# Contents

# Lesson 1: Types of Numbers

**Sets of Numbers**

- **Natural numbers**: the counting numbers, starting with 1.
  Examples:
- **Integers**: the natural numbers with their negatives and 0.
  Examples:
- **Rational numbers**: fractions involving integers, which includes repeated decimals.
  Examples:
- **Real numbers**: any number with a decimal representation.
- **Irrational numbers**: real numbers which are not fractions.
  Examples:
- **The imaginary number**: $i$ is the imaginary number, $i = \sqrt{-1}$. (More later on this)
- **Complex numbers**: numbers of the form $a + bi$, where $a, b$ are real. (More later on this)

**Example.** Identify which of the following real numbers belong to the listed sets.

$$-4, 10, 0, \frac{4}{2}, \frac{6}{5}, \sqrt{16}, \frac{1}{\pi}, \sqrt[3]{-8}, \sqrt[3]{2}$$

Natural numbers ______________________________

Integers ______________________________

Rational numbers ______________________________

Irrational numbers ______________________________

Real numbers ______________________________

## Intervals

The set consisting of all real numbers $x$ such that $a < x < b$ is the **open interval** $(a, b)$.

The set consisting of all real numbers $x$ such that $a \leq x \leq b$ is the **closed interval** $[a, b]$.

**Example.** Use the number line to describe the following intervals:

- $[1, 2]$
- $(0, 3)$
- $[1, 2)$
- $(1, 2]$
- $(-\infty, 2]$
- $(3, \infty)$
- $(-\infty, -.001)$
- $(-\infty, \infty)$
- $(-\infty.0] \cup [2, \infty)$
- $(-\infty, 0] \cup (2, 3]$

**Absolute Value.**

The **absolute value** of a real number $x$ is its **distance** from zero on the number line.

**Example.** Simplify the following:

- $|-14|$
- $|10|$
- $|4-\pi|$
- $|2-e|$
- $|x-3|$, $x \leq 3$
- $|x-4|$, $x \geq 4$

The **distance** between $a$ and $b$ on the number line is $|a-b|$ or $|b-a|$.

**Example.** Express each of the following sets using absolute value:

- The distance between $x$ and $y$ is 2.
- The distance between $y$ and $x$ is 2.
- The distance between $x$ and 4 is $a$.
- The distance between $x$ and $z$ is strictly less than 2.
- The distance between $a$ and $b$ is $c$ or greater.

### Solving Equations

**Example.** Solve the following equations for $x$:

- $12x - 4 = 68$
- $2[3 - (2x - 1)] = 4.$
- $\dfrac{x}{2} + \dfrac{2x}{3} = 2.$
- $a = \dfrac{x}{3 + x}$

# Lesson 2: Exponents

**Integer Exponents**
Suppose $a$ is a positive number.

| Property | Example |
|---|---|
| $a^0 =$ | $2^0 =$ |
| $a^1 =$ | $2^1 =$ |
| $a^{-1} =$ | $2^{-1} =$ |
| $a^m \cdot a^n =$ | $2^3 \cdot 2^4 =$ |
| $(a \cdot b)^m =$ | $(3 \cdot 5)^2 =$ |
| $\dfrac{a^m}{a^n} =$ | $\dfrac{2^7}{2^3} =$ |
| $\left(\dfrac{a}{b}\right)^m =$ | $\left(\dfrac{3}{5}\right)^2 =$ |
| $(a^m)^n =$ | $\left(2^3\right)^4 =$ |

**Example.** Distribute and making all exponents positive:

- $(a^2b)^3$
- $(a^3b^2)^3(a^4b)^2$
- $\left(a^4b^{-2}\right)^{-1}\left(a^{-1}b^2\right)^{-3}$
- $\left(\dfrac{a^2}{b^{-3}}\right)^{-2}\left(a^{-2}\cdot\dfrac{b^{-5}}{c^3}\right)^3$

**Exponents and the $n^{\text{th}}$ root.**

| We say that $a = \sqrt[n]{b}$ if and only if $a^n = b$, and $a$ can be written as $a = b^{1/n}$. |
|---|

**Example.** Remove the radicals.

- $\sqrt[3]{8}$
- $\sqrt[3]{16}$
- $\sqrt[3]{\dfrac{1}{64}}$
- $\sqrt{\dfrac{1}{81}}$
- $\sqrt[5]{64}$
- $\sqrt[3]{a^4b^2c^9}$

**Example.** Rationalize (remove square roots from) the denominator in the following:

- $\dfrac{1}{\sqrt{2}}$
- $\dfrac{2}{\sqrt{x}}$
- $\dfrac{1}{\sqrt{2}-1}$

## Rational Exponents

The <u>expression $a^{m/n}$</u> is shorthand for

- raising $a$ to the power $m$ and
- taking the $n$th root in some order.

The <u>expression $a^{-m/n}$</u> is shorthand for

- raising $a$ to the power $m$,
- taking the $n$th root, and
- taking the reciprocal in some order.

**Example.** Rewrite the following without using factional or negative exponents.

- $a^{2/5}$
- $a^{-2/5}$
- $8^{2/3}$
- $9^{-3/2}$

**Example.** Rewrite without radicals.

- $\sqrt[4]{w\sqrt{x}}$
- $\sqrt[5]{\dfrac{a^2\sqrt[3]{b}}{c^{4/3}}}$

**Example.** Rewrite using fractional exponents, without using root symbols or any other fractions.

- $\sqrt{\sqrt{x}}$
- $\sqrt{w^2\sqrt[3]{x}}$
- $\sqrt{\dfrac{a^2b^3}{\sqrt[3]{c}}}$
- $\sqrt{\dfrac{\sqrt[3]{x}\sqrt[4]{y^3}}{\sqrt[5]{z^4}}}$

# Lesson 3: Factoring and Simplification

**Factoring Techniques.**

1. Finding a common factor:

- $2x^3 - 12x^2 + 6x =$
- $18a^4b^2 - 30a^3b^3 =$

2. A difference of squares:

- $x^2 - a^2 =$
- $4x^2 - \dfrac{1}{9} =$

3. Difference/Sum of Cubes:

- $x^3 - a^3 =$
- $8x^3 - 27 =$
- $x^3 + a^3 =$
- $x^3y^3 + \frac{1}{8} =$

## Factoring Techniques (cont.)

4. Grouping:

- $x^3 - x^2 + x - 1$
- $2x^3 - 4x^2 - x + 2$

5. Trial and Error:

- $x^2 - 2x - 3$
- - $2x^2 - 9x + 10$

A little better method:

If you want to factor $ax^2 + bx + c$,

1. Multiply $a$ and $c$ together
2. Find factors of $ac$ which add to $b$
3. Split the middle term accordingly and factor by grouping

**Example.** Factor the following quadratics:

- $x^2 - 2x - 3 =$
- $2x^2 - 9x + 10 =$
- $6x^2 - 5x - 6 =$
- $6x^2 - 35x - 6 =$

**Example.** Factor the following expressions:

- $a^3b - ab^3$
- $6x^5\left(x+1\right)^3 + 3x^6\left(x+1\right)^2$
- $x^2\sqrt{x^2+4} - \left(x^2+4\right)^{3/2}$
- $\dfrac{x^2}{\left(x^2+1\right)^{2/3}} + \sqrt[3]{x^2+1}$

### Reducing Fractions

**Example.** Reduce the following fractions:

- $\dfrac{x^3 - 9x}{x^2 + 6x + 9}$
- $\dfrac{x^2 + 2x - 3}{x^2 + 4x + 4} \cdot \dfrac{x^2 - 4}{x^2 + 4x - 5}$
- $\dfrac{\left(\dfrac{1}{a} - 2\right)}{\left(\dfrac{1}{a^2} - 4\right)}$
- - $\dfrac{\left(\dfrac{1}{ab} + \dfrac{2}{ab^2}\right)}{\left(\dfrac{3}{a^3b} - \dfrac{4}{ab}\right)}$

## The Zero-Product Property

| If $pq = 0$, then $p = 0$ or $q = 0$. |
|---|

**Example.** Solve for all real solutions to the following equations using factoring and the zero-product property:

- $x^2 - 2x - 3 = 0$
- $x^3 - 9x = 0$
- $x^3 + x^2 + x + 1 = 0$
- $x(6x - 13) = -6$

## Equations with Absolute Value

| |
|---|
| Suppose $a$ is not negative. |
| Then the equation $\lvert x\rvert = a$ means that either $x = a$ or $x = -a$. |
| Make sure to **check your answers!**. |

**Example.** Solve the following equations with absolute value.

- $\lvert x + 4\rvert = 1$
- $\lvert 2x - 3\rvert + 2 = 0$

An alternative approach (The squaring method).

| |
|---|
| Observe that $(\lvert x\rvert)^2 = x^2$. |

**Example.** Solve the following equations with absolute value:

- $\lvert x - 3\rvert = 2$
- $\lvert x + 4\rvert = \lvert x - 1\rvert$

# Lesson 4: The Cartesian Coordinate Plane

**Rectangular Coordinates**.

The horizontal number line is called the ***x* axis**, unless otherwise specified.

The vertical number line is called the ***y*-axis**, unless otherwise specified.

The four regions separated by the axes are called **quadrants**.

The quadrants are numbered using **roman numerals**.

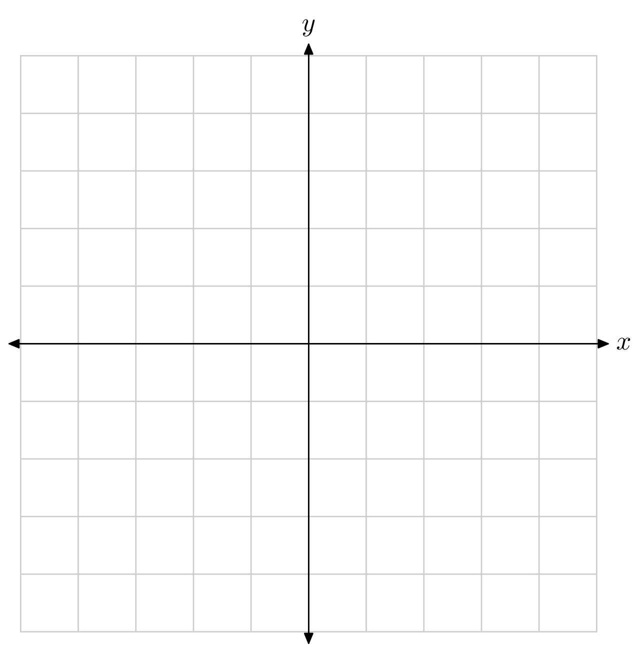
The point $(a, b)$ is found by

**going to** $a$ **on** $x$**-axis**, then

then **moving parallel to the** $y$**-axis** $b$ **units**.

**Example.** Plot the following points:

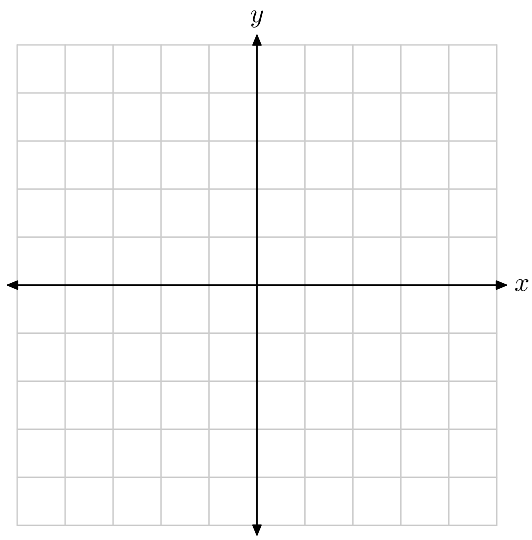
- $A(3, 2)$
- $B(-1, 1)$
- $C(1, -4)$
- $D(-4, -5)$
- $E(0, 4)$
- $F(-2, 0)$



## Symmetry

## The Midpoint Formula

**Example.** Find the midpoint between $(-3, 2)$ and $(3, 4)$.



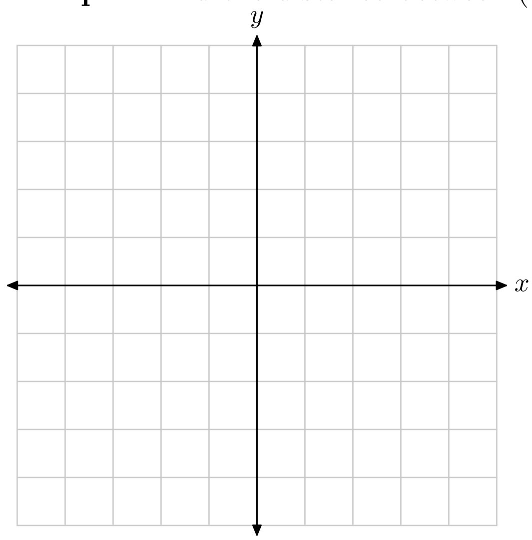

> The **midpoint** between $(x_1, y_1)$ and $(x_2, y_2)$ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
> The midpoint can be thought of as the **average** of two points.

**Example.** Find the midpoint between the following points:

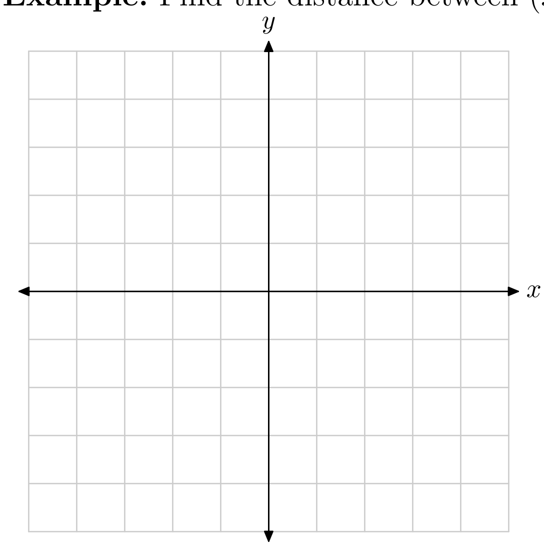
- $(-1, 1)$ and $(2, 3)$
- $(1, -3)$ and $(5, -3)$
- $(a, 0)$ and $(0, b)$
- $(a, b)$ and $(b, a)$

**The Distance Formula**.

**Example.** Find the distance between $(-2, -3)$ and $(4, 5)$.




**Example.** Find the distance between $(x_1, y_1)$ and $(x_2, y_2)$.




The **<u>distance</u>** between $(x_1, y_1)$ and $(x_2, y_2)$ is $\underline{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$.

Note: The order of the points does not affect the distance.

# Lesson 5: Relations and Graphs of Equations

The **graph** of an equation in two variables is the set of all points $(a, b)$ such that **the coordinates satisfy the given equation**.

**Example.** Determine whether the following points are on the graph of the given equation:

- $(1, \frac{5}{4})$, $x + 8y = 11$
- $(\sqrt{2}, -\sqrt{3})$, $x^2 - y^2 = 5$

**Example.** What value(s) of $a$ make $(2a, a+3)$ be a point on the graph of the equation $2x - 3y = 10$?

**Intercepts**.

If a point of the graph is on the $x$-axis, its $x$-coordinate is called an **$x$ intercept**.
Note, the $y$-coordinate of this point is **0**.

If a point of the graph is on the $y$-axis, its $y$-coordinate is called a **$y$-intercept**.
Note, the $x$-coordinate of this point is **0**.

**Example.** Find the intercepts for the following equations:

- $4x - 6y = 12$

- $\sqrt{2-x} + 1 = y$

- $y^2 = x^3 - x$

**Example.** Find the intercepts for the following equations:

- $2x + 3y = 7$

- $x^2 + 6y = y^2$

- $xy = x^2 + 1$

Determine symmetry.

# Lesson 6: Three Interesting Curves

Recall the **distance** between $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

**Equations of circles**.

A **circle** with center $(h, k)$ and radius $r$ is the set of points $(x, y)$ such that **the distance from** $(h, k)$ **to** $(x, y)$ **is** $r$.

**Example.** Find the equation for the circle with center $(1, 2)$ and radius 3.

The equation for a circle with center $(h, k)$ and radius $r$ is

$$\sqrt{(x-h)^2 + (y-k)^2} = r \text{ or } (x-h)^2 + (y-k)^2 = r^2.$$

**Example.** Find the equation for the circle with center $(-1, 3)$ and radius $\sqrt{2}$.

**Completing the square**.

**Example.** Multiply out the following squares:

- $(x-1)^2 =$

- $(x-3)^2 =$

- $(x+\frac{1}{2})^2 =$

- $(x+3)^2 =$

**Note**: The coefficient of the middle term in the above examples was always twice that of the second term in the binomial.

**Example.** For each of the following, what needs to be added to make a perfect square?

- $x^2 - 2x$

- $x^2 - 6x$

- $x^2 + x$

- $x^2 + 6x$

To make $x^2 + bx$ a perfect square, one must add $\underline{\left(\frac{b}{2}\right)^2}$.

Start with $b$, **divide by 2**, then **square the result**.

**Identifying Circles.**

**Example.** Find the center and radius of the circles whose equation is given below:

- $x^2 + y^2 - 6x = 7$

- $x^2 + y^2 - 4x + 6y = -12$

- $4x^2 + 4y^2 - 4x + 8y = 11$

**Example.** Find the intercepts of the circle $x^2 + y^2 - 7x + 8y = 12$.

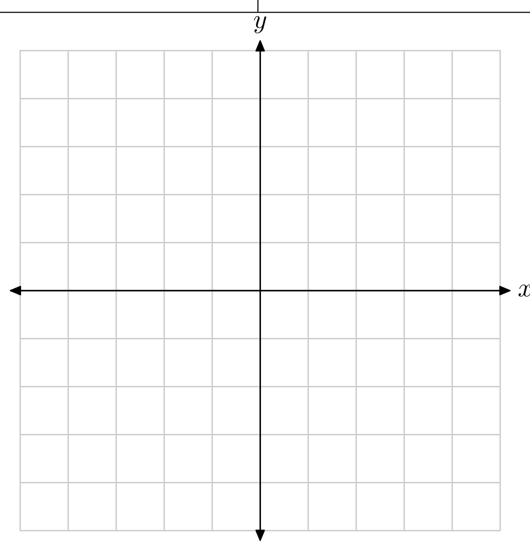
**Example.** Find the equation of a circle whose diameter is the segment $\overline{AB}$, where $A = (-3, 1)$ and $B = (5, -5)$.

## Parabolas

**Example.** Graph the point $(0, 1)$ and the line $y = -1$. Find the distance from each of the following points to both the point $(1, 0)$ and the line $y = -1$, then plot them:

Note: The way to compute the distance from a point to a horizontal line is to find the absolute value of the difference between the $y$-coordinate of the point and the value of the line.

| Point | Distance to $y = -1$ | Distance to $(0, 1)$ |
|---|---|---|
| $(0, 0)$ | | |
| $(2, 1)$ | | |
| $(-2, 1)$ | | |
| $(4, 4)$ | | |
| $(4, -4)$ | | |




Note: The above points are **equidistant** from the point $(0, 1)$ and the line $y = -1$.

A **parabola** is a set of points which are equidistant from a fixed point and a line.

The fixed point is called the **focus**. The line is called the **directrix**.

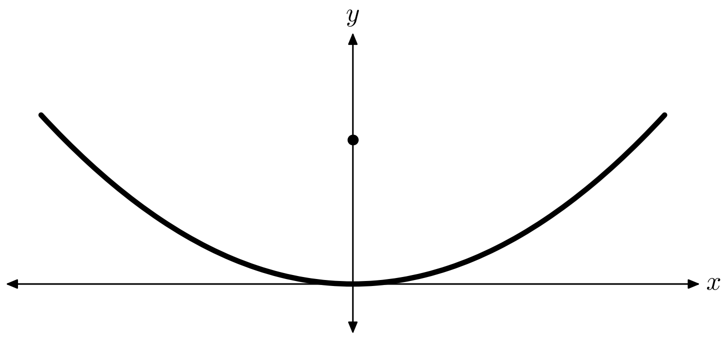
**Example.** Find the equation of the parabola with focus $(0, 1)$ and directrix $y = -1$.

**Example.** Find the equation of the parabola with focus $(0, p)$ and directrix $y = -p$.
(The length $p$ is referred to as the **focal length** of the parabola.)

## Reflective Properties of Parabolas

If a beam leaves the focus of a parabola, its reflection will be parallel to the axis of symmetry of the parabola.

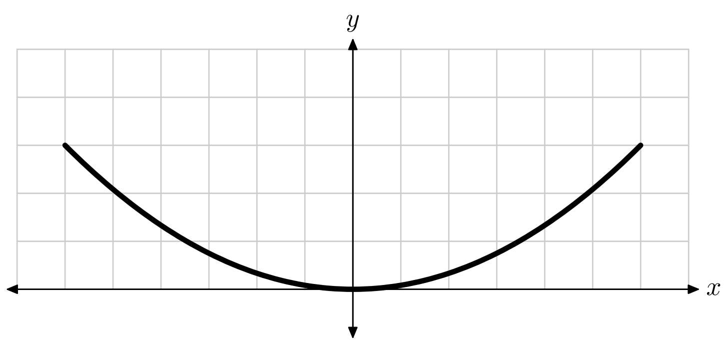
If beams going parallel to the axis of symmetry hits the parabola, their reflection will always go through the focus.




**Example.** Satellite dishes are designed in the shape of parabolas; when signals come in from space, the shape of the dish directs all of the signal to the receiver, which is located at the focus of the parabola. Suppose the shape of a particular satellite dish is one foot wide and is modeled by the equation

$$y = \frac{1}{12}x^2, \ -6 \leq x \leq 6 \text{ (in inches)}$$

If the receiver snapped off the satellite dish, how long should the new one be? (i.e. What is the focal length of the parabola?)



# Lesson 7: Introduction to Functions and Function Notation

## Definition of Functions

**Example.** For each of given numbers, find an expression for the result of the given directions:

- For 2, square it, then add 1.

- For 2, add 1, then square it.

- For $x$, divide by 2, then square.

- For $x$, square, then divide by two.

A **function** is
**a rule, a definition, or set of directions for getting from one value to another value**.

The initial value is called the **input** of the function.
A variable acting as the input is called an **independent variable**.

The end value after applying the function is called the **output** of the function.
A variable acting as the output is called a **dependent variable**.

The set of all possible inputs for a function is called the **domain**.
The set of all possible outputs from a function is called the **range**.

## Notation

> Let $x$ be the input for a function. Let $f$ be the name given to a function.
>
> The result of applying $f$ to $x$ will be denoted as $f(x)$ **(pronounced "$f$ of $x$")**.

