similarly, $b \geq a$ means that b is greater than or equal to a i.e. either $b > a$ or else $b = a$. Thus when $a < b$, a number [1]

$$c \text{ is between } a \text{ and } b \iff a < c < b.$$

Sometimes we insert the word **strictly** for emphasis: a is strictly less than b means that $a < b$ (not just $a \leq b$).

Order. The **order relation** just described is characterized by the following.

(Trichotomy) Every real number is either positive, negative, or zero (and no number satisfies two of these conditions).

(Sum) The sum of two positive numbers is positive.

(Product) The product of two positive numbers is positive.

This characterization together with the notation explained in the previous paragraph implies the following:

[1]The notation $\iff$ is an abbreviation for "if and only if".

(i) Either $a < b$, $a = b$, or $a > b$.

(ii) If $a < b$ and $b < c$, then $a < c$.

(iii) If $a < b$, then $a + c < b + c$.

(iv) If $a < b$ and $c > 0$, then $ac < bc$.

(v) If $a < b$ and $c < 0$, then $ac > bc$.

(vi) If $0 < a < b$, then $0 < \dfrac{1}{b} < \dfrac{1}{a}$.

Interval Notation. The **open interval** (a, b) is the set of all real numbers x such that $a < x < b$, and the **closed interval** $[a, b]$ is the set of all real numbers x such that $a \le x \le b$. Thus

$$x \text{ is in the set } (a, b) \iff a < x < b$$

and

$$x \text{ is in the set } [a, b] \iff a \le x \le b.$$

These notations are extended to include **half open intervals** and **unbounded intervals** as in

$$\begin{aligned} x \text{ is in the set } (a, b] \quad &\iff \quad a < x \le b, \\ x \text{ is in the set } (a, \infty) \quad &\iff \quad a < x, \\ x \text{ is in the set } (-\infty, a] \quad &\iff \quad x \le a, \qquad \text{etc.} \end{aligned}$$

The **union** symbol $\cup$ is used to denote a set consisting of more than one interval as in

$$x \text{ is in the set } (a, b) \cup (c, \infty) \iff \text{ either } a < x < b \text{ or else } c < x.$$

The symbol ∞ is pronounced **infinity** and is used to indicate that an interval is unbounded. It is not a number so we never write $(c, \infty]$.

Example 1.2.1. Which is bigger: π or $\sqrt{10}$? (Don't use a calculator.)
Solution. $\pi = 3.14\ldots < 3.15$. and

$$3.15^2 = (3 + 0.15)^2 = 3^2 + 2 \times 3 \times 0.15 + 0.15^2 = 9 + 0.90 + 0.0225 = 9.9225 < 10$$

so $\pi < 3.15 < \sqrt{10}$.

1.2.1 Exponents

The proof of the following theorem requires a more careful definition of the set of real numbers than we have given and is best left for more advanced courses.

Theorem 1.1. Suppose that a is a positive real number. Then there is one and only one way to define a^x for all real numbers x such that

(i) $a^{x+y} = a^x \cdot a^y, \quad a^0 = 1, \quad a^1 = a, \quad 1^x = 1.$

(ii) If $a > 1$ and $x < y$ then $a^x < a^y$.

(iii) If $a < 1$ and $x < y$ then $a^x > a^y$.

With this definition, the laws of exponents in Paragraph **??** continue to hold when a and b are positive real numbers and m and n are arbitrary real numbers. The number a^x is positive (when a is positive) regardless of the sign of x.

