Appendix A

The Laws of Algebra Proved

A.1 The Laws of Algebra

The Laws of Algebra. There are four fundamental operations which can be performed on numbers.

1. Addition. The **sum** of a and b is denoted $a+b$.

2. Multiplication. The **product** of a and b is denoted ab.

3. Reversing the sign. The **negative** of a is denoted $-a$.

4. Inverting. The **reciprocal** of a (for $a \neq 0$) is denoted by a^{-1} or by $\dfrac{1}{a}$.

These operations satisfy the following laws.

Associative	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Commutative	$a+(b+c)=(a+b)+c$	$a(bc)=(ab)c$
Identity	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$
Inverse	$a+(-a)=(-a)+a=0$	$a\cdot a^{-1}=a^{-1}\cdot a=1$
Distributive	$a(b+c)=ab+ac$	$(a+b)c=ac+bc$

The operations of **subtraction** and **division** are then defined by

$$a-b := a+(-b) \qquad\qquad a \div b := \frac{a}{b} := a\cdot b^{-1} = a\cdot\frac{1}{b}.$$

All the rules of calculation that you learned in elementary school follow from the above fundamental laws. In particular, the Commutative and Associative Laws say that you can add a bunch of numbers in any order and similarly you can multiply a bunch of numbers in any order. For example,

$$(A+B)+(C+D)=(A+C)+(B+D), \qquad (A\cdot B)\cdot(C\cdot D)=(A\cdot C)\cdot(B\cdot D).$$

Because both addition and multiplication satisfy the commutative, associative, identity, and inverse laws, there are other analogies:

(i)	$-(-a) = a$	$(a^{-1})^{-1} = a$
(ii)	$-(a+b) = -a-b$	$(ab)^{-1} = a^{-1}b^{-1}$
(iii)	$-(a-b) = b-a$	$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$
(iv)	$(a-b)+(c-d) = (a+c)-(b+d)$	$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$
(v)	$a-b = (a+c)-(b+c)$	$\frac{a}{b} = \frac{ac}{bc}$
(vi)	$(a-b)-(c-d) = (a-b)+(d-c)$	$\frac{a/b}{c/d} = \frac{a}{b}\cdot\frac{d}{c}$

Here are the proofs. Notice how each proof of an addition rule is immediately followed by the proof for the corresponding multiplication rule.

Theorem A.1. Inverses are unique: $b+a=0 \implies b=-a$; similarly $b\cdot a = 1 \implies b = a^{-1}$.

Proof. Assume that $b+a=0$. Then

step	by	with
$b = b+0$	$A = A+0$	$A = b$
$= b+\big(a+(-a)\big)$	$A+(-A) = 0$	$A = a$
$= \big(b+a\big)+(-a)$	$A+(B+C) = (A+B)+C$	$A=b$, $B=a$, $C=-a$
$= 0+(-a)$	$b+a=0$	by hypothesis
$= -a$	$0+A = A$	$A = -a$

Similarly assume that $b\cdot a^{-1} = 1$. Then

step	by	with
$b = b\cdot 1$	$A = A\cdot 1$	$A = b$
$= b\cdot\big(a\cdot a^{-1}\big)$	$A\cdot A^{-1} = 1$	$A = a$
$= \big(b\cdot a\big)\cdot a^{-1}$	$A\cdot(B\cdot C) = (A\cdot B)\cdot C$	$A=b$, $B=a$, $C=a^{-1}$
$= 1\cdot a^{-1}$	$b\cdot a = 1$	by hypothesis
$= a^{-1}$	$1\cdot A = A$	$A = a^{-1}$

□

Theorem A.2. $-(-a) = a$ and similarly $(a^{-1})^{-1} = a$.

Proof.

step	by	with
$a + (-a) = 0$	$A + (-A) = 0$	$A = a$
$(-a) + a = 0$	$A + B = B + A$	$A = a,\ B = -a$
$a = -(-a)$	$A + B = 0 \implies B = -A$	$A = -a,\ B = a$

step	by	with
$a \cdot a^{-1} = A$	$A \cdot A^{-1}$	$A = a$
$a^{-1} \cdot a = 1$	$A \cdot B = B \cdot A$	$A = a,\ B = a^{-1}$
$a = -(-a)$	$A \cdot B = 1 \implies B = A^{-1}$	$A = a^{-1},\ B = a$

□

Here are some further identities which are proved using the distributive law.

(i) $a \cdot 0 = 0$

(ii) $-a = (-1)a$

(iii) $a(-b) = -ab$

(iv) $(-a)(-b) = ab$

(v) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + cb}{bd}$

(vi) $(a + b)(c + d) = ab + ad + bc + bd$

(vii) $(a + b)^2 = a^2 + 2ab + b^2$

(viii) $(a + b)(a - b) = a^2 - b^2$