**Example.** Let $f(x) = \dfrac{x^2}{x-4}$. Find

- $f(2)$
- $f(-1)$
- $f(4)$
- $f(a+2)$

**Example.** Let $f(x) = x^2 - x - 2$. Evaluate

- $f(x-2)$
- $f(2x)$
- $f(x+h)$
- $f(x+h) - f(x)$

### Domain

The **implied domain** of a function are all values which are permissible inputs.

At this stage, there are primarily two bad things that can go wrong when trying to compute a value:

- **dividing by zero**, and
- **taking the even root of a negative number**.

To find implied domain, determine which real numbers do not cause the above events to occur.

**Example.** Find the domain of the following functions:

- $f(x) = x^2 - x$
- $f(x) = \sqrt{1 - 4x}$
- $f(x) = \dfrac{1}{x - 1}$
- $f(x) = \dfrac{1}{x^2 + 4}$

**Example.** Find the domain and range for $y = f(x) = \dfrac{x}{x+1}$.

## Combining Functions

Below is some notation on how we can combine two functions $f(x)$ and $g(x)$:

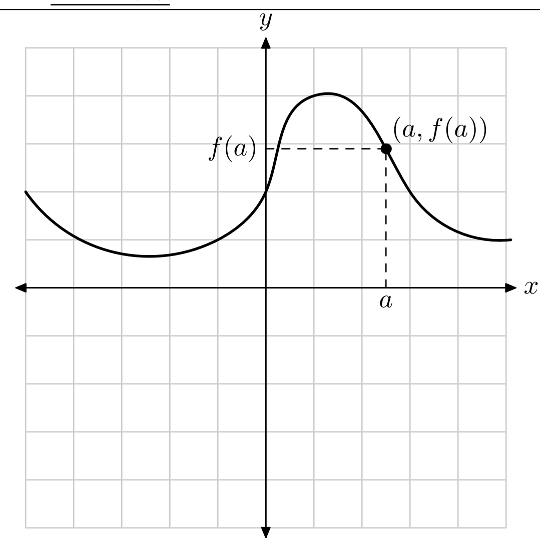
- $(f+g)(x) = \underline{f(x)+g(x)}$.
- $(f-g)(x) = \underline{f(x)-g(x)}$.
- $(fg)(x) = \underline{f(x)\cdot g(x)}$.
- $(f/g)(x) = \underline{f(x)/g(x) = \dfrac{f(x)}{g(x)}}$, provided that $\underline{g(x)}$ **<u>is not zero</u>**.

**Example.** Let $f(x) = 1 - x^2$ and $g(x) = 2x + 1$. Find

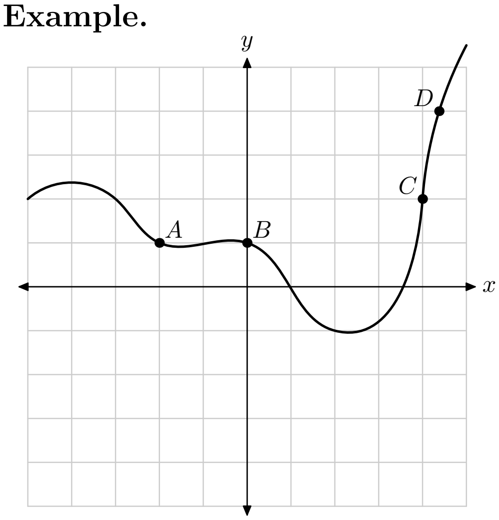
- $(f+g)(x) =$
- $(f-g)(x) =$
- $(fg)(x) =$
- $(f/g)(x) =$

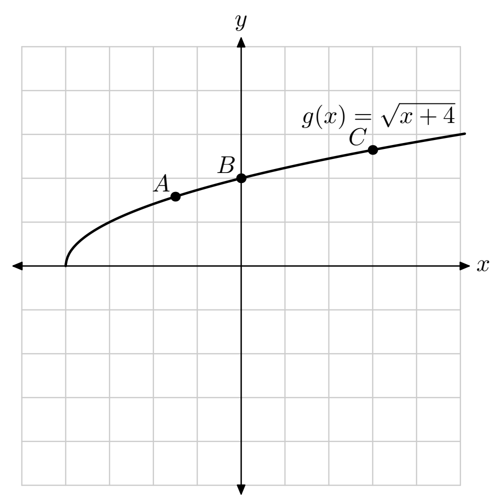
# Lesson 8: Graphs of Functions

> The **graph of a function** $f$ in the $x$-$y$ plane consists of those points $(x, y)$ such that
>
> $\underline{x \textbf{ is in the domain of } f}$ and $\underline{y = f(x)}$.




**Example.**




Evaluate the following:

- $f(0)$ _______
- $f(-2)$ _______
- $f(4)$ _______
- $f(4.5)$ _______

Find the coordinates for the following:

- $A$ ____________________
- $B$ ____________________
- $C$ ____________________

**When does a graph represent a function?**.

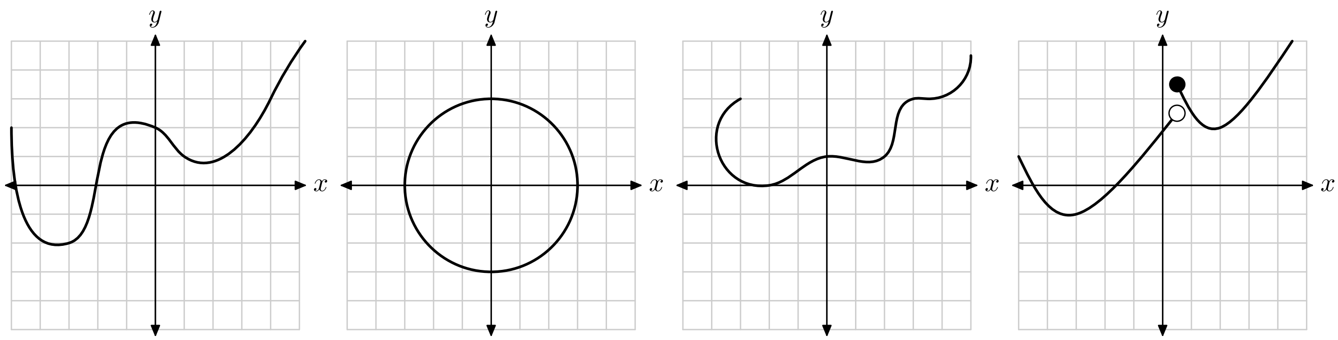
A **function** is a rule which identifies to each input exactly one output.

If a graph is to represent a function $f$, then for any $a$ in the domain, **there must be exactly one point on the graph whose $x$-coordinate is $a$.**.

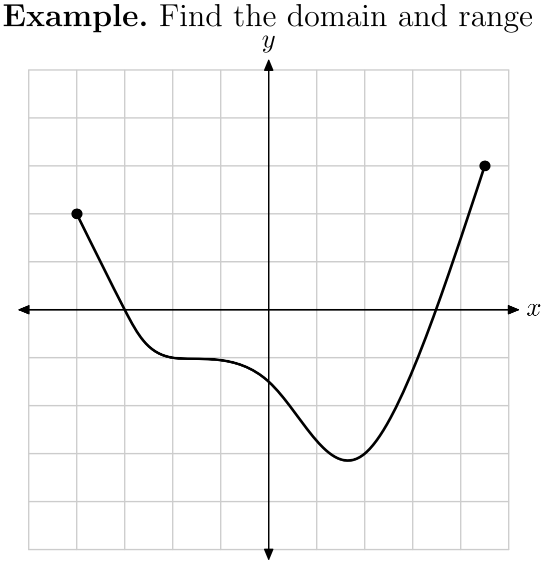
**The Vertical Line Test.**

A graph in the $x$-$y$ plane represents a function $y = f(x)$ provided that **any vertical line intersects the graph in at most one point.**.

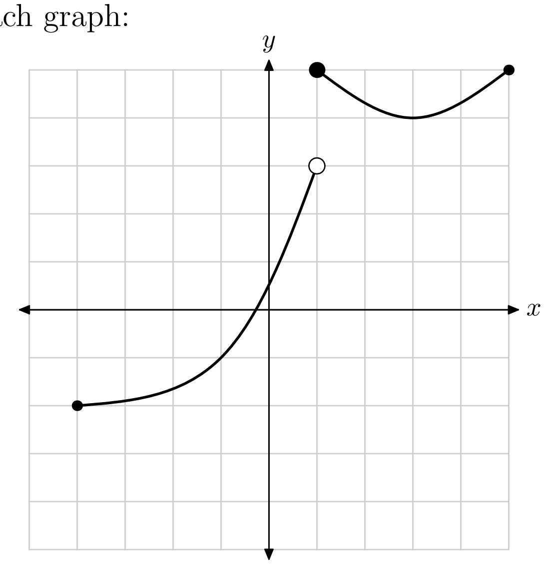
**Example.** Do these graphs represent a function?




**Example.** Find the domain and range of $f$ in each graph:



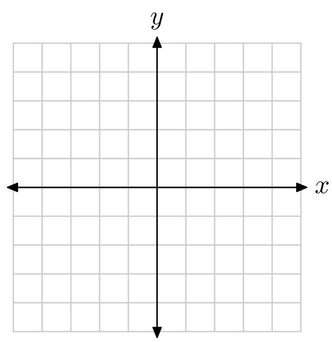

- Domain:
- Range:
- $f(-2) =$
- $f(4) =$




- Domain:
- Range:
- $f(1) =$
- $f(3) =$

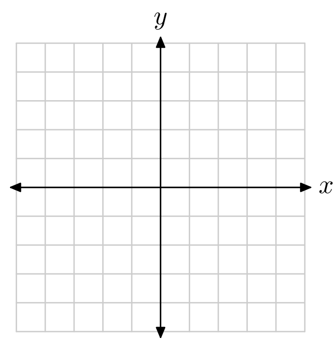
## The Six Basic Functions

### Identity function



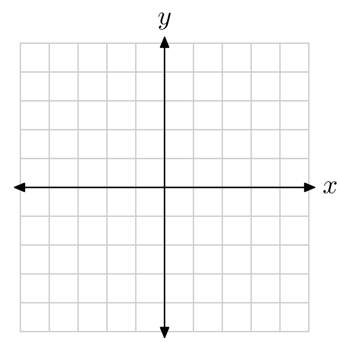

- Domain:
- Range:

### Squaring function



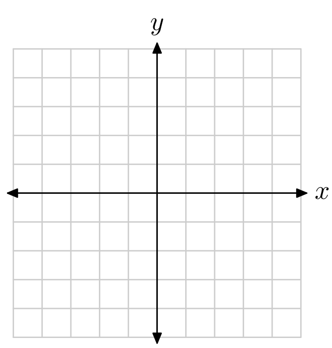

- Domain:
- Range:

### Cubing function



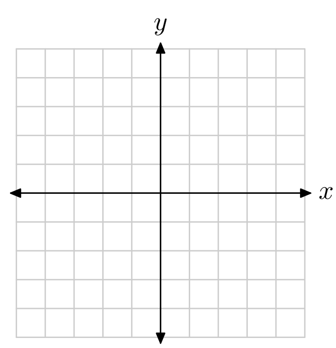

- Domain:
- Range:

### Reciprocal function



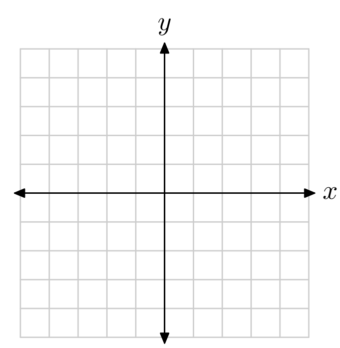

- Domain:
- Range:

### Square root function




- Domain:
- Range:

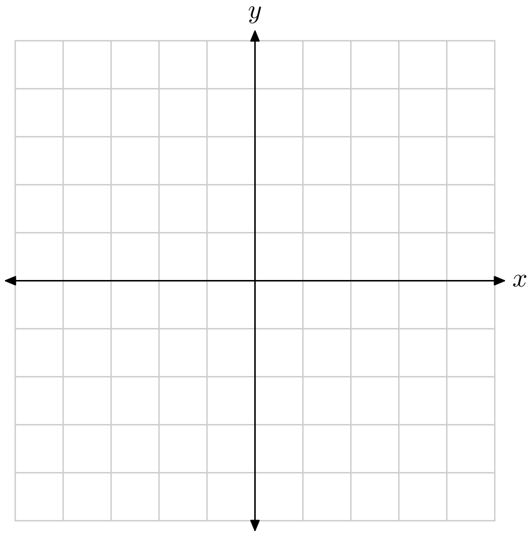
### Absolute value function

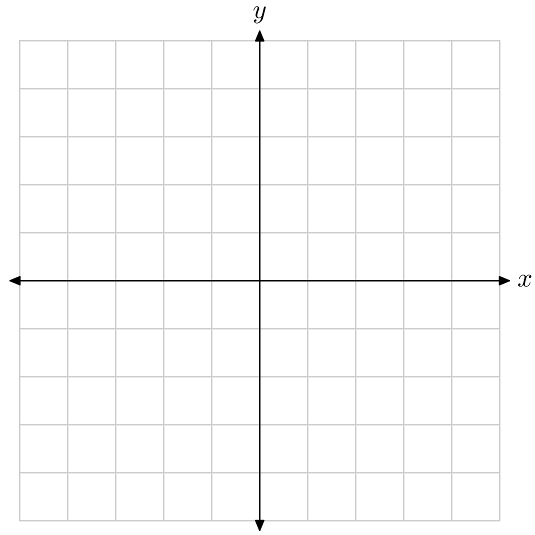



- Domain:
- Range:

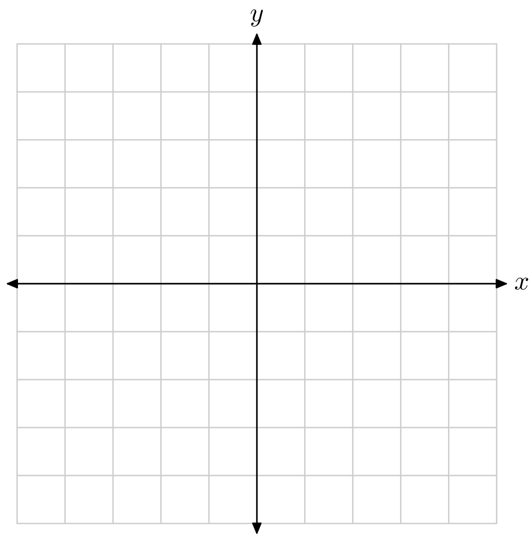
## Piecewise Functions

**Example.** Graph $y = |x|$ and $y = \sqrt{x}$ in the two graphs below:





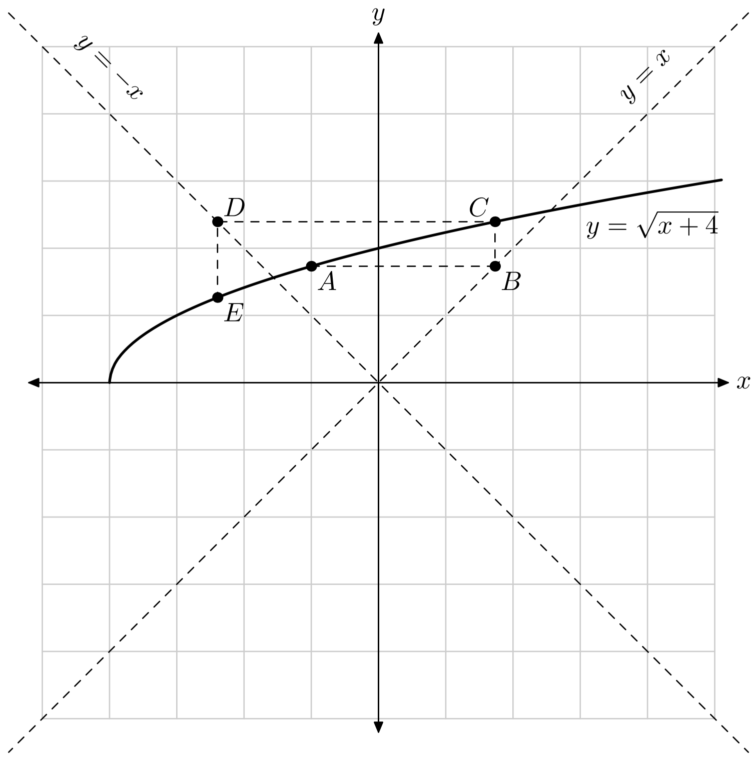
Now, graph the piecewise function $g(x) = \begin{cases} |x| & -2 < x \leq 2 \\ \sqrt{x} & 2 < x \leq 4 \end{cases}$



Evaluate the following:

- $g(-1)$
- $g(2)$
- $g(-2)$
- $g(4)$
- $g(3)$
- $g(0)$

**Example.** Consider the following graph:




Find the coordinates for the the above 5 points.

### Shapes of Graphs

A function is **increasing** on an interval if for each $a, b$ in the interval, if $a < b$, then $\underline{f(a) < f(b)}$.

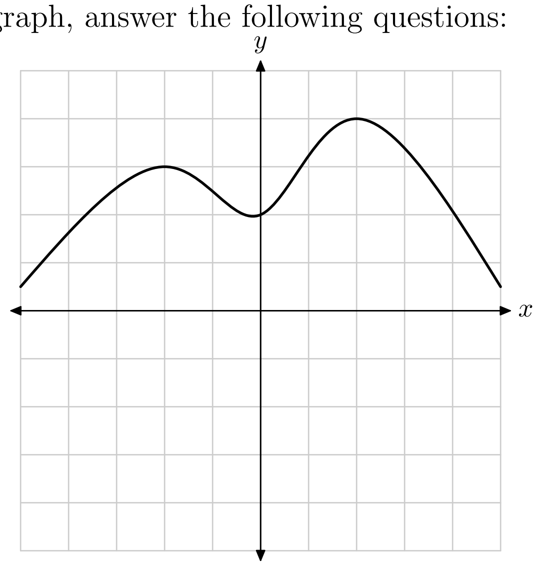
A function is **decreasing** on an interval if for each $a, b$ in the interval, if $a < b$, then $\underline{f(a) > f(b)}$.

A value $M$ is the **maximum value of** $f$ if $\underline{M = f(x_0) \text{ for some } x_0 \text{ in the domain}}$ $\underline{\text{and if for every } x \text{ in the domain of } f,\ f(x) \leq M}$.

A value $m$ is the **minimum value of** $f$ if $\underline{m = f(x_0) \text{ for some } x_0 \text{ in the domain}}$ $\underline{\text{and if for every } x \text{ in the domain of } f,\ f(x) \geq m}$.

A **turning point** is a point where a graph **changes between increasing and decreasing**.

**Example.** Using the below graph, answer the following questions:




- What are the turning points?

- Where is $f$ increasing?

- Where is $f$ decreasing?

- What is the maximum and minimum?

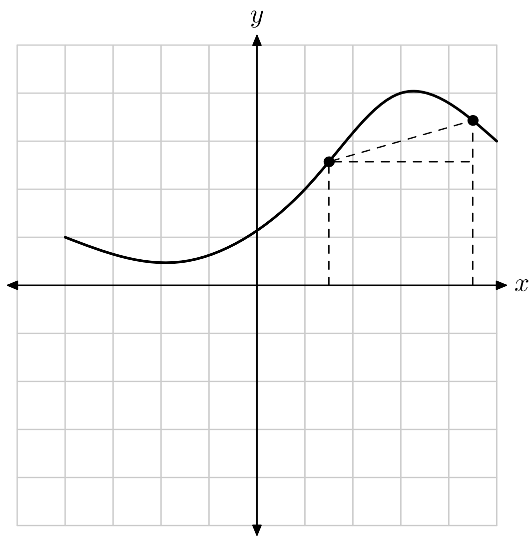
## The Average Rate of Change

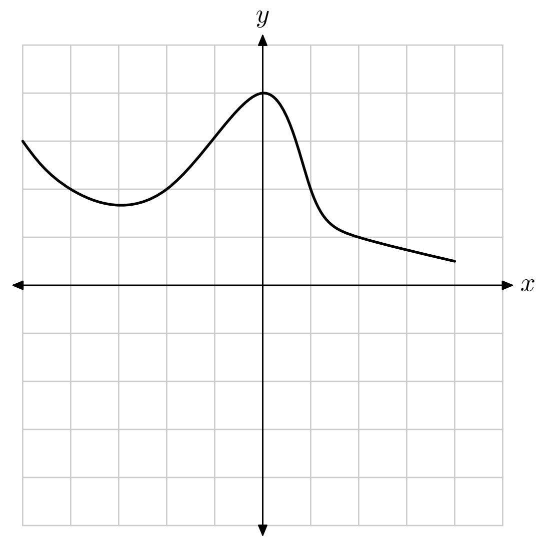
The **average rate of change** of a function $f$ on the interval $[a, b]$ is **the slope of the line connecting the points of the graph whose $x$-coordinates are $a$ and $b$.**

The point whose $x$-coordinate is $a$ is $\underline{(a, f(a))}$.

The point whose $x$-coordinate is $b$ is $\underline{(b, f(b))}$.

The slope of the line connecting these would be $\dfrac{f(b) - f(a)}{b - a}$.




**Example.** Find the average rates of change of $f(x)$ in the above right graph over the given intervals:

- $[-2, 0]$
- $[0, 1]$
- $[0, 2]$
- $[2, 5]$

**Example.** Find the average rate of change of the given function over the given interval:

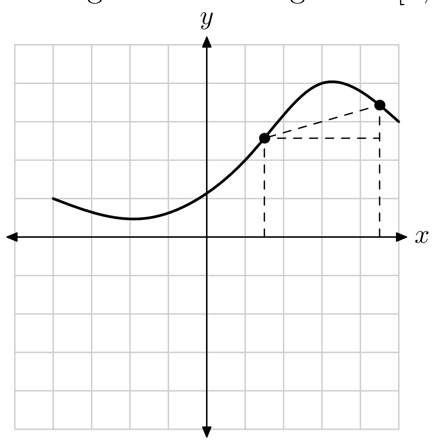
- $f(x) = x^2 - x,\ [0, 2]$
- $f(x) = x^2 - x,\ [1, 3]$
- $g(t) = |t - 1|,\ [0, 2]$
- $h(t) = t^2,\ [a, b]$
- $f(t) = \sqrt{t},\ [4, a]$
- $h(z) = \dfrac{1}{z},\ [1, 1 + x]$

**The Difference Quotient**.

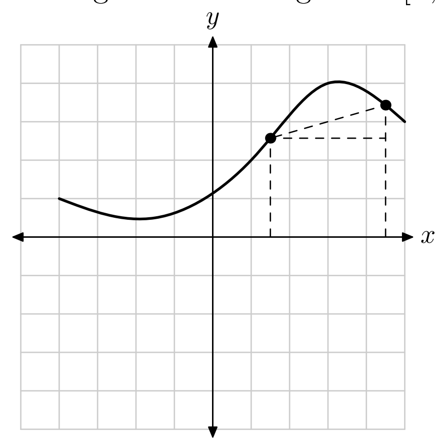
The difference quotient is an algebraic expression giving **average rate of change**.