In particular by property (v) in Paragraph **??** we have $(a^x)^y = a^{xy}$ so $(a^{m/n})^n = a^m$ and $(a^m)^{1/n} = a^{m/n}$. Hence for positive numbers a and b we have

$$b = a^{m/n} \iff b^n = a^m.$$

When $m = 1$ and n is a natural number the number $a^{1/n}$ is called the nth **root** (**square root** if $n = 2$ and **cube root** if $n = 3$) and is sometimes denoted

$$\sqrt[n]{a} := a^{1/n}.$$

When n is absent, $n = 2$ is understood:

$$\sqrt{a} := a^{1/2}.$$

A number b is said to be an n**th root** of a iff $b^n = a$. When n is odd, every real number a has exactly one (real) nth root and this is denoted by $\sqrt[n]{a}$. When n is even, a positive real number a has two (real) nth roots (and $\sqrt[n]{a}$ denotes the one which is positive) but a negative number has no real nth roots. (In trigonometry it is proved that every nonzero complex number has exactly n distinct complex nth roots.)

The equation $b^2 = 9$ has two solutions, namely $b = 3$ and $b = -3$ and each is “a” square root of 9 but only $b = 3$ is “the” square root of 9. However -2 is the (only) real cube root of -8 because $(-2)^3 = -8$. The number -9 has no real square root (because $b^2 = (-b)^2 > 0$ if $b \neq 0$) but does have two complex square roots (because $(3i)^2 = (-3i)^2 = -9$). For most of this book[1] we only use real numbers and we say that

[1] More precisely until Chapter **??**

$$\sqrt{a} \text{ is undefined when } a < 0$$

and that

you can't take the square root of a negative number.

Also $\sqrt{a}$ always denotes the nonnegative square root: thus $(-3)^2 = 3^2 = 9$ but $\sqrt{9} = 3$ and $\sqrt{9} \neq -3$.

1.2.2 Absolute Value

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. You may have been taught that $|x|$ is the distance from the real number x to the 0 on the number. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line.

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Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

Definition 1.2. The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

In Definition **??**, we define $|x|$ using a piecewise-defined function. (See page **??** in Section **??**.) To check that this definition agrees with what we previously understood as absolute value, note that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$. This is one of the times when it's best to interpret the expression '$-x$' as 'the opposite of x' as opposed to 'negative x.' Before we embark on studying absolute value functions, we remind ourselves of the properties of absolute value.

THEOREM 1.2. **Properties of Absolute Value:** Let a, b, and x be real numbers and let n be an integer.[a] Then

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided $b \neq 0$
- **The Triangle Inequality:** $|a + b| \leq |a| + |b|$
- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $x = -c$.
- For $c < 0$, $|x| = c$ has no solution.

[a] Recall that this means $n = 0, \pm 1, \pm 2, \ldots$.

The proof of the Product and Quotient Rules in Theorem **??** boils down to checking four cases: when both a and b are positive; when they are both negative; when one is positive and the other is negative; when one or both are zero. For example, suppose we wish to show $|ab| = |a||b|$. We need to show this equation is true for all real numbers a and b. If a and b are both positive, then so is ab. Hence, $|a| = a$, $|b| = b$, and $|ab| = ab$. Hence, the equation $|ab| = |a||b|$ is the same as $ab = ab$ which is true. If both a and b are negative, then ab is positive. Hence, $|a| = -a$, $|b| = -b$, and $|ab| = ab$. The equation $|ab| = |a||b|$ becomes $ab = (-a)(-b)$, which is true. Suppose a is positive and b is negative. Then ab is negative, and we have $|ab| = -ab$, $|a| = a$ and $|b| = -b$. The equation $|ab| = |a||b|$ reduces to $-ab = a(-b)$ which is true. A symmetric argument shows the equation $|ab| = |a||b|$ holds when a is negative and b is positive. Finally, if either a or b (or both) are zero, then both sides of $|ab| = |a||b|$ are zero, and so the equation holds in this case, too. All of this rhetoric has shown that the equation $|ab| = |a||b|$ holds true in all cases. The proof of the Quotient Rule is very similar, with the exception that $b \neq 0$. The Power Rule can be shown by repeated application of the Product Rule. The last three properties can be proved using Definition **??** and by looking at the cases when $x \geq 0$, in which case $|x| = x$, or when $x < 0$, in which case $|x| = -x$. For example, if $c > 0$, and $|x| = c$, then if $x \geq 0$, we have $x = |x| = c$. If, on the other hand, $x < 0$, then $-x = |x| = c$, so $x = -c$. The remaining properties are proved similarly and are left as exercises.

1.2.3 Solving Equations

DEFINITION 1.3. A number a is called a **solution** of an equation containing the variable x if the equation becomes a true statement when a is substituted for x. A solution of an equation is sometimes also called a **root** of the equation. Two equations are said to be **equivalent** iff they have exactly the same solutions. We will sometimes use the symbol $\iff$ to indicate that two equations are equivalent.

Usually two equations are equivalent because one can be obtained from the other by performing an operation to both sides of the equation which can be reversed by another operation of the same kind. For example, the equations $3x + 7 = 13$ and $x = 2$ are equivalent because

$$\begin{aligned} 3x + 7 = 13 \iff & \quad 3x = 6 \text{ (subtract 7 from both sides)}, \\ \iff & \quad x = 2 \text{ (divide both sides by 3)}. \end{aligned}$$

The reasoning is reversible: we can go from $x = 2$ to $3x = 6$ by multiplying both sides by by 3 and from $3x = 6$ to $3x + 7 = 13$ by adding 7 to both sides.

We use the symbol $\implies$ when we want to assert that one equation **implies** another but do not want to assert the converse. The guiding principal here is

> *If an equation E' results from an equation E by performing the same operation to both sides, then $E \implies E'$, i.e. every solution of E is a solution of E'.*

If the operation is not "reversible" as explained above, there is the possibility that the set of solutions gets bigger in which case the new solutions are called **extraneous solutions**. (They do not solve the original equation.) The simplest example of how an extraneous solution can arise is

$$x = 3 \implies x^2 = 9 \qquad \text{(square both sides)}$$

but the operation of squaring both sides is not reversible: it is incorrect to conclude that $x^2 = 9$ implies that $x = 3$. What *is* correct is that $x^2 = 9 \iff x = \pm 3$, i.e. *either* $x = 3$ *or else* $x = -3$. When solving an equation you may use operations which are not reversible provided that you

$$\boxed{\textit{Always check your answer!}}$$

In addition to catching mistakes, this will show you which – if any – of the solutions you found are extraneous.

Here are two ways in which extraneous solutions can arise:

(i) Squaring both sides of an equation.

(ii) Multiplying both sides of of an equation by a quantity not known to be nonzero.

As an example of (i) consider the equation

$$\sqrt{10 - x} = -x - 2.$$

Squaring both sides gives the quadratic equation $10 - x = x^2 + 4x + 4$ which has two solutions $x = -6$ and $x = 1$. Now $\sqrt{10 - (-6)} = -(-6) - 2$ but $\sqrt{10 - 1} \neq -1 - 2$ (Remember that $\sqrt{\ }$ means the positive square root.) Thus $x = -6$ is the only solution of the original equation and $x = 1$ is an extraneous solution.

As an example of (ii) consider

$$\frac{1}{x-1} = 2 + \frac{1}{x-1}.$$

This equation has no solution: if it did we would subtract $(x-1)^{-1}$ from both sides and deduce that $0 = 2$ which is false. But if we multiply both sides by $x - 1$ we get $1 = 2(x-1) + 1$ which has the (extraneous) solution $x = 1$.

1.2.4 Exercises

1. Use the properties of exponents to rewrite the given expression as a simple fraction with only positive exponents.

   (a) $\left(\frac{k^{-4}s^2}{k^9 s^{-7}}\right)^9$

2. Reduce the fraction $\frac{s+t}{sx^4 + tx^4}$ to lowest terms.

1.2.5 Answers

1. (a) $\left(\frac{k^{-4}s^2}{k^9 s^{-7}}\right)^9 = \left(\frac{s^9}{k^{13}}\right)^9 = \frac{s^{81}}{k^{117}}$.

2. $\frac{s+t}{sx^4 + tx^4} = \frac{s+t}{x^4(s+t)} = \frac{1}{x^4}$.