The two primary difference quotients are constructed using the following conventions:

Average rate of change over $[a, b]$




Average rate of change over $[x, x+h]$



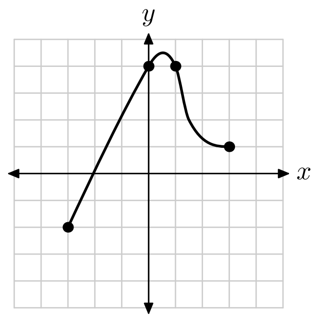

Note: You will be expected to be able to simplify these difference quotients.

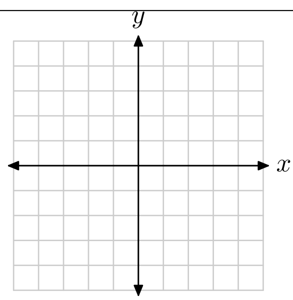
**Example.** Simplify each difference quotient for the following functions:

- $f(x) = \sqrt{x}$, $[a, b]$
- $f(x) = \dfrac{1}{x}$, $[x, x+h]$

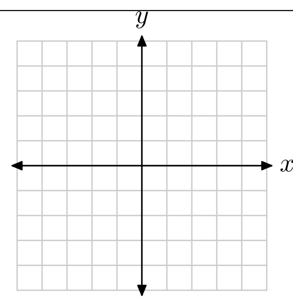
# Lesson 9: Transformations

Use the following graph and preform the given operations:

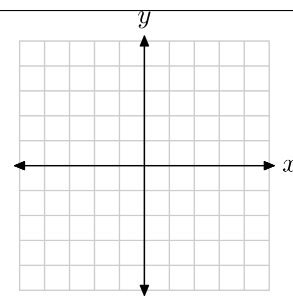


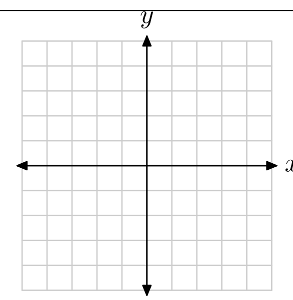
right 3 units



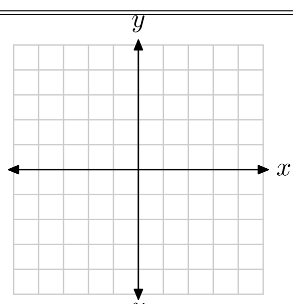
down 1 unit

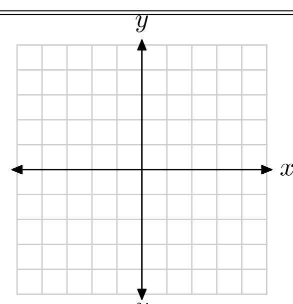


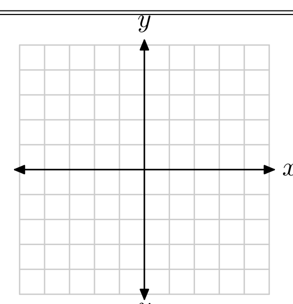
reflect over $y$-axis

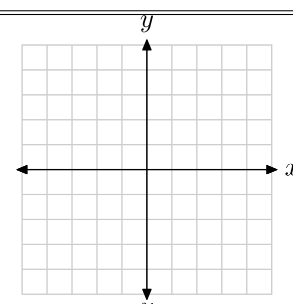


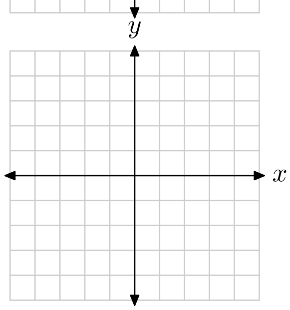
reflect over $x$-axis



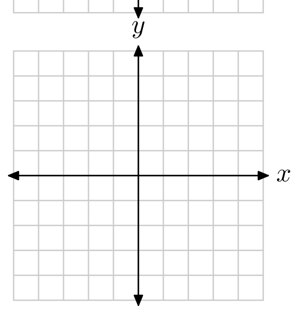




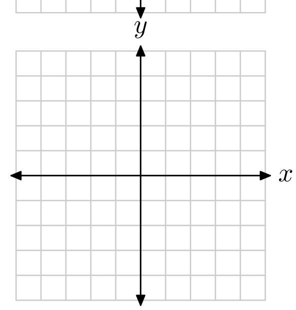




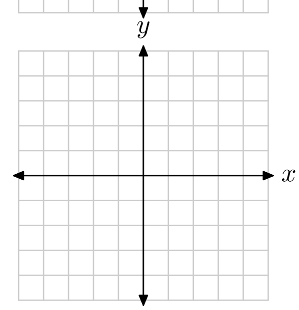
reflect over $x$, left 2



left 2, reflect over $x$



left 1, reflect over $y$

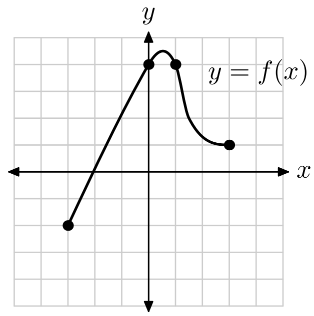


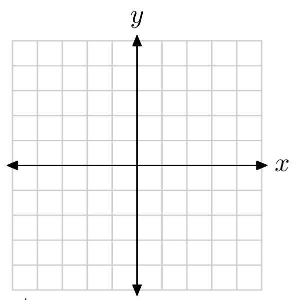
reflect over $y$, left 1

Observation: In some cases, the order of the movements **affects where the graph goes**.

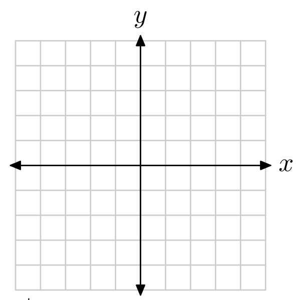
## Translating and Reflecting functions

**Example.** Complete each of the charts belows and use your findings to plot the functions.

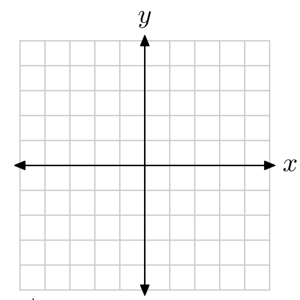


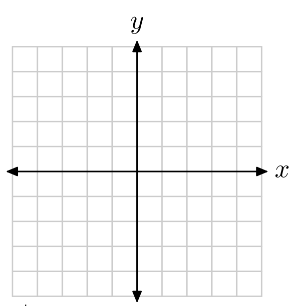
| $x$ | $g(x) = f(x) + 1$ |
|---|---|
| $-3$ | |
| 0 | |
| 1 | |
| 3 | |



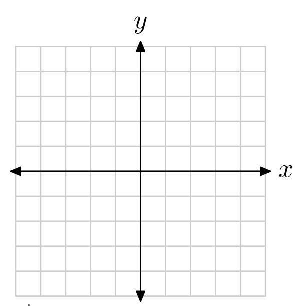
| $x$ | $g(x) = f(x) - 2$ |
|---|---|
| $-3$ | |
| 0 | |
| 1 | |
| 3 | |



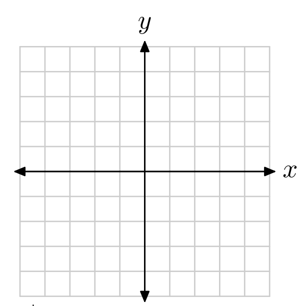
| $x$ | $g(x) = -f(x)$ |
|---|---|
| $-3$ | |
| 0 | |
| 1 | |
| 3 | |



| $x$ | $g(x) = f(x - 2)$ |
|---|---|
| $-1$ | |
| 2 | |
| 3 | |
| 5 | |



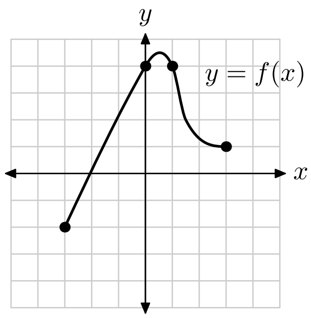
| $x$ | $g(x) = f(x + 1)$ |
|---|---|
| $-4$ | |
| $-1$ | |
| 0 | |
| 2 | |



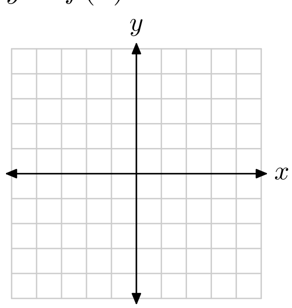
| $x$ | $g(x) = f(-x)$ |
|---|---|
| 3 | |
| 0 | |
| $-1$ | |
| $-3$ | |

## Summary

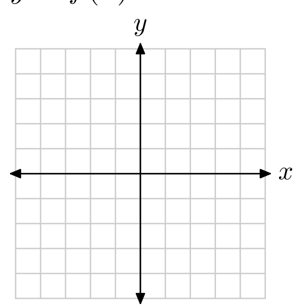
Given the graph of a function $y = f(x)$ and $c$ positive,



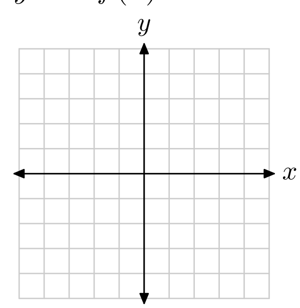

- $y = f(x) + c$



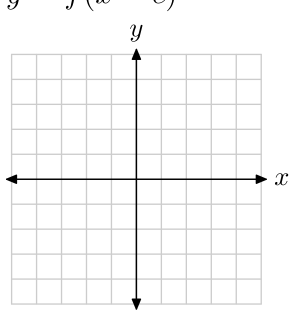

- $y = f(x) - c$



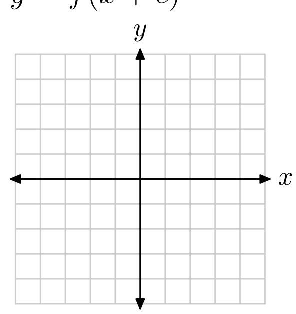

- $y = -f(x)$



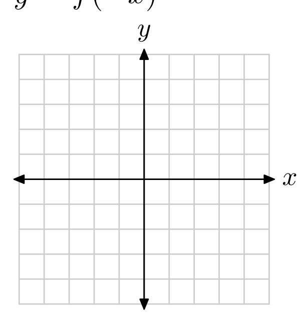

- $y = f(x - c)$



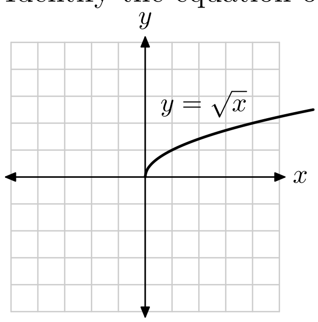

- $y = f(x + c)$

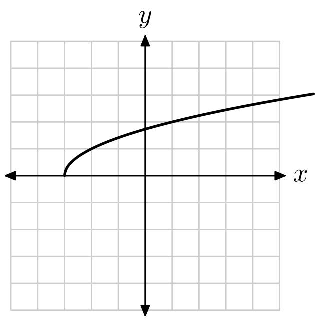


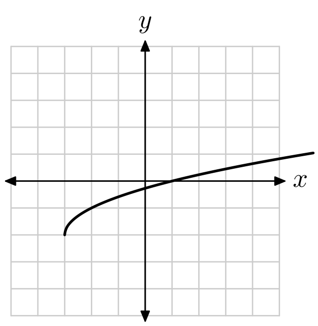

- $y = f(-x)$

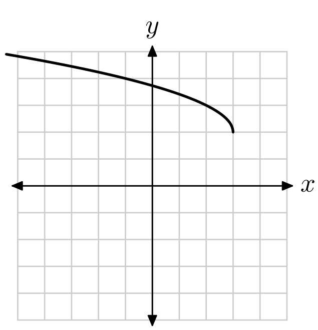


**Example.** Each of the following is a translation and/or a reflection of the given graph of $y = f(x)$. Identify the equation of each graph in terms of $f$.



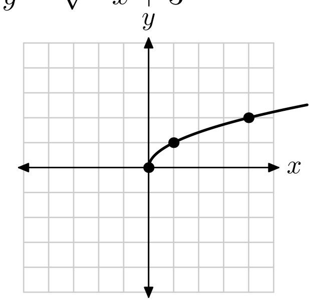



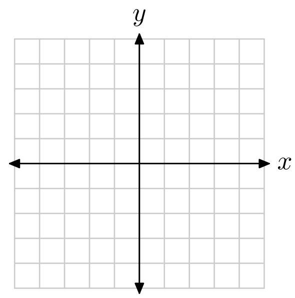



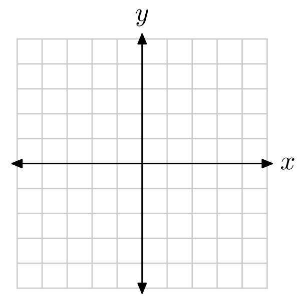



**Example.** Graph the following translates of $y = \sqrt{x}$:

$y = \sqrt{-x+3}$

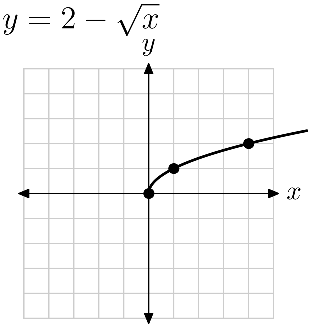


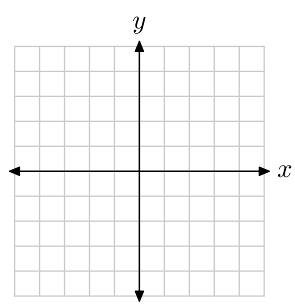



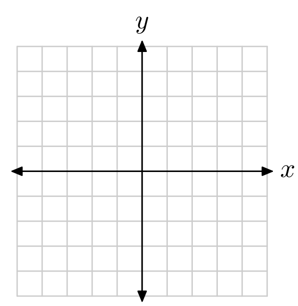




Original Graph

$y = 2 - \sqrt{x}$

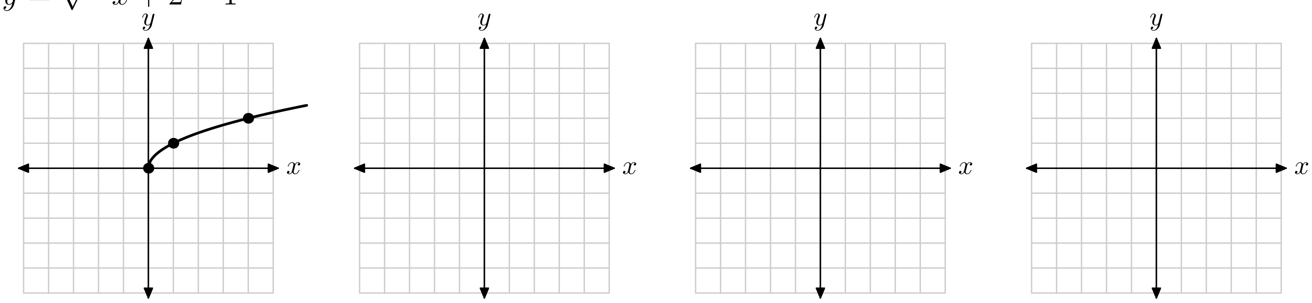


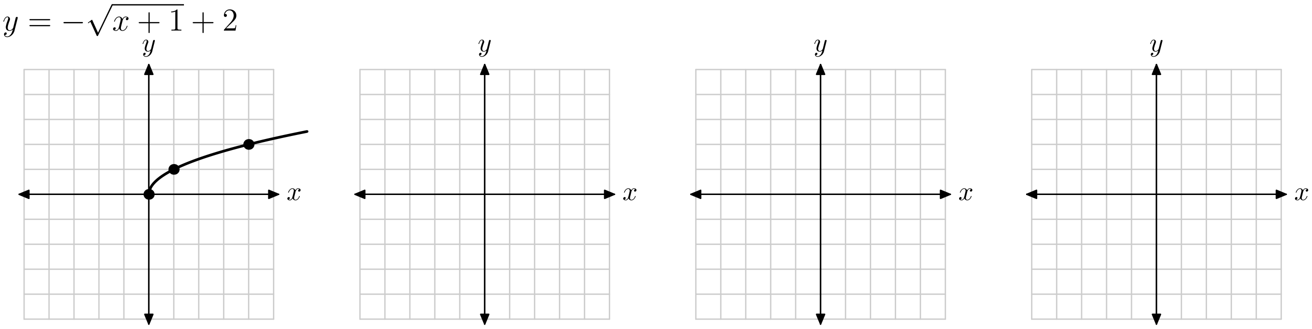

Original Graph

$y = \sqrt{-x+2} - 1$




Original Graph

$y = -\sqrt{x+1} + 2$



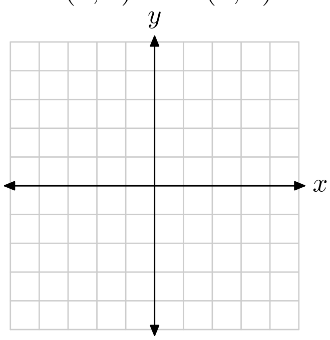

Original Graph

# Lesson 10: Linear Functions and Absolute Value Functions

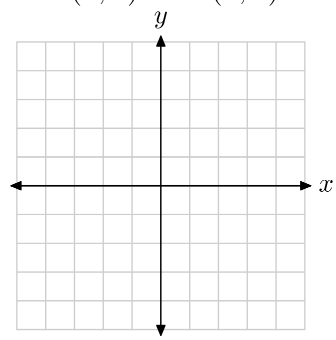
The **slope** of a line going through $(x_1, y_1)$ and $(x_2, y_2)$ is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$.

**Example.** Find the slope of the line going through the following points:

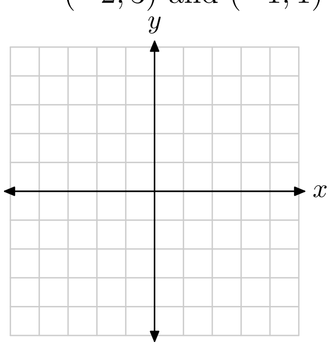
- $(1, 1)$ and $(3, 5)$



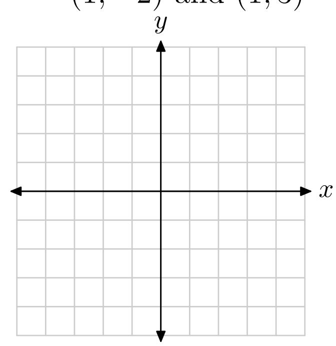

- $(3, 4)$ and $(0, 4)$



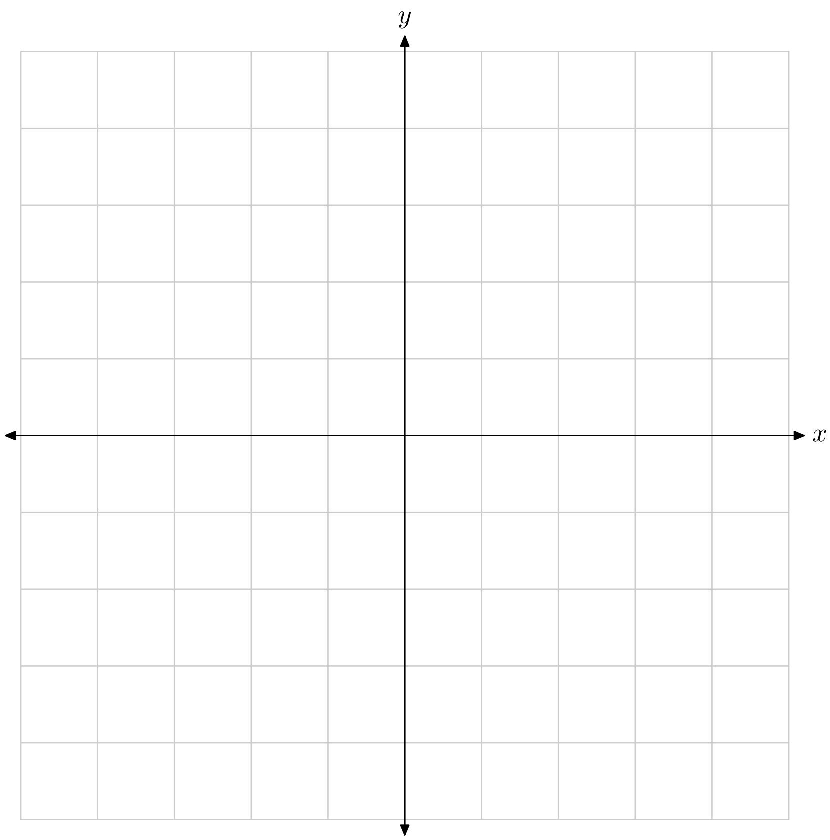

- $(-2, 3)$ and $(-1, 1)$




- $(1, -2)$ and $(1, 3)$



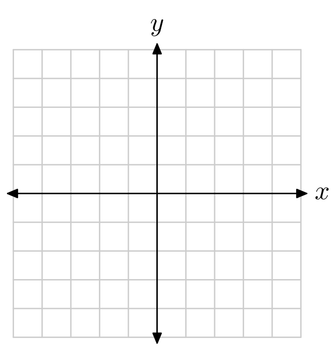
**Comparing Slopes**.




The slope of a **decreasing** line is **negative**.

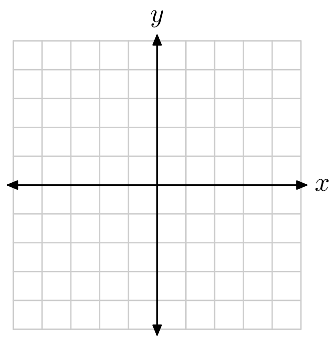
The slope of an **increasing** line is **positive**.

**Example.** Suppose a line has slope and goes through $(2, -1)$ and $(x, y)$. What is the slope of the line?



**Point-Slope Form**.

**Example.** A line has slope 4 and goes through the point $(2, -1)$. What if $(x, y)$ is on the line? Find an equation relating $x$ and $y$.



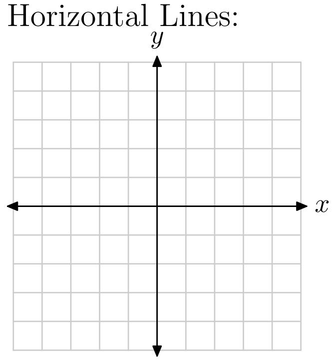
The **point-slope formula** for a line with slope $m$, going through $(x_1, y_1)$ is

$$m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y - y_1 = m(x - x_1).$$

This can be used to find the equation of any line that has a **defined slope**.

Special Cases.

Horizontal Lines:

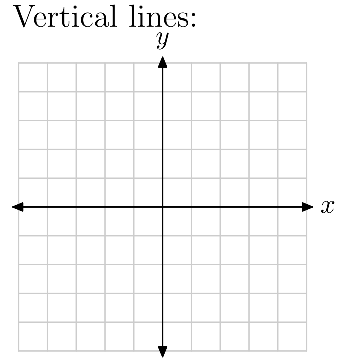


Horizontal lines have a slope of 0.

General equation of a horizontal line: $y = b$.

In other words, every point on a horizontal line has the same $y$-coordinate, namely $b$.

Vertical lines:



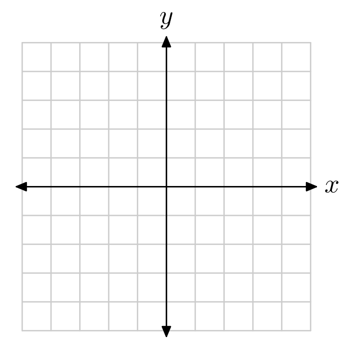
Vertical lines have a slope which is undefined.

General equation of a vertical line: $x = a$.

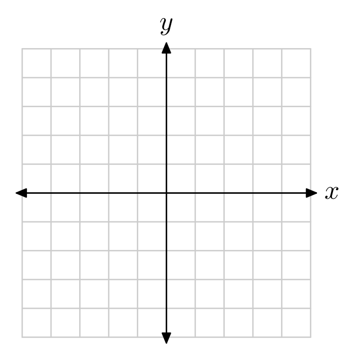
In other words, every point on a vertical line has the same $x$-coordinate, namely $a$.

**Example.** Find the equation for the following lines:

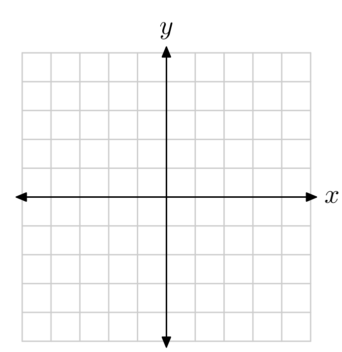
- a vertical line through $(2, -1)$



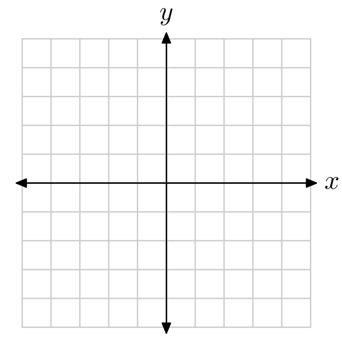

- a horizontal line through $(3, \frac{1}{2})$




- $m = \frac{1}{2}$, goes through $(1, 3)$

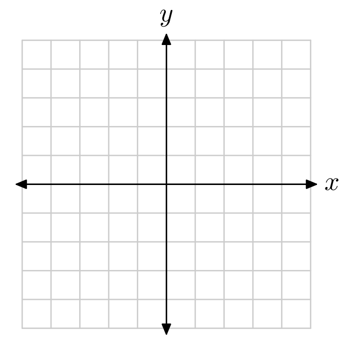


- Goes through $(1,2)$ and $(-3,4)$



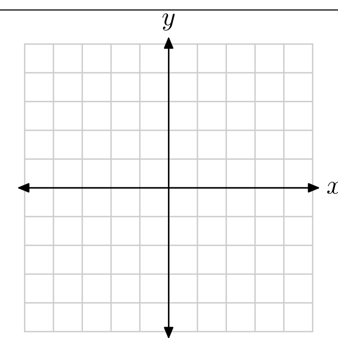
## Slope-Intercept

**Example.** Find the equation of the line of slope $-\frac{1}{3}$ going through $(0, 4)$.

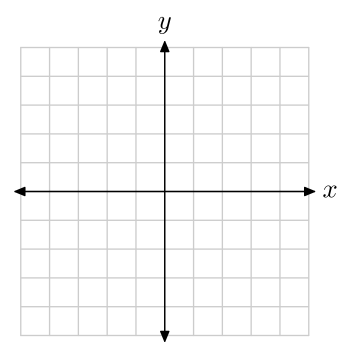



If a line has slope $m$ and a $y$-intercept $b$, then the **slope-intercept** formula for the line is

$$\underline{y = mx + b}.$$




**Example.** Find the equation for the line going through $(1, 3)$ with slope $\frac{1}{2}$ in slope-intercept form.



## Standard Form of a Line

The equation of any line can be expressed as $\underline{Ax + By + C = 0}$, where $A$**,** $B$**, and** $C$ **are real numbers and** $A$ **and** $B$ **are both not zero.**

In most cases, this is acheived by **subtracting everything from one side of the equation**.

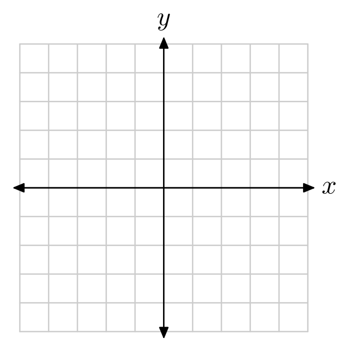
## Graphing Lines

To graph lines, a simple approach is to **plot and connect the intercepts**.

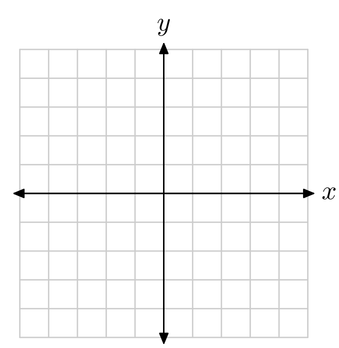
If there is only one intercept (the origin), then plot any other point.

**Example.** Graph the following lines:

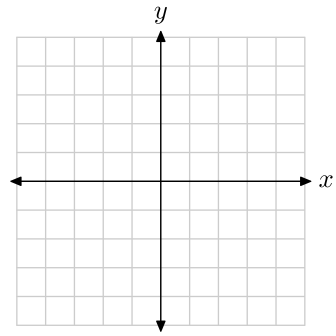
- $3x + 4y - 6 = 0$




- $y = \frac{1}{2}x + 2$



- $y - 2 = \frac{3}{2}(x + 2)$



## Parallel and Perpendicular Lines

Two lines are parallel if **their slopes are the same**.
In other words, if our two **parallel** lines have slopes $m_1$ and $m_2$, then $\underline{m_1 = m_2}$.

Two lines are perpendicular if **their slopes are inverse reciprocals of one another**.
In other words, if our two **perpendicular** lines have slopes $m_1$ and $m_2$, then $m_1 = -\dfrac{1}{m_2}$.

**Example.** Find the equation of the following lines:

- A line parallel to $3x - 4y = 12$ going through $(1, 2)$

- A line perpendicular to $x + y = 6$ going through $(3, 4)$.

## Linear Functions

A **linear function** is a function defined by an equation of the form $\underline{f(x) = m \cdot x + b}$, where $\underline{m}$ and $\underline{b}$ are constants.

**Example.** Find the linear function satisfying:

- $f(0) = 2,\ f(2) = 5$

- $f(2) = 10$, the graph $y = f(x)$ is parallel to $x - 2y = 4$.

- The graph $y = f(x)$ passes through $(1, 5)$ and $(-1, -3)$.

**Example.** You can buy a new car (a nice one) for \$30,000, and after 5 years, you can sell it for \$18,000. Assume the depreciation can be modeled by a linear function in terms of time.

- Find the linear function $V(t)$ which models the value of the car $t$ years after purchase, taking into account the depreciation, with $0 \leq t \leq 5$.

- Find the value of the car after 3 years.

**Example.** The distance (in mi) that a casino ship is from land after $t$ hours of starting its engines is modeled by the linear function

$$d(t) = 5 + 10t$$

- How fast is the ship going?

- How far are they after 3 hours?

- What is the $y$ intercept in the graph $y = d(x)$?

- What does the $y$-intercept tell us in the context of this problem?

**Example.** In 1917, Camp Randall was rebuilt with concrete stands to hold approximately 10,000 people. By 1951, the capacity of Camp Randall was 51,000. Assume that the stadium's growth can be modeled by a linear function of time. Let $C(t)$ be the capacity of the stadium $t$ years after its inauguration in 1917.

- Find $C(t)$.

- Project how big the stadium would be in 2002.

- The current capacity of Camp Randall has 80,321 seats. How good is the prediction?

### Cost Functions and Marginal Cost

A function giving the cost $C(x)$ for producing $x$ units of a commodity is called a **cost function**.

The additional cost to produce one unit is called the **marginal cost**.

**Example.** You open a business making cheesy macaroni art. The initial cost for starting up is \$50 (the foldup table to set up on the corner of the road) and the marginal cost is \$0.10 per artwork for materials.

- Find the linear function $C(a)$, the cost for starting up shop and making $a$ pieces of art.

- What is the cost of making 2000 macaroni pieces?

- What would be the cost of making 2001 macaroni pieces?

In a linear cost function, the value of $m$ is the **marginal cost**.

The value of $b$ is the **initial cost**.

**Example.** The cost (in dollars) of making $x$ whoopie cushions can be modeled by the linear function

$$C(x) = 5000 + 0.15x$$

- Find the cost of making 1000 whoopie cushions.

- What is the marginal cost for making a whoopie cushion?

- Find the cost of making 1002 whoopie cushions.

- What is the $y$-intercept on the graph $y = C(x)$? What does it mean?

**Example.** Robin Banks left a bank at 52 mph. Officer Willie Catchup arrived at the bank 15 minutes behind Robin leaving and chases him at 60 mph. Let $t$ denote the time elapsed from when Willies arrives at the bank.

- How far away from the bank is Robin before Willie shows up to the bank?

- Find $R(t)$, the distance that Robin is from the bank after $t$ hours.

- Find $W(t)$, the distance that Willie is from the bank, $t$ hours after he arrives at the bank.

- When does Willie catch up to Robin?

- The state line is 100 miles away. Will Robin get caught (does Willie catch up in time)?

# Lesson 11: Defining Functions

**Example.** Suppose you have two numbers which add to 12.

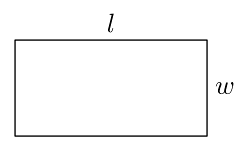
- If one of the numbers is 2, what is the sum of the squares of the two numbers?

- If one of the numbers is $x$, then find a function $f(x)$ for the sum of the squares of the two numbers.

In finding functions, you may also want to remember the following facts:

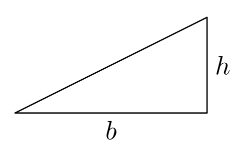
- A rectangle with dimensions $l$ and $w$ has area ____________

  and perimeter ____________ .



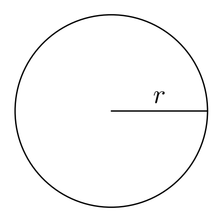

- A right triangle with dimensions $b$ and $h$ has area ____________

  and hypothenuse ____________




- A circle with radius $r$ has area ____________

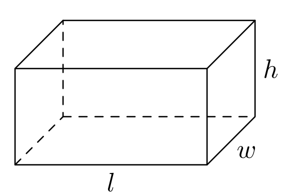
  and circumference ____________




- A rectangular box with dimensions $l$, $w$, and $h$ has

  volume _________ and surface area _____________________ .

  The surface area with no top would be _____________________ .

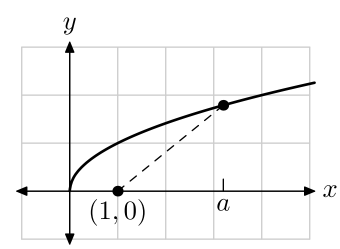



- The distance between $(x_1, y_1)$ and $(x_2, y_2)$ is ____________________________________ .

- The midpoint between $(x_1, y_1)$ and $(x_2, y_2)$ is ____________________________________ .

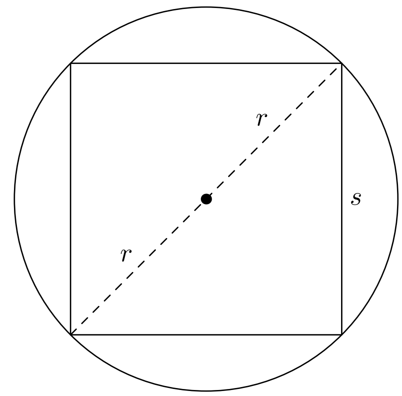
**Example.** The sum of two numbers is 12. Find a function $f(x)$ which computes the sum of the cubes of the two numbers, where $x$ is one of the two numbers.

**Example.** Let $f(x) = \sqrt{x}$. Find the function $d(a)$, which computes the distance from the point $(1, 0)$ to the point on the graph of $y = f(x)$ whose $x$-coordinate is $a$.



**Example.** A square is inscribed inside a circle of radius $r$.

- Find a formula for the diameter of the circle in terms of $r$.

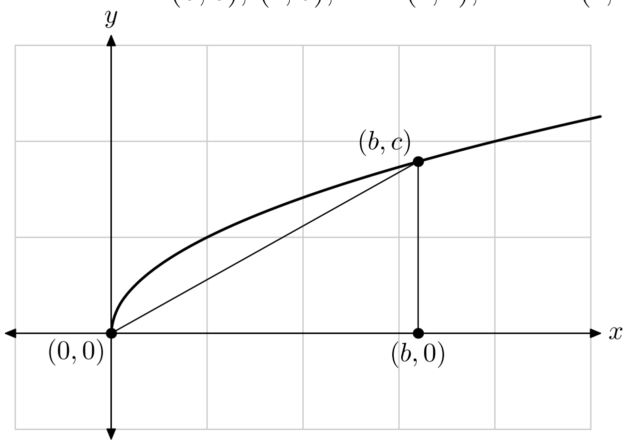



- Find a formula for the length of the side of the square in terms of the diameter, then the radius.

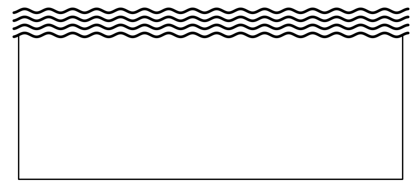
- Find a function $P(r)$ which computes the perimeter of the inscribed square.

- Find a function $A(r)$ which computes the area of the inscribed square.

**Example.** Let $f(x) = \sqrt{x}$. Find a function $A(b)$ which computes the area of a triangle whose vertices are $(0,0)$, $(b,0)$, and $(b,c)$, where $(b,c)$ is on the graph of $y = f(x)$.



**Example.** You own land along side a river, which you intend to fence. You have 400 feet of fence which you intend to fence three sides of a rectangular plot, allowing the river to compose the final side.



- If the side parallel to the river is to be 100 feet, find the area of the resulting plot of land.

- Let $x$ be the length of fence parallel to the river. Find the function $A(x)$ which computes the area of the enclosed rectangular area.

**Example.** A piece of wire which is 20cm long is to be cut in to two smaller wires.

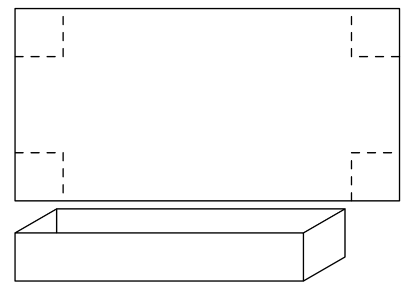
- If one length is 8cm (and thus the other is 12cm) and each is bent into the shape of a square, what would be the enclosed area?

- Let $x$ be one of the cut wire lengths. Find the function $A(x)$ which computes the enclosed area if each length of wire is shaped into a square.

- Let $x$ be one of the cut wire lengths. Find the function $B(x)$ which computes the enclosed area if each length of wire is shaped into a circle.

**Example.**

An open-top box is created by taking a $8 \times 12$ in$^2$ piece of cardboard and cutting squares from the corners, then folding up the sides.



- If you cut out $2 \times 2$ squares from each corner, what would be the volume of the resulting box?

- Let $x$ be the length of the side of a cut-out square. Find the function $V(x)$ which computes the resulting volume of the box.

# Lesson 12: Quadratic Functions

## Lecture 11: Quadratic Equations

### Completing the square ... again.

To make $x^2 + bx$ a perfect square, for example $x^2 - 6x$, first divide $b$ by 2, then square the result.

### Quadratic Equations

A **quadratic equation** is an equation of the form $a \cdot x^2 + b \cdot x + c = 0$.

The solutions to a quadratic equation are called **roots**.

**Example.** Solve the following quadratic equations by completing the square:

- $x^2 - 2x - 3 = 0$

- $x^2 - 2x - 4 = 0$

**Example.** Solve the equation $3x^2 + 6kx + 4 = 0$ (in terms of $k$)

## The Quadratic Formula

**Example.** Solve the equation $ax^2 + bx + c = 0$ (in terms of $a, b, c$).

The general solutions to a quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This is referred to as the **quadratic formula**.

**Example.** Solve for all real roots in the following quadratic equations:

- $x^2 - 12x + 35 = 0$

- $x^2 - 12x + 36 = 0$

- $x^2 - 12x + 37 = 0$

| Observation: The number of distinct real solutions to a quadratic equation can be 0, 1, or 2. |
|---|

**The Discriminant.**

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is $b^2 - 4ac$, which is **the quantity under the square root in the quadratic formula**.

The discriminant is used to find **the number of real solutions of a quadratic equation**.

- If the discriminant is **positive**, then there are two (2) distinct real solutions,
- If the discriminant is **zero (0)**, there is exactly one (1) real solution, and
- If the discriminant is **negative**, then ther are zero (0) real solutions.

Note: The discriminant does NOT tell you the SOLUTIONS of a quadratic equation. It tells you **HOW MANY** solutions it will have.

**Example.** Find the number of real solutions for $2x^2 - 3x - 1 = 0$.

**Example.** Find the value(s) of $k$ which make $4x^2 + kx + 9 = 0$ have exactly 1 real solution.

## Quadratic Functions

A **quadratic function** is a function of the form $f(x) = a \cdot x^2 + b \cdot x + c$, where $a$, $b$, and $c$ are constants, and $a \neq 0$.

Note: If $a = 0$, then our function would be **linear**.

## Graph properties of a quadratic function

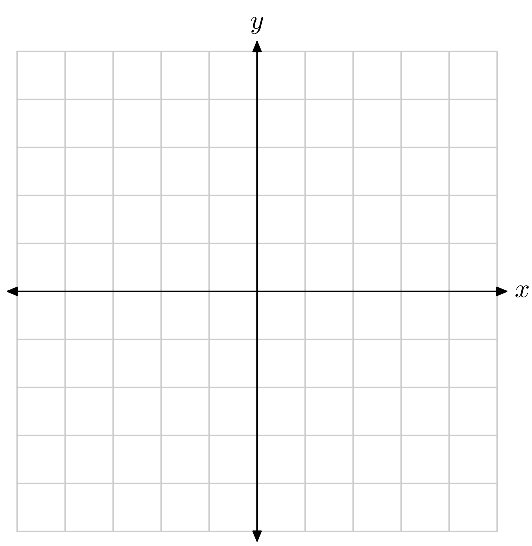
The graph of a quadratic function is a **parabola**.

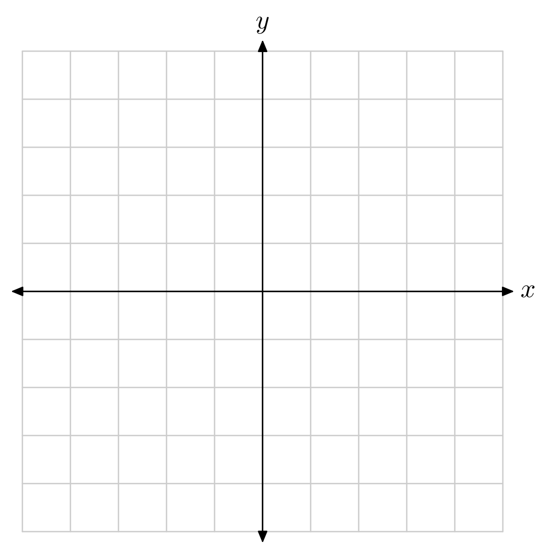
- When $a < 0$, the parabola will open **downward**.
- When $a > 0$, the parabola will open **upward**.

The turning point on the parabola is called the **vertex**.

The vertical line passing through the vertex is called the **axis of symmetry**, because the parabola is **symmetric about that line**.

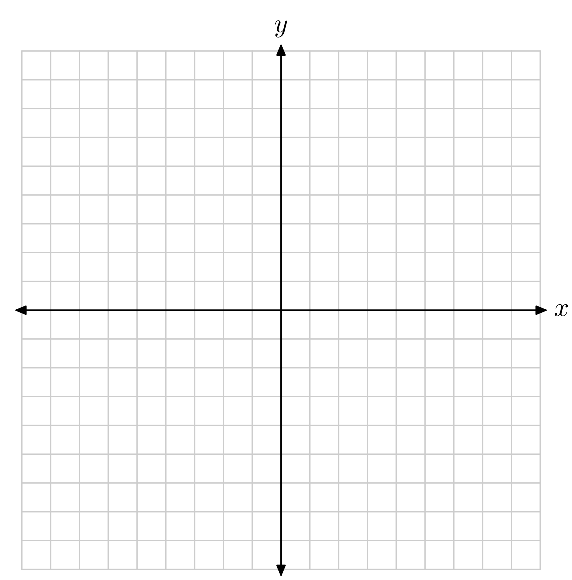
**Example.** Sketch a parabola which opens downward and one which opens upward. Label the vertex and axis of symmetry.



**Example.** Graph the function using translation of $y = x^2$ by completing the square. Find the vertex, axis of symmetry, and intercepts.

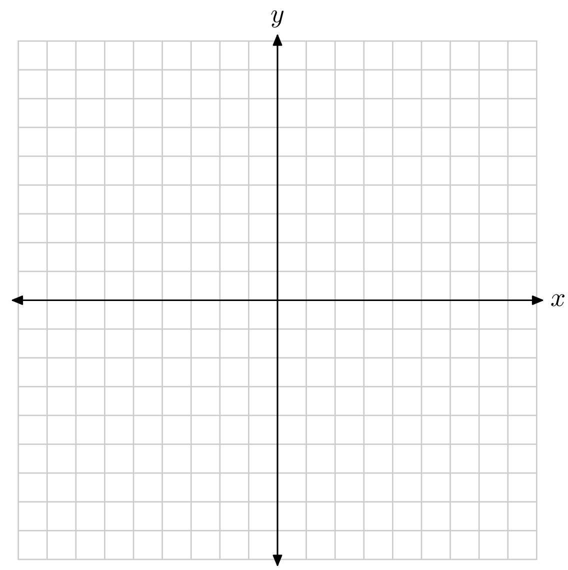
$$y = x^2 - 6x + 8$$



## The Graph of $y = ax^2$

**Example.** Let $f(x) = x^2$, $g(x) = \frac{1}{2}x^2$, $h(x) = 2x^2$, and $j(x) = -2x^2$. Complete the chart below and graph each quadratic function.

| $x$ | $f(x)$ | $g(x)$ | $h(x)$ | $j(x)$ |
|---|---|---|---|---|
| $-2$ | | | | |
| $-1$ | | | | |
| $0$ | | | | |
| $1$ | | | | |
| $2$ | | | | |

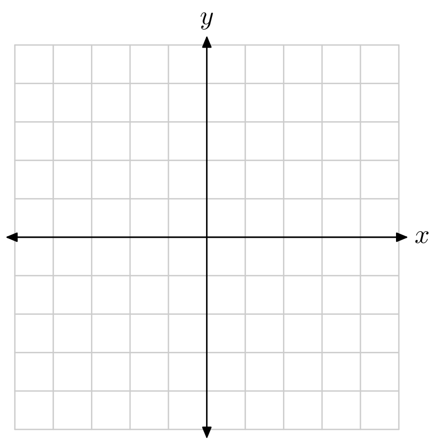


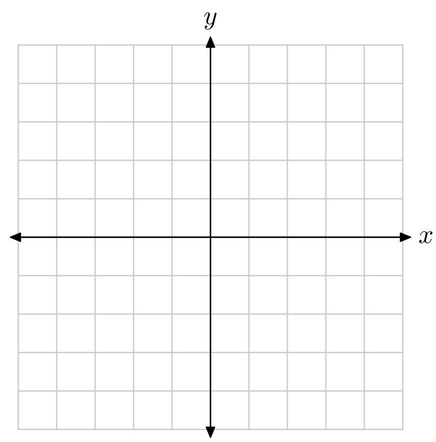

To summarize, $y = ax^2$ is a parabola, similar to $y = x^2$, and

- If $a < 0$, then the graph opens **downward**.
- If $a > 0$, then the graph opens **upward**.
- If $|a| > 1$, then the graph opens **narrower than** $y = x^2$.
- If $|a| < 1$, then the graph opens **wider than** $y = x^2$.

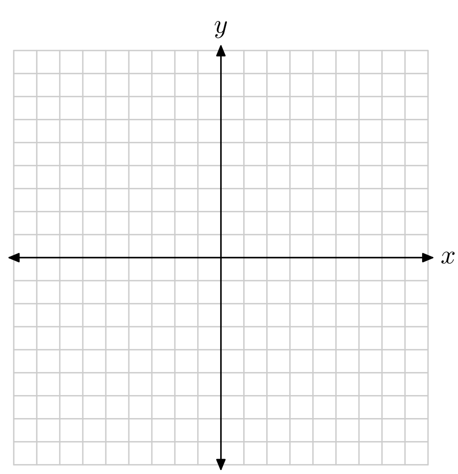
**Example.** Graph the following as translations of $y = ax^2$:

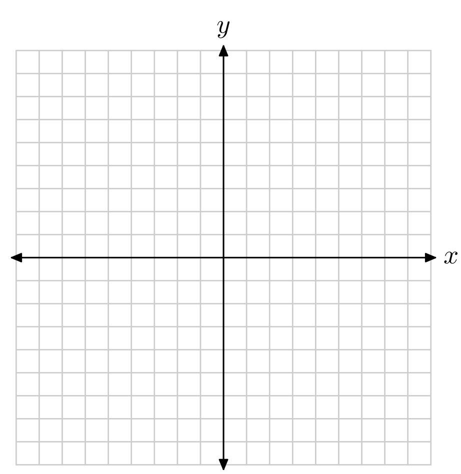
- $y = (x-1)^2 - 2$



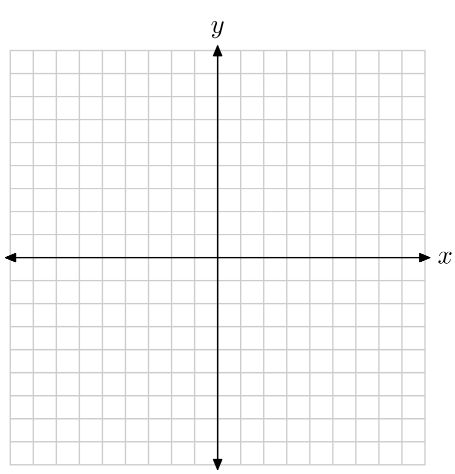

- $y = 3(x+1)^2 + 4$

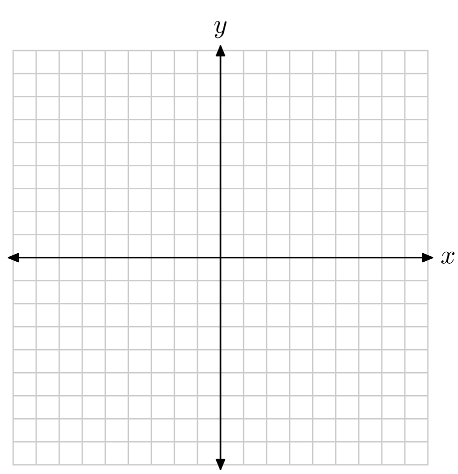


**Example.** Graph each of the following as translates of $y = ax^2$ by completing the square.

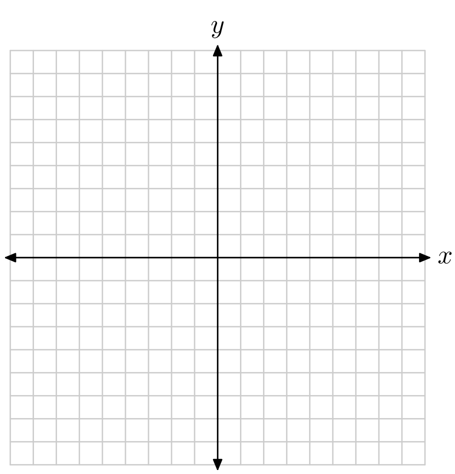
$$y = -2x^2 - 4x + 5$$

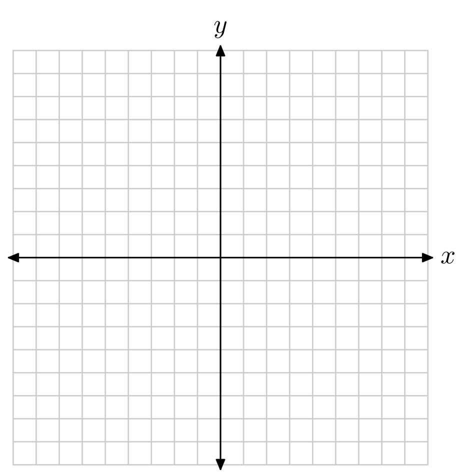




**Example.** Graph the following as translates of $y = ax^2$ by completing the square.

$$y = \frac{1}{2}x^2 - 2x + 3$$



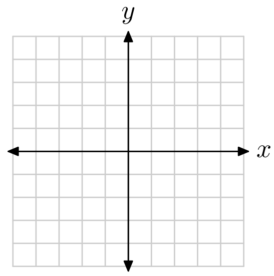



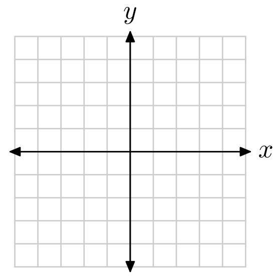
## Extreme Values

A quadratic function will have a **maximum output** when $a$ **is negative**.

A quadratic function will have a **minimum output** when $a$ is positive.

This will always happen at the **vertex**.




The **$x$-coordinate of the vertex** is the input which yields the maximal/minimal output.

The **$y$-coordinate of the vertex** is the maximal/minimal output.

## Shortcut for finding the Vertex

The **$x$-coordinate** of the vertex of $y = a \cdot x^2 + b \cdot x + c$ is $-\dfrac{b}{2a}$.

**Example.** Find the maximum / minimum output for the following functions:

- $y = f(x) = x^2 - 4x + 3$

- $y = f(x) = -2x^2 + 6x - 9$

## The Vertex Form of a Quadratic Function

The equation of the parabola $y = ax^2 + bx + c$ can always be rewritten as $\underline{y = a(x-h)^2 + k}$, where the $\underline{(h, k)}$ is the **vertex**.

**Example.** Find the quadratic function which passes through the point $(-2, 3)$ and has a vertex of $(1, 5)$.

**Example.** For what value of $c$ will the minimum value of $f(x) = x^2 - 4x + c$ be -7?

**Example.** For what value of $c$ will the maximum value of $f(x) = -3x^2 + 6x + c$ be 12?

# Lesson 13: Optimization of Quadratic Functions

**Example.** If two numbers add to 12, what is the largest their product can be?

A Rough Sketch for How to Solve Optimization problems

1. Identify that which you want to optimize (I'll call it $A$ for the time being). Determine a way (a formula) to, in general, compute $A$.
2. If your formula gives $A$ in terms of more than one quantity (variable), then you must find a relationship (a **constraint**) between your variables used to find $A$
3. For this course, these problems should yield a quadratic function. Find the vertex, and determine whether the vertex gives the maximal or minimal output.
4. Answer the question that is asked in the problem.

**Example.** What is the largest possible area of a right triangle if the lengths of the two legs add up to 80 in.?

**Example.** A ball is thrown in the air with a velocity of 29 ft/s.
Its height is modeled by the function

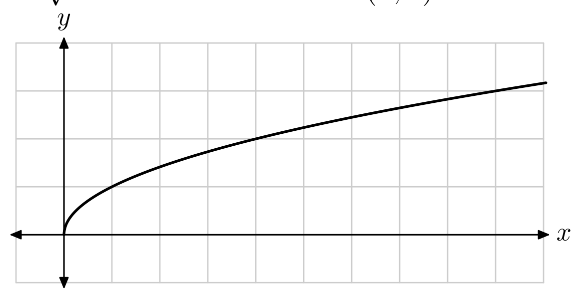
$$H(t) = -16t^2 + 29t + 6$$

- When does the ball hit the ground?

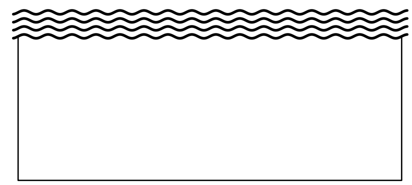
- When does it reach its maximum height?

- How high does it go?

**Example.** Find the point on the curve $y = \sqrt{x}$ which is nearest to $(8, 0)$.



**Example.** A farmer wants to erect a fence for cattle using the nearby river as a border. He has 600 feet of fence. What dimensions would maximize the enclosed area?

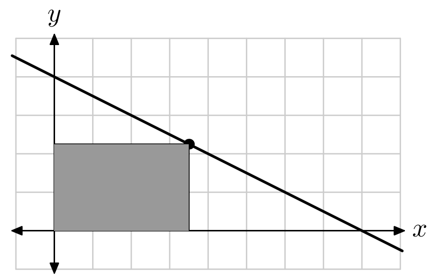


**Example.** The revenue for making $x$ units of methylchloroisothiazolinone is modeled by

$$R(x) = -0.1x^2 + 20x - 100$$

Find the maximum revenue, assuming you can sell partial units.

**Example.** Find the maximum area that a rectangle can have being bordered by the $x$-axis, the $y$-axis, and the graph of $y = 4 - \frac{1}{2}x$.



# Lesson 14: Inequalities

## Properties of Inequalities

| Property | Example |
|---|---|
| ▸ Suppose $a < b$.<br>Then $\underline{a + c < b + c}$ and $\underline{a - c < b - c}$. | ▸ |
| ▸ Suppose $a < b$ and $c$ is positive.<br>Then $\underline{a \cdot c < b \cdot c}$ and $\underline{a/c < b/c}$. | ▸ |
| ▸ Suppose $a < b$ and $c$ is positive.<br>Then $\underline{a \cdot c > b \cdot c}$ and $\underline{a/c > b/c}$. | ▸ |
| ▸ Suppose $a < b$ and $b < c$. Then $\underline{a < c}$. | ▸ |

**Example.** Solve the following inequalities. Give your answer in interval notation.

- $4x + 3 < -5$
- $2x - 1 \leq 7(x + 2)$
- $\dfrac{1}{2} \leq \dfrac{2 - x}{3} \leq 2$

## Inequalities with Absolute Value

| Property | Example |
|---|---|
| ▸ $\lvert u\rvert < a$ means $\underline{-a < u < a}$. | ▸ |
| ▸ $\lvert u\rvert \le a$ means $\underline{-a \le u \le a}$. | ▸ |
| ▸ $\lvert u\rvert > a$ means $\underline{u > a \text{ or } u < -a}$. | ▸ |
| ▸ $\lvert u\rvert \ge a$ means $\underline{u \ge a \text{ or } u \le -a}$. | ▸ |

Note: When handling cases like $|u| > a$, you cannot solve it as a chain of inequalities like the previous example.

$u > a$ or $u < -a$ does NOT mean $-a \ge u \ge a$. This would imply that $-a \ge u$ AND $u \ge a$.

Make sure not to make this mistake.

**Example.** Solve the following inequalities:

- $|x - 1| \le 2$
- $\left|-\frac{1}{2}x + 3\right| < 2$
- $|2x - 3| \ge 1$
- $\left|\dfrac{-x + 3}{-2}\right| > 2$

**Example.** Solve the following inequalities:

- $\left|\dfrac{3(x-2)}{4}+\dfrac{4(x-1)}{3}\right| \leq 2$

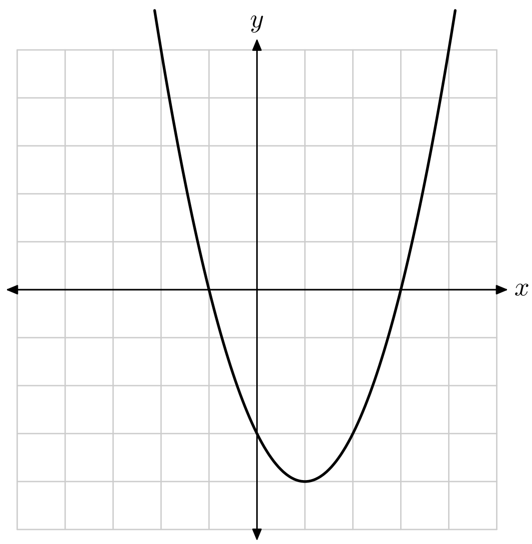
- $|-x+3| < -2$

- $|(x+h)^2 - x^2| < 3h^2$
  (Solve for $x$, $h$ positive)

- $\left|-\frac{1}{2}x+3\right| > 0$

## Polynomial Inequalities

**Example.** The graph of $y = x^2 - 2x - 3$ is given.
Use the graph to determine when




- $x^2 - 2x - 3 < 0$

- $x^2 - 2x - 3 \geq 0$

## How to solve polynomial inequalities without graphing:

If you have an inequality that looks like

$$\text{polynomial} < 0, \text{ (or using } >, \geq, \text{ or } \leq)$$

you must find **when the polynomial equals zero**. We call these solutions the **key values**.

Then, determine whether the polynomial is **positive** or **negative** between the key values.

If the polynomial is positive at one value of $x$ between two key values, then it must be so for all values $x$ between them. Similarly, if the polynomial is negative at one value of $x$ between two key values, then it must be so for all values $x$ between them.

**Example.** Solve the following inequality:

$$x^2 - 2x - 3 < 0$$

- Find all values of $x$ for which $x^2 - 2x - 3 = 0$.

  These values are the **key values** for this inequality.

- Identify the intervals which are separated by the key values.

- Now, test a value from each interval and each key value; See if they satisfy the inequality.

| Interval | Test Number | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

**Example.** Solve the following inequality:

$$x^3 - 4x \geq 0$$

- 

- 

| Interval | Test # | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |

**Example.** Solve $x^3 - 4x < x^2 - 5x + 1$

- 

- 

- 

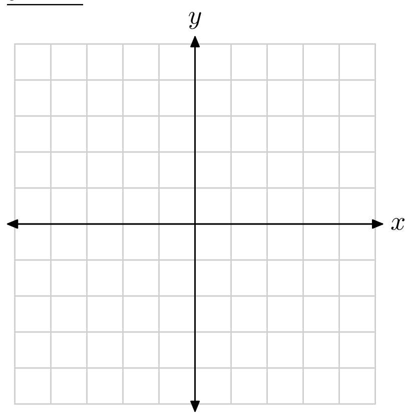
| Interval | Test # | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |

# Lesson 15: Graphs of Polynomials

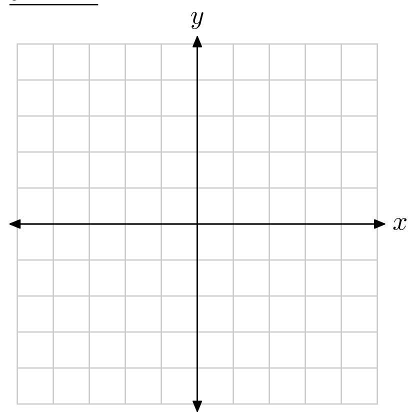
A **power function** is a function of the form $f(x) = x^n$.

**Graphs of the power functions $y = x^n$:**

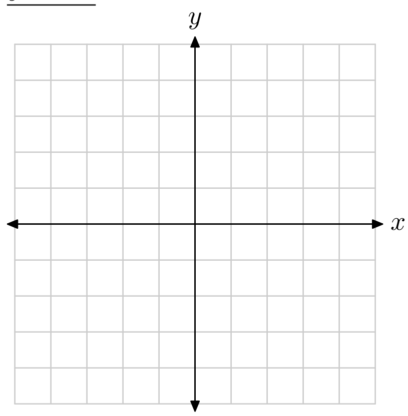
$y = x$



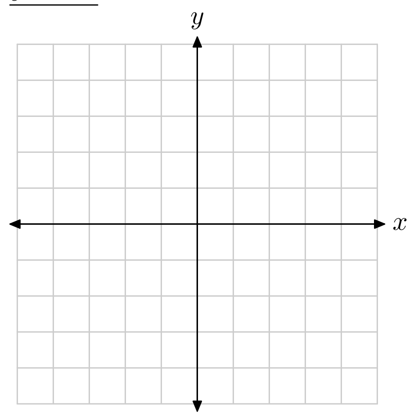

$y = x^2$



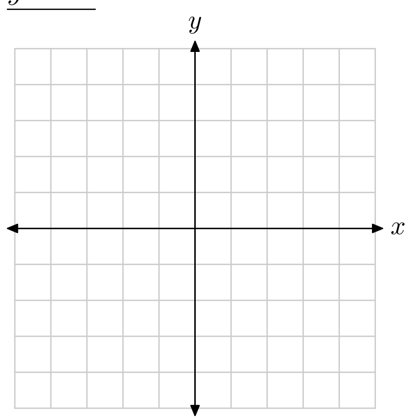

$y = x^3$



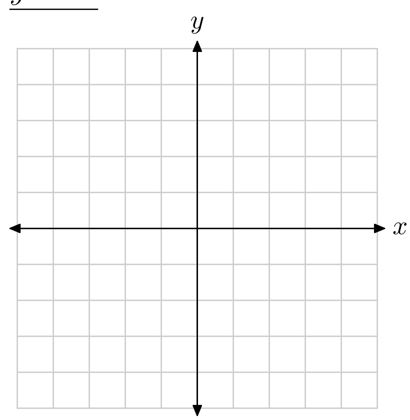

$y = x^4$




$y = x^5$




$y = x^6$



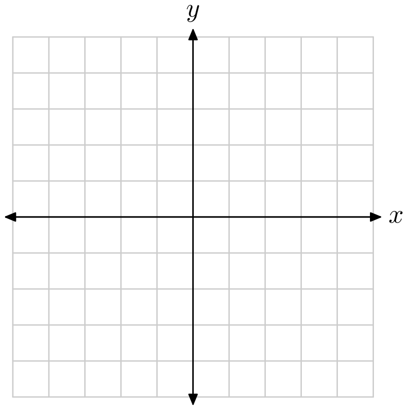
## Summary

The graph $y = x^n$ goes through the points

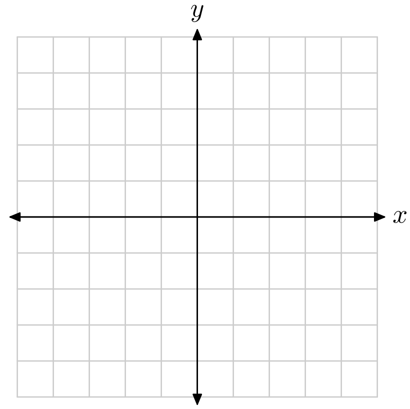
$(1, 1)$, $(0, 0)$, and $(-1, -1)$ when **$n$ is odd**, or

$(1, 1)$, $(0, 0)$, and $(-1, 1)$ when **$n$ is even**.

$y = x^n$ ($n$ odd)

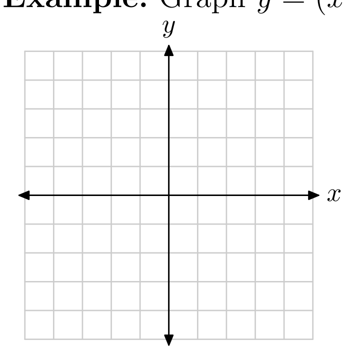


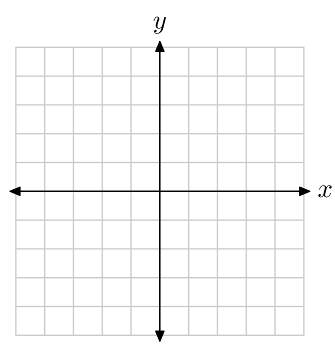

$y = x^n$ ($n$ even)



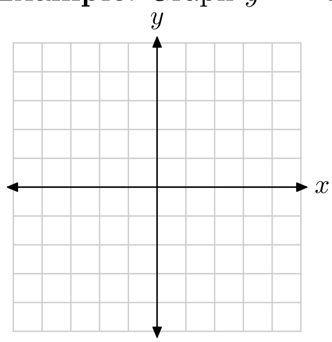

## Translating Graphs of Power Functions

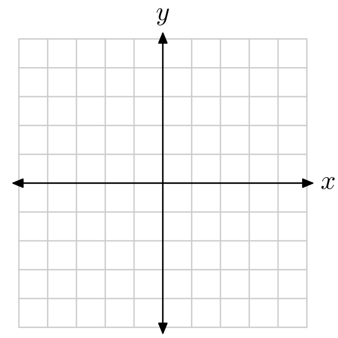
**Example.** Graph $y = (x + 1)^3$.



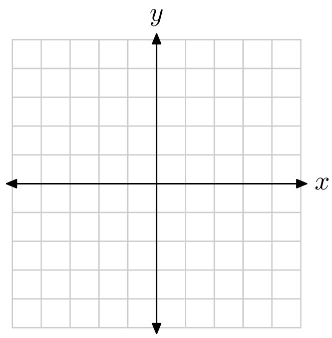
**Example.** Graph $y = -2x^3$.

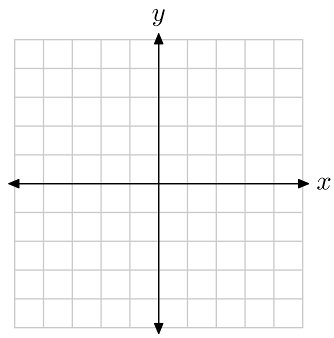




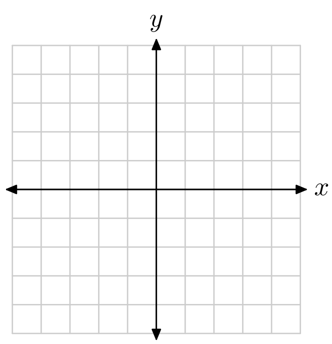
**Example.** Graph the following functions:

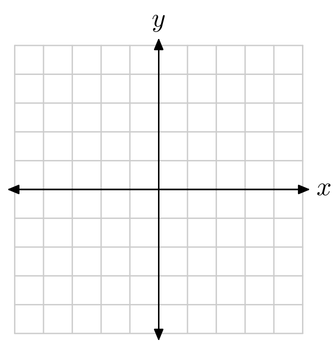
- $y = -(x-2)^4$

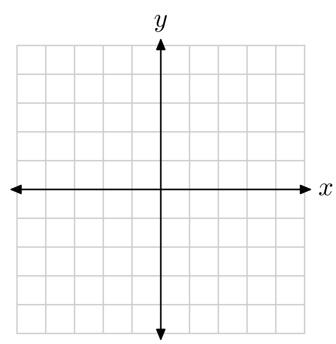


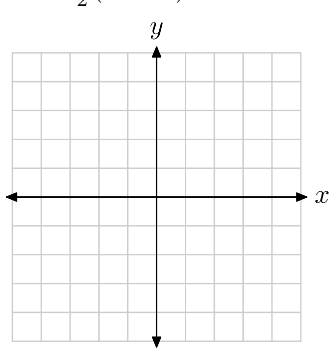

- $y = 2(-x-4)^5$

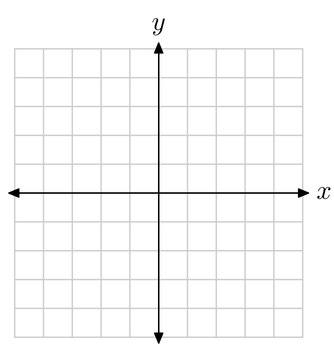


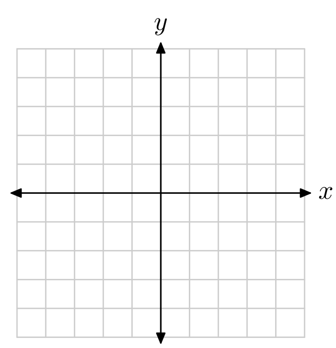




- $y = -\frac{1}{2}(x+1)^6 - 2$



## Polynomial Functions

A **polynomial function** is a function defined by an equation of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

**(a product/sum made from various power functions)**

The value of $\underline{n}$ is called the **degree of** $p(x)$.

The quantities $\underline{a_n, a_{n-1}, \ldots, a_1, a_0}$ are called the **coefficients** of $p(x)$.

**Example.** Determine if each of the following functions are polynomial functions. If so, identify the degree of the polynomial.

- $f(x) = x^3 - 4x^2 + 1$
- $g(x) = x - x^5$
- $h(x) = 4$
- $f(x) = \sqrt{x} + 2$
- $g(x) = \frac{1}{x} + x^2 - 2$
- $h(x) = x^2 - \frac{x}{a} + \sqrt{a^\pi}$

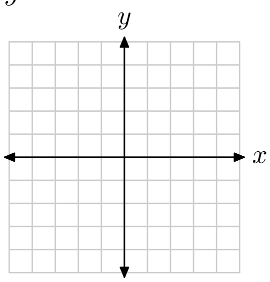
**Example.** Identify the $\underline{x^2}$ coefficient (the coefficient in front of the squared input variable for each of the following:

- $f(x) = 1 - x^2$
- $g(y) = y + 3y^2$
- $h(z) = z - z^3$
- $j(q) = abq - pq^2rs$

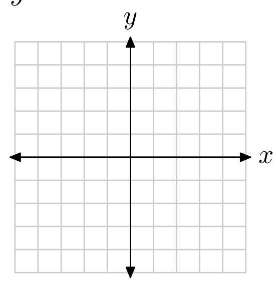
## Properties of Graphs of Polynomials

- If a polynomial is degree 2 or more, the graph is **curved and smooth** (no kinks or breaks).

$y = x^2$

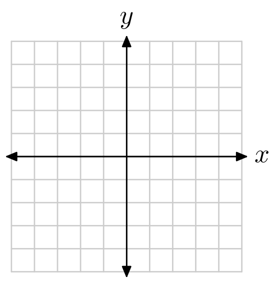



$y = x^3$

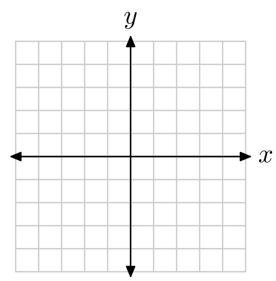



- A polynomial of degree $n$ has at most $n-1$ turning points.

$y = x - x^3$




$y = x^4$

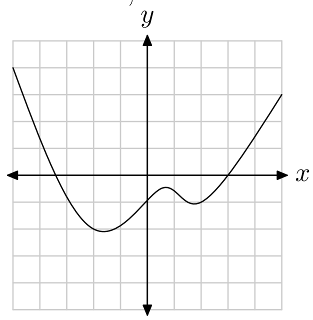


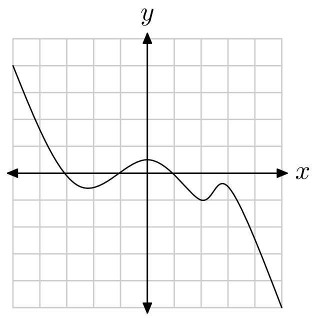

- If the polynomial has even degree, then the end behaviors of the graph **agree**.
- If the polynomial has odd degree, then the end behaviors of the graph **differ**.
- If the leading coefficient is positive, the graph goes to $+\infty$ as $x$ gets bigger.
- If the leading coefficient is positive, the graph goes to $-\infty$ as $x$ gets bigger.

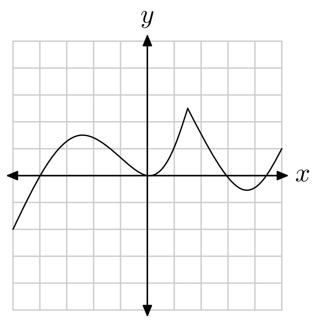
**Example.** Describe the end behavior of the following polynomial functions.

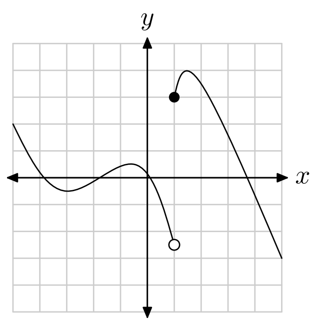
- $f(x) = x^4 + 3x^2$
- $f(x) = x - \pi x^3$
- $f(x) = -4x^{1000}$

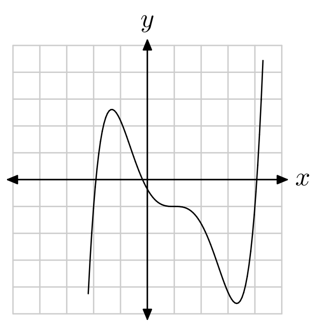
**Example.** For each of the following graphs, determine if the function could be a polynomial function. If so, determine the possible degree.



### Procedure for graphing completely-factored polynomials

In order to graph a polynomial function $f(x)$ (if $f$ can be completely factored):

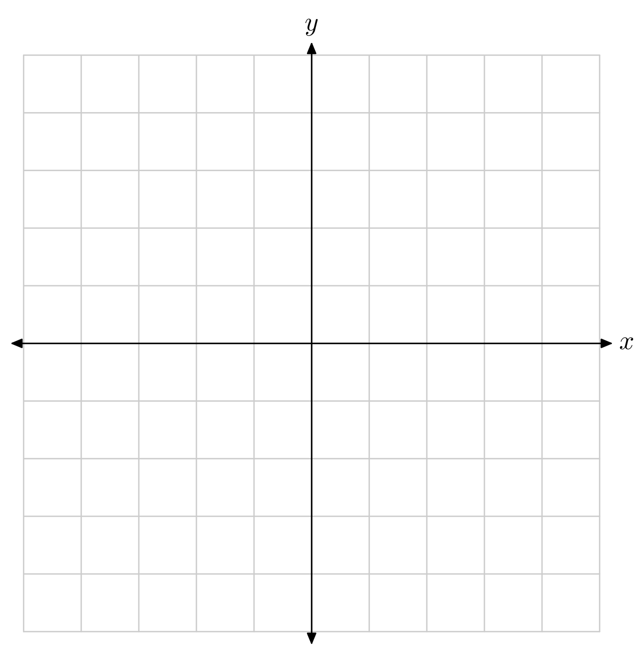
1. Find the $x$-intercepts of the graph of the polynomial and graph them.

2. Make a sign chart and use test values to determine when $f(x)$ is positive or negative

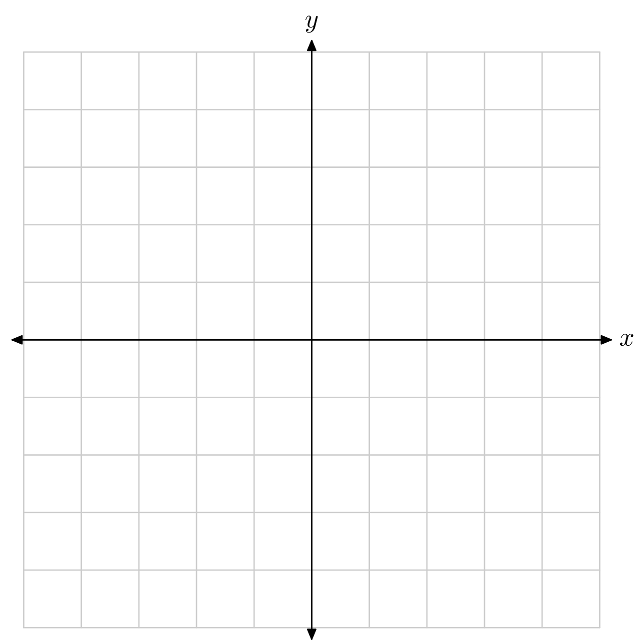
3. (optional) Shade excluded regions, i.e. shade regions in the plane in which the graph cannot pass, based on the sign chart

4. Connect the intercepts and depicts the correct end behavior.

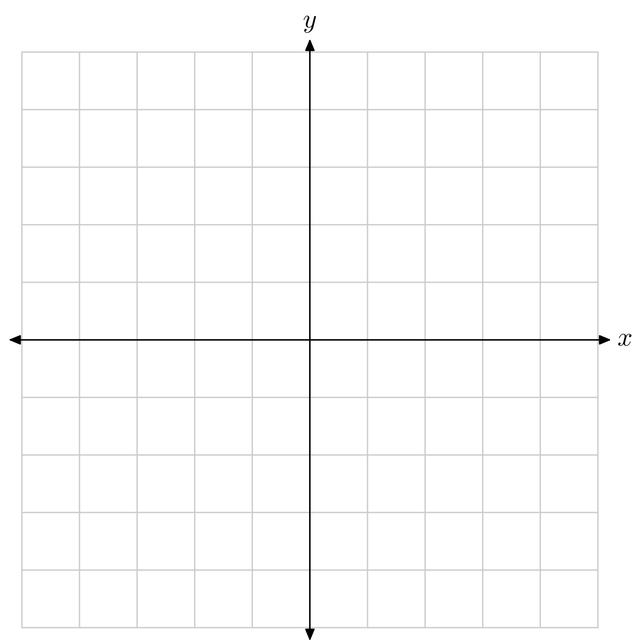
**Example.** Graph $y = \frac{1}{2}(x+2)(x-3)^2$



**Example.** Graph $y = -2x(x+2)^2(x-1)(x-3)^3$



**Example.** Graph $y = -3x(x-1)(x-2)(x-3)$



# Lesson 16: Polynomial Division

**Example.** Using long division, compute $502 \div 3$ and $502 \div 13$.

**Definitions.**

When doing division,

- the **dividend** is the expression into which is being divided (Ex. 502),
- the **divisor** is the expression which is being used to divide (ex. 3, 13),
- the **quotient** is the expression which is the result of the division (ex. 167, 38), and
- the **remainder** is the expression left over that cannot be further divided.

Notice that the procedure involves finding a number to go on top, multiplying it by the divisor, subtracting from the dividend, then repeating.

When performing polynomial long division, essentially the same procedure holds.

**Example.** Find the quotient of $\dfrac{3x^2 - 2x + 5}{x - 3}$

**Example.** Find the quotient and remainder of $\dfrac{x^3 - x + 1}{x + 1}$.

**Example.** Find the quotient and remainder of $\dfrac{x^4 - 3x^3 + x - 1}{x^2 + 1}$

**Example.** Find the quotient and remainder of

- $\dfrac{x^3 - 3x^2 + 2x + 1}{x^2 - 2x + 3}$

- $\dfrac{x^2 - 10x + 3}{x - 4}$

## Synthetic Division

**Example.** Recall from the previous example that $\dfrac{x^2 - 10x + 3}{x - 4}$ is $x - 6 = 1 \cdot x + (-6)$ with a remainder of $-21$. Follow the following instructions using this division example.

- Write out the coefficients (including zeroes for the dividend) and identify $r$, where $x - r$ is the divisor.

- Set up a grid with three rows; the first row will be $r$, followed by your coefficients.

  The second and third row are to be separated by a line.

- Drop down the first coefficient and write it in the third row. Then multiply it by $r$. Put that result under the second coefficient in the second row. Then add that result to the second coefficient, and put the sum in the third row.

- Repeat this procedure until the last entry in the third row is filled.

- The last entry in the third row is the remainder of dividing by $x - r$. The rest of the third row are the coefficients of the quotient. This method is called **synthetic division**.

**Example.** Find the following quotients and remainders:

- $\dfrac{x^3 - 2}{x + 2}$

- $\dfrac{x^4 - 3x^3 + x^2 - 1}{x - i}$

- $\dfrac{x^2 - 5x + 2}{2x - 1}$

- $\dfrac{x^3 - 7x^2 + x - 3}{x^2 - 1}$

**Example.** Find the quotient and remainder of $\dfrac{x^4 - 13a^3x + 12a^4}{x - a}$.

# Lesson 17: The Factor and Remainder Theorems

## The Remainder Theorem

**Example.** Find the quotient and remainder for each of the following divisors, then rewrite $f(x) = 2x^2 + 6x - 5$ in terms of the quotient and remainder.

- $\dfrac{2x^2 + 6x - 5}{x - 1}$
- $\dfrac{2x^2 + 6x - 5}{x - 2}$
- $\dfrac{2x^2 + 6x - 5}{x - 3}$

Fill in the chart below. In the second column, write out $f(x)$ as $(x - r)q(x) + R$. When evaluating $f(r)$, use the formula in the second column.

| $r$ | $f(x) = 2x^2 + 6x - 5 =$ | Remainder | $f(r)$ |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |

**The Remainder Theorem**. If $f(x)$ is a polynomial, then the remainder of $\dfrac{f(x)}{x - r}$ is $\underline{f(r)}$.

**Example.** Find only the remainder in the following division examples.

- $\dfrac{x^2 - 10x + 3}{x - 4}$
- $\dfrac{x^6}{x - 2}$

**Property of Integers.**

Suppose $a$ and $B$ are positive integers. Then $a$ is a factor of $B$ if the remainder of $\frac{B}{a}$ is **zero (0)**.

**Example.** Determine if $a$ is a factor of $B$:

- $a = 2,\ B = 26$
- $a = 9,\ B = 112$
- $a = 13,\ B = 117$

**Factors of Polynomials**

Suppose $a(x)$ and $B(x)$ are polynomials.
Then $a(x)$ is a factor of $B(x)$ if $\frac{B(x)}{a(x)}$ has a remainder of **zero (0)**.

**Example.** Determine if $a(x)$ is a factor of $B(x)$.

- $a(x) = x - 1,\ B(x) = x^3 - 1$
- $a(x) = x + 2,\ B(x) = x^3 - 2x + 3$
- $a(x) = x - 3,\ B(x) = x^4 - 9x - 54$
- $a(x) = x - 2,\ B(x) = x^9 - 4x^7 + 2$

**The Factor Theorem.** $x - r$ is a factor of a polynomial $f(x)$ if and only if $f(r) = 0$.

**Example.** Determine if $d(x)$ is a factor of $p(x)$. If $d(x)$ is a factor, rewrite $p(x) = (x - r) \cdot q(x)$.

- $p(x) = x^3 - 2x + 1$, $d(x) = x - 1$.

- $p(x) = x^4 + 2x^2 - 7x - 9$, $d(x) = x - 2$.

- $p(x) = x^4 - 2x^3 - 7x^2 - 2x - 8$, $d(x) = x^2 + 1$.

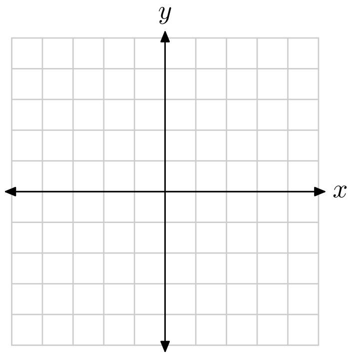
- $p(x) = x^3 + 4x^2 - 3x - 14$, $d(x) = x + 2$.

**Example.** Determine the value of $k$ which makes $x - 3$ be a factor of $f(x) = x^3 - 4x^2 - kx + 3$.

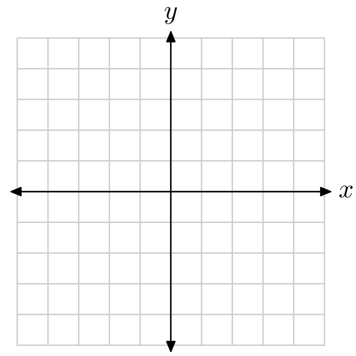
# Lesson 18: Introduction to Rational Functions

**Graphing** $f(x) = \dfrac{1}{x^n}$

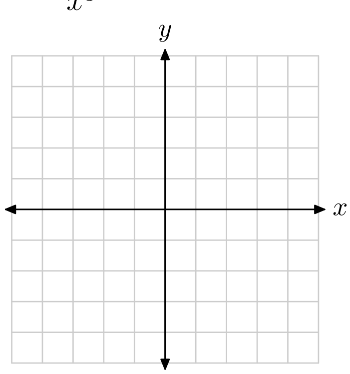
$y = \dfrac{1}{x}$



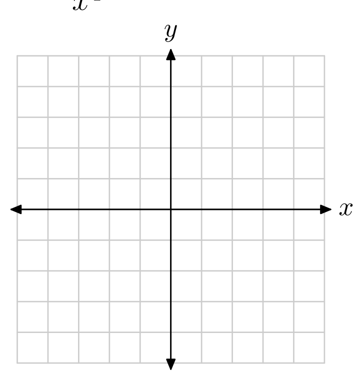

$y = \dfrac{1}{x^2}$



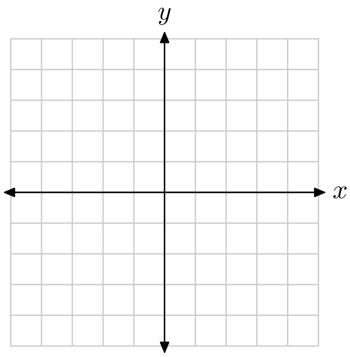

$y = \dfrac{1}{x^3}$




$y = \dfrac{1}{x^4}$



$y = \dfrac{1}{x^n}$, $n$ odd



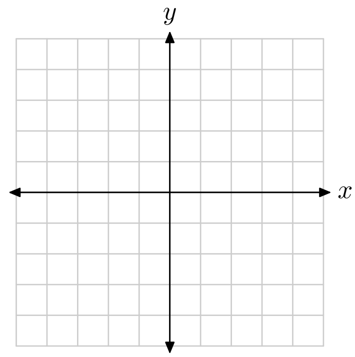

The graph has asymptotes

____________________

Contains points

___________________________

$y = \dfrac{1}{x^n}$, $n$ even




The graph has asymptotes

____________________

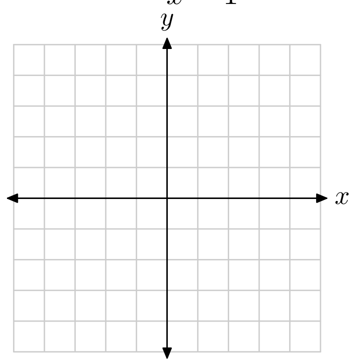
Contains points

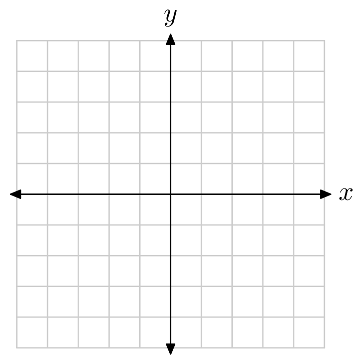
___________________________

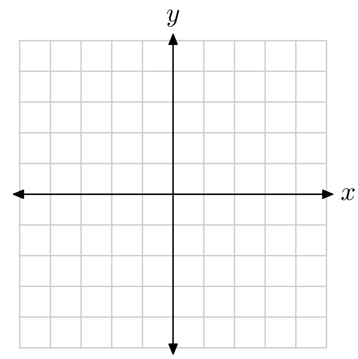
## Graphing Translates and Scales

**Example.** Graph the following translates of reciprocal power functions.

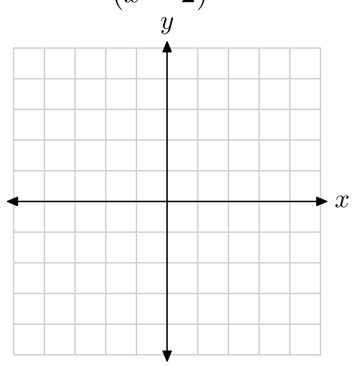
- $y = 2 - \dfrac{1}{x-1}$

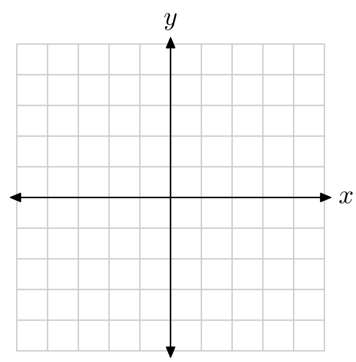


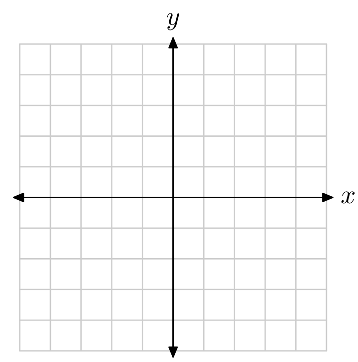




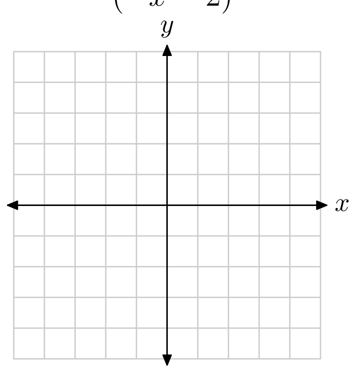

- $y = \dfrac{4}{(x-2)^2} + 1$

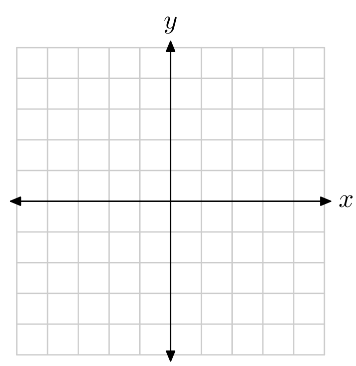


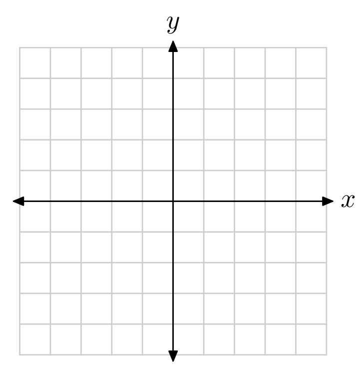





- $y = \dfrac{1}{(-x-2)^3}$







A ________________________ is a function which is the ________________________

of two ________________________ . So a rational function $r(x)$ is of the form

$$r(x) =$$

For this course, we will assume that $p(x)$ and $q(x)$ have no factors in common.

For example, functions like $f(x) = \dfrac{x^2}{x}$ will not be discussed.

**Example.** Identify if each of the following is a rational function. If so, identify $p(x)$ and $q(x)$.

- $f(x) = \dfrac{x^2 + 2x - 3}{x^3 - 1}$
- $g(t) = \dfrac{t^2 + 1}{t^2 - 1}$
- $l(x) = \dfrac{1}{x} + x$
- $m(x) = \dfrac{x + 1}{\sqrt{x^2 + 1}}$
- $h(y) = \dfrac{1}{y^3 + 1}$
- $k(s) = s^2 - 10s + 7$

## Properties of Rational Functions

- The domain of a rational function $r(x) = \dfrac{p(x)}{q(x)}$ is __________________________ .

**Example.** Find the domain of the following rational functions.

- $f(x) = \dfrac{1}{x^2 - 1}$
- $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

## Vertical Asymptotes.

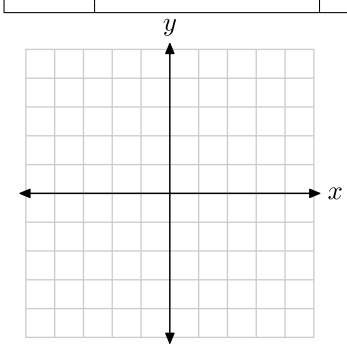
An __________________________ is a line which describes the behavior of a graph as it leaves the viewable area.

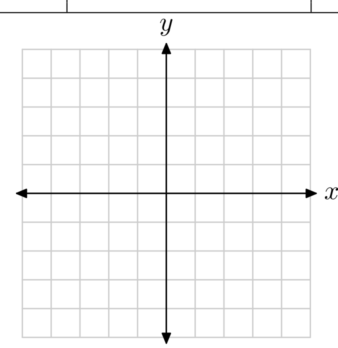
A line is ____________________ to a graph if the graph gets arbitrarily close the to the line as it leaves the viewable area of the graph.

**Example.** Evaluate the function chart below for $f(x) = \dfrac{1}{x-2}$ and $g(x) = -\dfrac{5}{(x-2)^2}$.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 3 | | |
| 2.1 | | |
| 2.01 | | |

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | | |
| 1.9 | | |
| 1.99 | | |




**Conclusion.**
When graphing a rational function near a value $a$ outside its domain,

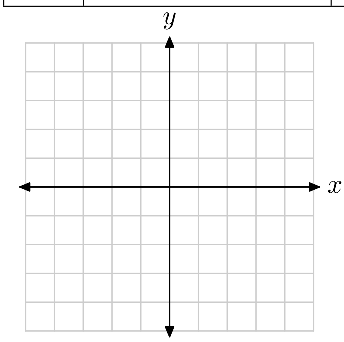
its graph goes to ______ or ______ , and is indicated using a vertical asymptote _____________ .

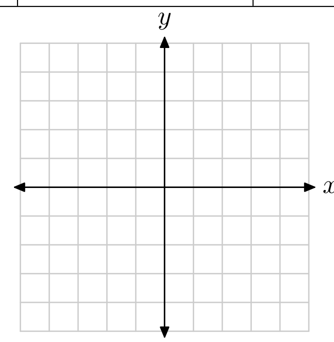
## Horizontal Asymptotes

**Example.** Evaluate the function chart for $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{1-x}{x-2}$.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 3 | | |
| 5 | | |
| 100 | | |

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | | |
| $-5$ | | |
| $-100$ | | |




**Conclusion.**

When these rational functions are evaluated at large inputs, outputs get close to a certain value.

When graphing, we indicate this with a ________________________ .

## How to Determine the Horizontal Asymptote

If $x$ gets very large, look at terms of __________________ in ____________ and ____________ .

Do this by ________________________________________________ .

| $f(x)$ | What is $f(x)$ when $\lvert x\rvert$ is big | Horizontal Asymptote |
|---|---|---|
| $\dfrac{1}{x-2}$ | $f(x) \approx$ ________________________ | ____________ |
| $\dfrac{1-x}{x-2}$ | $f(x) \approx$ ________________________ | ____________ |

**Summary**

A rational function $r(x) = \dfrac{p(x)}{q(x)}$ has

- a vertical asymptote at ____________ when ____________ .
- a horizontal asymptote at ____________ when ____________________________________ .

**Example.** Find vertical and horizontal asymptotes for the following:

- $f(x) = \dfrac{x-2}{x^2-9}$

- $g(t) = \dfrac{x^2-6x}{x^2-2x-3}$

- $f(x) = \dfrac{x(2-x)(4x+3)^3}{4+x^5}$

- $g(t) = \dfrac{x^2+10000x-1}{2x^2-x-1}$

# Lesson 19: Graphs of Rational Functions and Inequalities

## Rational Inequalities

When solving rational inequalities, simplify your inequality to look like

$$\frac{\text{polynomial}}{\text{polynomial}} < 0 \text{ (or } >, \geq, \leq)$$

Again, find key values; values of $x$ where

______________________________________________________________ .

With these key values, do the same interval testing as before to determine your interval of solutions.

**Example.** Solve $\dfrac{x-1}{x+2} \leq 0$

- 

- 

| Interval | Test Value | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |
| | | | | | |

**Example.** Solve $\dfrac{3}{x+2} \geq x$

- 

- 

| Interval | Test Value | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |
| | | | | | | | |

## Graphing Rational Functions

If we are to graph $y = r(x) = \dfrac{p(x)}{q(x)}$, we go through the following procedure:

1.

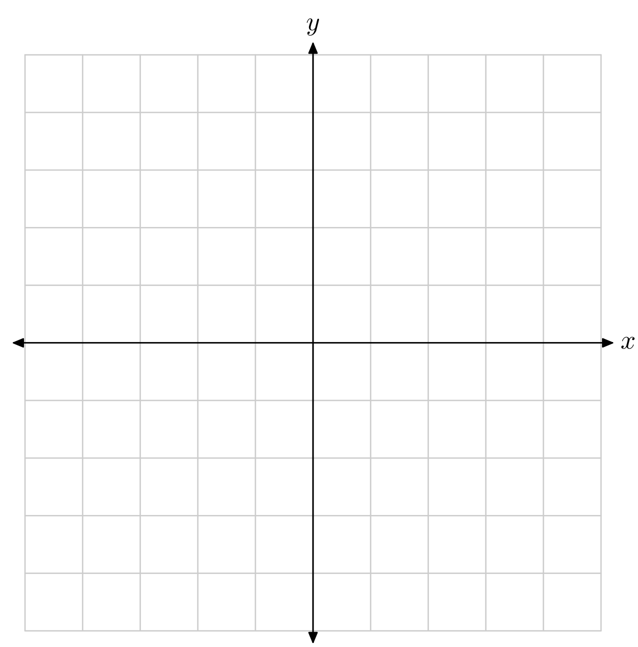
2.

3.

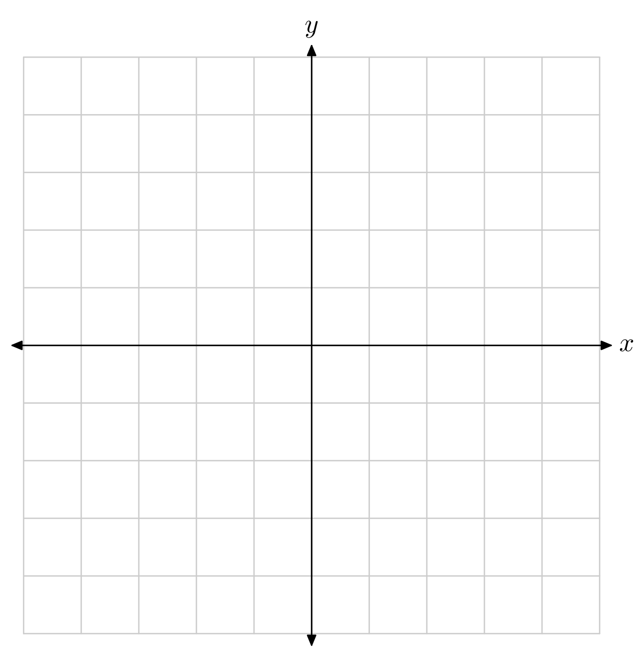
4.

5.

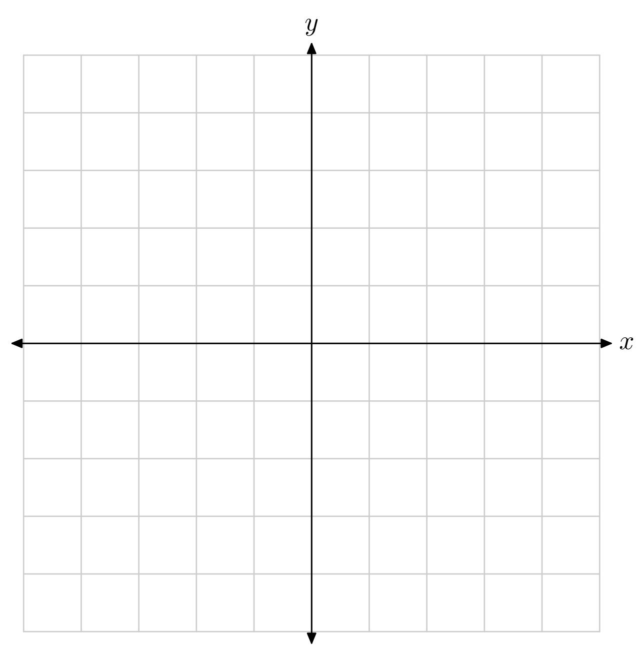
**Example.** Graph $y = \dfrac{x+1}{x-1}$.



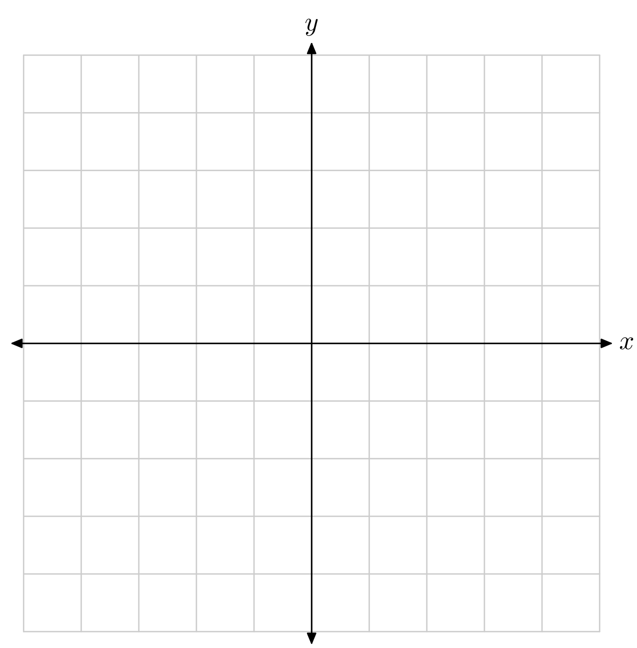
**Example.** Graph $y = \dfrac{x-1}{(x+1)(x-3)}$



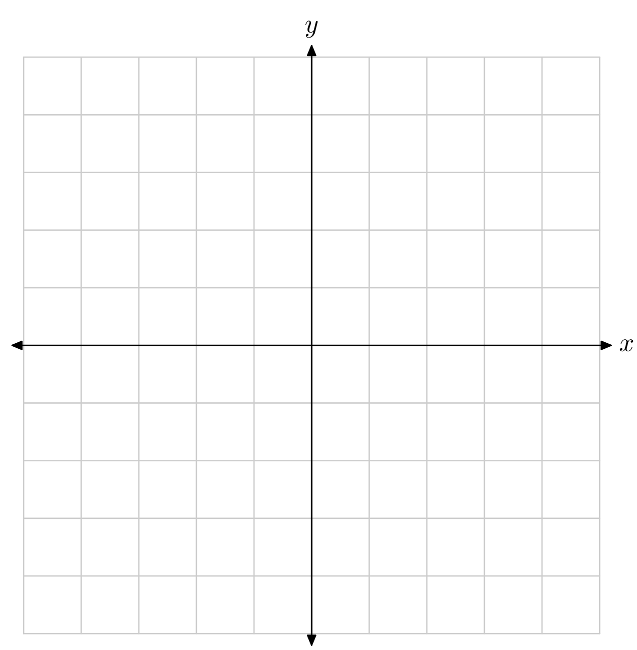
**Example.** Graph $y = \dfrac{1-x}{x^2-4}$



**Example.** Graph $y = \dfrac{-x^2}{x^2 - x - 2}$



**Example.** Graph $y = \dfrac{x^2 + 2x}{x^2 - 2x + 1}$



# Lesson 20: Function Composition

If we have two functions, we can create a new function by taking an input $x$, plugging it into $g$, then taking that result and plugging it directly into $f$:

$$x \longrightarrow \boxed{g} \xrightarrow{\quad g(x) \quad} \boxed{f} \longrightarrow f(g(x))$$

This is called the **composition $f$ of $g$ (or $f \circ g$) of $x$**, also written as $f(g(x))$.

**Example.** Let $f(x) = 1 - x^2$ and $g(x) = 2x + 1$. Evaluate

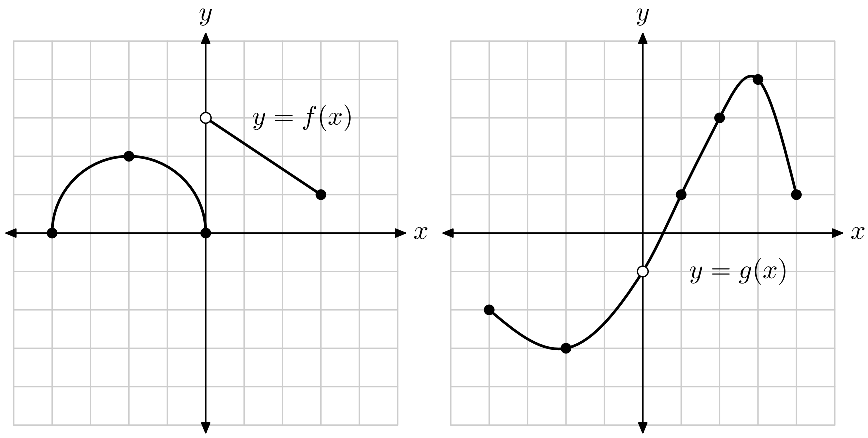
- $f[g(2)] =$

- $g[f(1)] =$

- $f[g(x)] =$

- $g[f(x)] =$

**Example.** Use the following graphs of the functions $f(x)$ and $g(x)$ to evaluate the following:




- $(f+g)(-2)$
- $(g^2)(-2)$
- $(g/f)(-4)$
- $(f \circ g)(2)$
- $(g \circ f)(-2)$
- $(g \circ f)(-4)$
- $(f \circ f)(0)$
- $(g \circ g)(3)$
- $(g \circ g)(2)$

**Example.** Using the following table, evaluate the quantities listed.

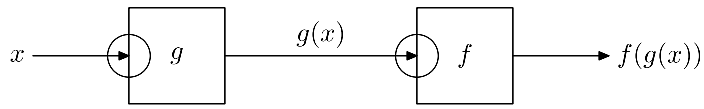
| $x$ | $s(x)$ | $t(x)$ |
|---|---|---|
| $-3$ | 3 | $-2$ |
| $-2$ | 1 | 0 |
| $-1$ | $-2$ | 2 |
| 0 | $-1$ | 3 |
| 1 | 0 | 1 |
| 2 | 2 | 0 |
| 3 | 4 | $-3$ |

- $(s+t)(-2)$
- $(st)(3)$
- $(s/t)(-1)$

- $(s \circ t)(0)$
- $(t \circ s)(2)$
- $(s \circ s)(1)$

## Domain of Composition Functions

When dealing with domains of composed functions, there are two bad things that could happen when evaluating at some number $x$:




- $x$ **may not be in the domain of** $g$, and
- $g(x)$ **may not be in the domain of** $f$.

To find the domain of $f \circ g$, we must identify all values of $x$ for which

- $x$ **is in the domain of** $g$, and
- $g(x)$ **is in the domain of** $f$.

**Example.** Let $f(x) = \sqrt{x-3}$ and $g(x) = x^2 + 6$. First, find

- $(f \circ g)(x) =$
- $(g \circ f)(x) =$
- the domain of $f \circ g$:
- the domain of $g \circ f$:

**Example.** Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \dfrac{x}{x+1}$. Find $f \circ g$, $g \circ f$, and their domains.

## Applications

**Example.** Krispy Kreme is making doughnuts at a rate of 60 doughnuts per minute. They sell the doughnuts for \$0.40 each. Let $t$ be the number of minutes elapsed since the machines were started. Let $N(t)$ be the number of doughnuts produced in $t$ minutes.

- Find $N(t)$.

- Let $R(N)$ be the revenue made if we sell $N$ doughnuts. Find $R(N)$.

  Find $(R \circ N)(t)$. Explain what the function computes.

**Example.** A spherical balloon is inflated in such a way that the radius (in cm) after $t$ minutes can be found as

$$r(t) = \frac{1}{2}t + 1$$

The surface area of a sphere is $S(r) = 4\pi r^2$.

- Find $(S \circ r)(t)$. What does $S \circ r$ compute?

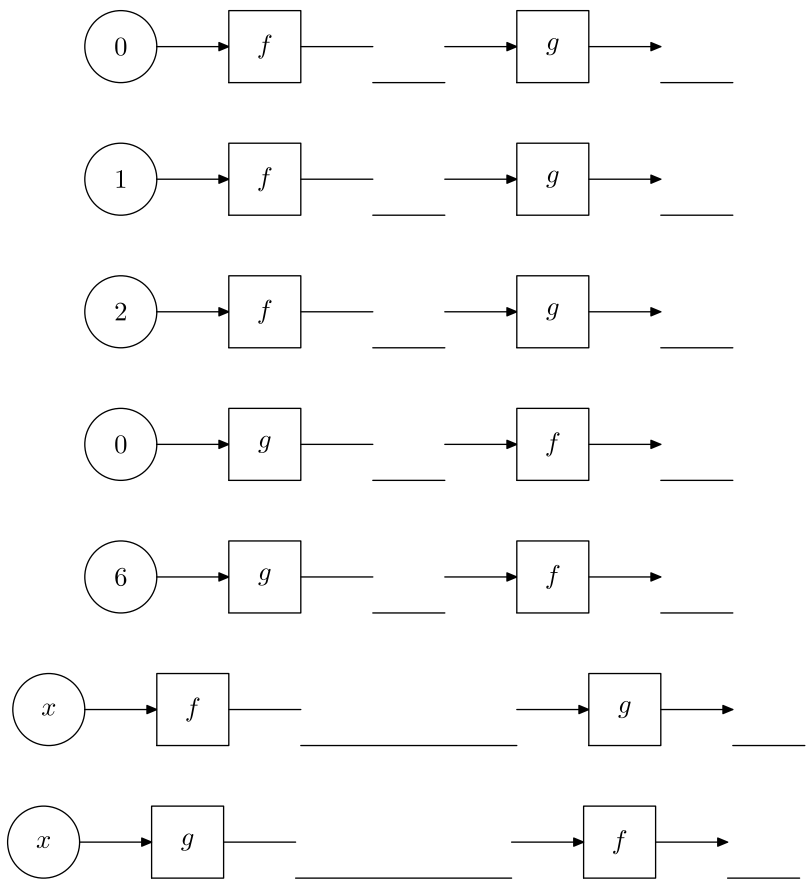
- What is the surface area of the balloon after 4 minutes?

**Example.** Let $f(x) = x^2 - 1$, $g(x) = \sqrt{x+1}$, and $h(x) = x + 1$. Find a composition of $f$, $g$, or $h$ to obtain the following functions:

- $x^2$
- $\sqrt{x+2}$
- $x^2 + 2x$
- $|x|$

# Lesson 21: Inverse Functions

**Example.** Let $f(x) = 3x + 6$ and let $g(x) = \frac{1}{3}x - 2$. Fill out the following pictures:




## Definition

Two functions $f$ and $g$ are **inverses** of one another provided that

- $\underline{f(g(x)) = x}$ for each $x$ in the domain of $g$, and
- $\underline{g(f(x)) = x}$ for each $x$ in the domain of $f$.

**Notation.**

Say $g$ is the **inverse** of $f$: $g(x) = \underline{f^{-1}(x)}$.

Say $f$ is the **inverse** of $g$: $f(x) = \underline{g^{-1}(x)}$.

**Caution!** This is NOT the same as $\dfrac{1}{f(x)}$ and $\dfrac{1}{g(x)}$!

**Example.** Determine if the following pairs of functions are inverses.

- $f(x) = 4x - 1,\ g(x) = \frac{1}{4}x + 1$

- $f(x) = 2x + 6,\ g(x) = \frac{1}{2}x - 3$

- $f(x) = x^2,\ g(x) = \sqrt{x}$

**Example.** Suppose that $f$ and $g$ are inverses.

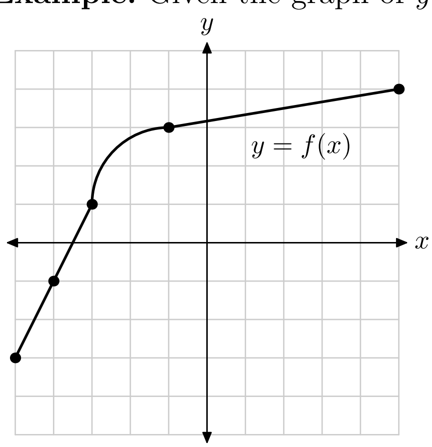
- If $f(2) = 5$, then $g(5) =$ ______

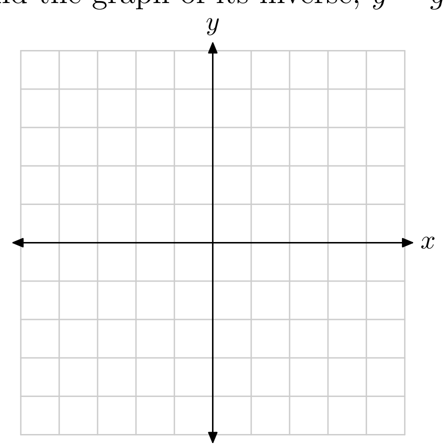
- If $g(7) = -1$, then $f(-1) =$ ______

**Summary**

- If $f(a) = b$, then $g(b) = \underline{a}$.
- The domain of $g$ is $\underline{\text{the range of } f}$.
- The range of $g$ is $\underline{\text{the domain of } f}$.

**Example.** Given the graph of $y = f(x)$ below, find the graph of its inverse, $y = g(x)$.



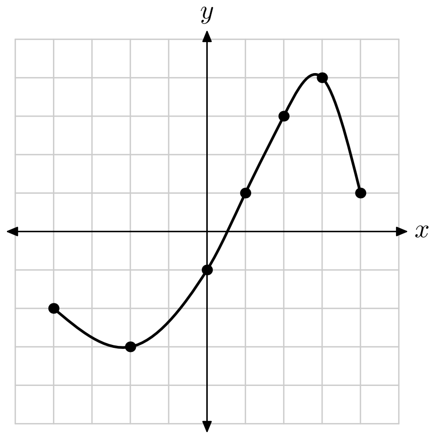



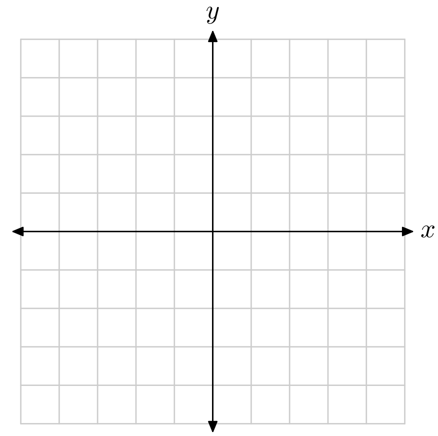

**Observation:**

The graph of $y = g(x) = f^{-1}(x)$ is the graph of $y = f(x)$ reflected over $\underline{\textbf{the line } y = x}$.

## Existence of an Inverse Function

Consider the function $f(x)$ given by the graph below. If this function was to have an inverse, graph the "inverse" on the adjacent graph:




Does the resulting graph represent a function?

Does $f$ have an inverse **function**?

## Requirements for a function to have an Inverse function

As in the example above, the resulting graph of the "inverse" must be a **function**.

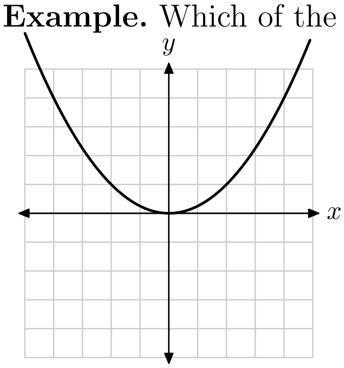
Therefore, the graph of the "inverse" must pass the **vertical line test**.

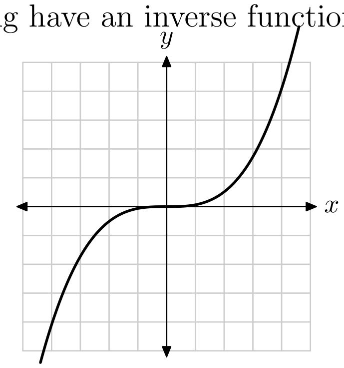
This is equivalent to the graph of the original function passing the **Horizontal line test**.

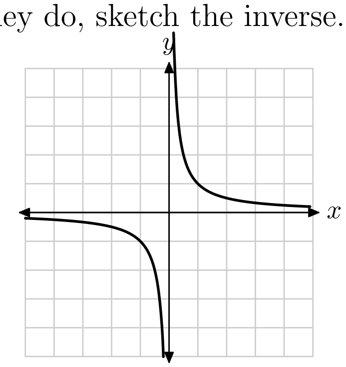
**The Horizontal Line Test**

If any horizontal line and the graph of a function $f$ intersect in at most one point, then $f$ **has an inverse function**. Functions which pass are said to be **one-to-one**.

**Example.** Which of the following have an inverse function? If they do, sketch the inverse.



## Finding Inverse Functions Algebraically

Remember if $f$ and $g$ are inverses and $f(a) = b$, then $g(b) = a$.

So if $y = f(x)$ and we solve for $x$ (get something like $x = \cdots$), we get $\underline{x = g(y)}$.

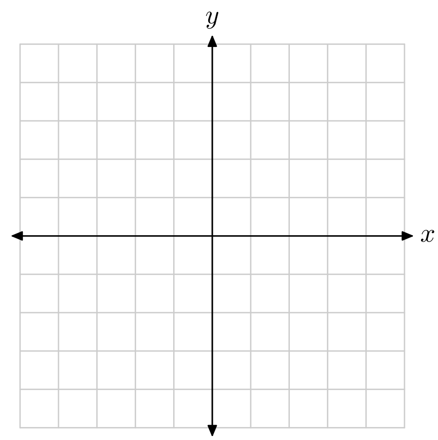
## Method to Find an Inverse

- Determine if your function can have an inverse (HLT), then
- when starting with $y = f(x)$, solve for $x$. The result will be $x = f^{-1}(y)$.

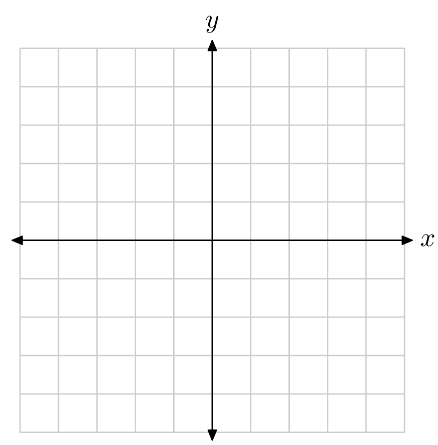
Note: You can sometimes determine whether a function has an inverse by trying to solve for the inverse; if while solving for $f^{-1}$ a problem arises (like getting an unresolved $\pm$), then you don't have an inverse.

**Example.** Determine if each of the following has an inverse. If it does, find the inverse.

- $f(x) = 4x - 6$

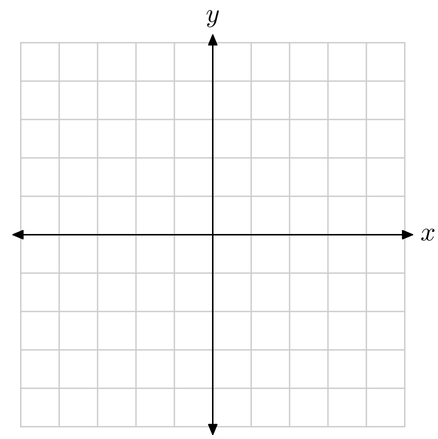



- $f(x) = x^2 - 4$

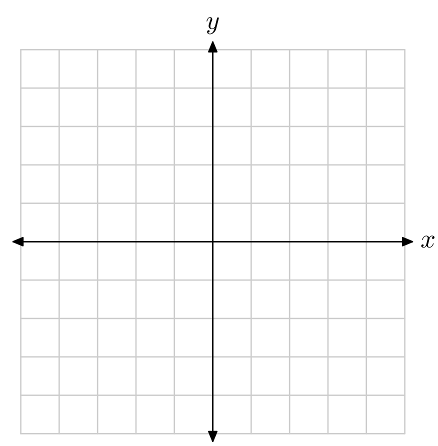


**Example.** If possible, find the inverse of each of the following functions:

- $f(x) = x^2 - 4$ with domain $[0, \infty)$




- $f(x) = \dfrac{3x-2}{x+1}$



# Lesson 22: Other Algebraic Functions

**$n^{\text{th}}$ roots.**

| |
|---|
| Let $x$ be a real number. |
| If $n$ is an even number, then $\sqrt[n]{x^n} = \lvert x \rvert$. |
| If $n$ is an odd number, then $\sqrt[n]{x^n} = x$. |

**Example.** Solve the following equations.

- $x^2 = 4$
- $x^2 = -4$
- $x^3 = 8$
- $x^3 = -8$
- $x^4 = 16$
- $x^4 = -16$

**When taking $n^{\text{th}}$ roots:**

- If $n$ is **<u>even</u>**, then <u>you may end up with zero or two real solutions</u>.
- If $n$ is **<u>odd</u>**, then <u>you will have exactly one real solution</u>.

## Fractional Exponents.

Rule: To solve a problem with fractional exponents, rewrite $x^{a/b}$ as $\sqrt[b]{x^a}$ or $\left(\sqrt[b]{x}\right)^a$.

Also, for any positive integer $n$, $\left(\sqrt[n]{x}\right)^n = x$.

Notice the order of the root and exponent are different from before. Make sure to check your answer afterward if you use this technique.

**Example.** Solve the following equations.

- $x^{\frac{2}{3}} = 4$
- $x^{\frac{2}{3}} = -4$
- $x^{\frac{3}{2}} = 8$
- $x^{\frac{3}{2}} = -8$

**Using substitution and factoring.**

**Example.** Solve the following equationss.

- $x^4 - 2x^2 - 3 = 0$
- $3t^{-2} - 2t^{-1} - 5 = 0$
- $6x - 5\sqrt{x} - 6 = 0$
- $t^{4/3} - 4t^{2/3} - 5 = 0$

**Solving with Radicals.** Isolate the radical and square it. (Make sure to check answers!)

**Example.** Solve the following equations.

- $\sqrt{x-2} = 10$
- $\sqrt{x+3} - 1 = x$
- $\sqrt{x-5} - \sqrt{x+4} + 1 = 0$

**Example.** Find the domain of the following functions:

- $f(x) = \sqrt{\dfrac{1}{x^2 - 1}}$

- $f(x) = \sqrt[3]{\dfrac{x-2}{x+1}}$

# Lesson 23: Introduction to Exponential Functions

Before we discuss exponential functions, let us first review the basic laws of exponents: Assume $a > 0$ and $m, n$ are real numbers.

- $a^m \cdot a^n =$
- $\dfrac{a^m}{a^n} =$
- $(a^m)^n =$
- $(ab)^m =$
- $\left(\dfrac{a}{b}\right)^m =$
- $a^0 =$
- $a^1 =$
- $a^{-1} =$
- $a^{-n} =$
- $(a+b)^m =$

The following fact is a result of exponential functions being one-to-one, which we'll see later:

- If $a^x = a^y$, then $\underline{x = y}$.

**Example.** Use the properties of exponents from above to rewrite each of the following with a minimal number of exponents.

- $(4^\pi)\left(4^{2-\pi}\right)$
- $\left(\left(\sqrt{12}\right)^3\right)^{-1/2}$
- $\dfrac{7^{a+1}}{7^{a-3}}$

**Example.** Solve for $x$:

- $3^x = 81$
- $5^{2t} = \left(\dfrac{1}{25}\right)^{3t-2}$
- $4^x = \dfrac{1}{64}$
- $\left(\dfrac{8}{27}\right)^y = \left(\dfrac{9}{4}\right)^{y+1}$

## Exponential Functions

An **exponential function** is a function defined by the equation $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

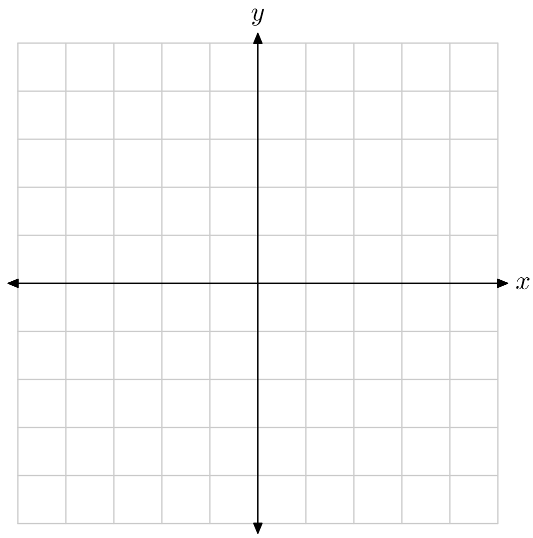
More generally, we say an exponential function can have the form $f(x) = a \cdot b^x$, where $a$ and $b$ are constants, $a \neq 0$, $b > 0$, and $b \neq 1$.

## Graphing Exponential Functions

The basic shape of the graph of an exponential function differs depending on the base, the cases being that the base $b$ is either $b > 1$, or $0 < b < 1$.

**Example.** Graph the functions $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ using the chart.

| $x$ | $f(x) = 2^x$ | $g(x) = \left(\frac{1}{2}\right)^x$ |
|---|---|---|
| -2 | | |
| -1 | | |
| 0 | | |
| 1 | | |
| 2 | | |




## Summary

The following are properties of all exponential functions $y = b^x$, regardless of base.

- Intercept(s): The graph $y = b^x$ will always have an intercept at $(0, 1)$.
- Asymptote: The graph $y = b^x$ will always have a horizontal asymptote of $y = 0$.
- Domain: The graph $y = b^x$ will have a domain of $(-\infty, \infty)$.
- Range: The graph $y = b^x$ will have a range of $(0, \infty)$.

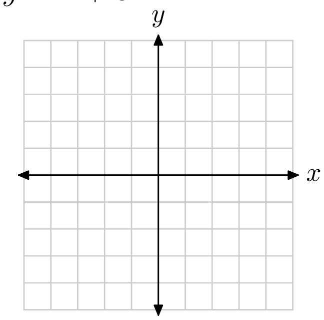
If $b > 1$, the graph will, in shape, resemble $y = 2^x$.

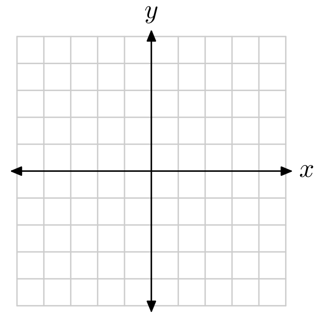
If $b < 1$, the graph will, in shape, resemble $y = (\frac{1}{2})^x$.

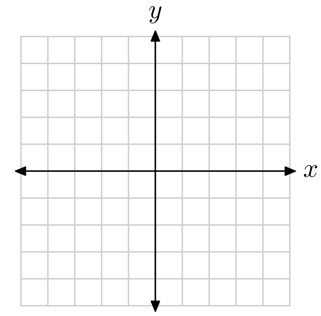
**Note:** There is no real value of $x$ for which $b^x =$ **zero** or **negative**.

**Example.** Graph the following functions by using translations and reflections.

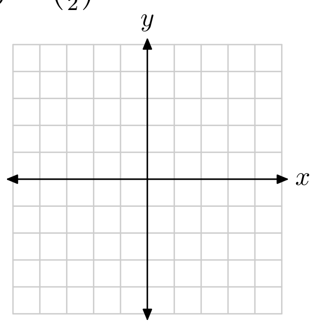
$y = 1 + 3^{-x}$

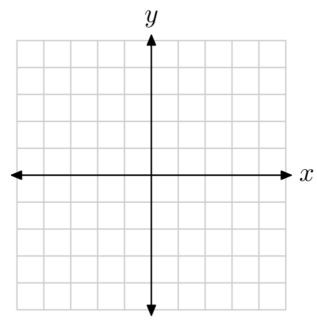


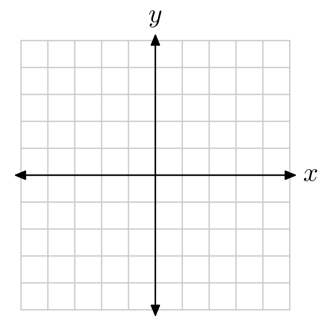




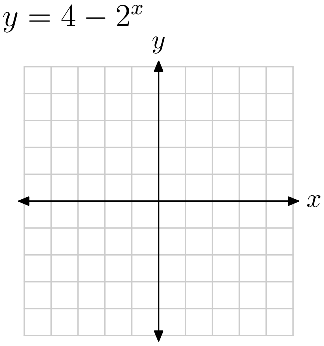

$y = \left(\frac{1}{2}\right)^{-x+2}$

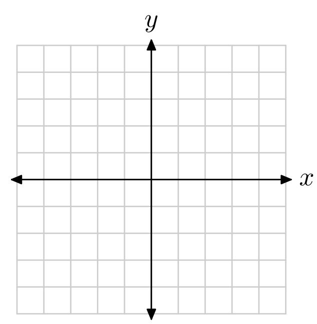


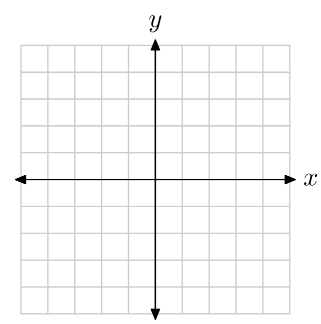





$y = 4 - 2^x$








**Note:** You should always identify and label ALL intercepts and asymptotes.

Remember to find $x$ intercepts, set $\underline{y = 0}$ and solve for $x$.

To find $y$ intercepts, set $\underline{x = 0}$ and solve for $y$.

**Example.** Solve for $x$:

- $x^2\left(2^x\right) - 2^x = 0$

- $x\left(\dfrac{1}{2}\right)^x = \dfrac{2x}{(x+1)2^x}$

## The exponential function $y = e^x$

The value of $e$ is 2.7182818284590 . . . .

Most notably, $e$ is a real number between 2 and 3.

It will not be truly apparent until later lessons, but we will use $e$ as our predominant choice of base in most applications.

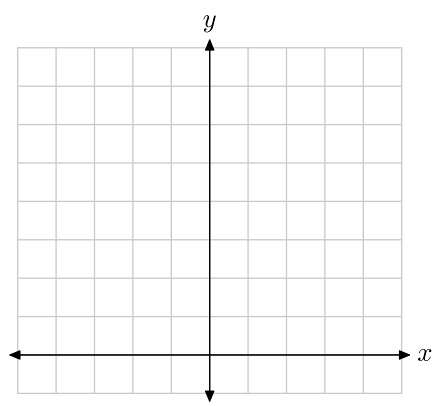
**Example.** Determine which quantity is larger. Indicate it with < or >.

$e$ ______ 2 $\sqrt{e}$ ______ 1.4

$e^2$ ______ 16 $e^4$ ______ 81

**Example.** Graph the function $f(x) = e^x$ using the chart below:

| $x$ | $f(x) = e^x$ |
|---|---|
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |




## Summary

The following are properties of the exponential function $f(x) = e^x$:

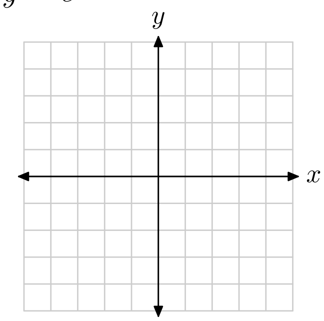
- Intercept(s): The graph $y = b^x$ will always have an intercept at $(0, 1)$.
- Asymptote: The graph $y = b^x$ will always have a horizontal asymptote of $y = 0$.
- Domain: The graph $y = b^x$ will have a domain of $(-\infty, \infty)$.
- Range: The graph $y = b^x$ will have a range of $(0, \infty)$.

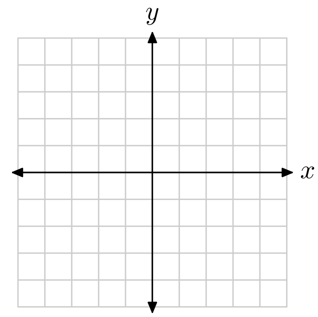
**Note:** These are the same as it would be for any graph $y = b^x$.

However, since $e > 1$, the graph has a shape similar to $y = 2^x$.

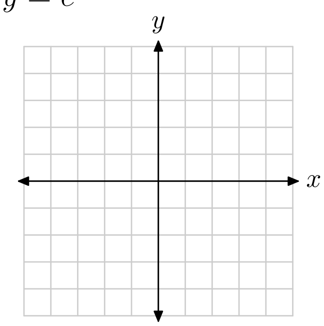
**Example.** Graph these translates.

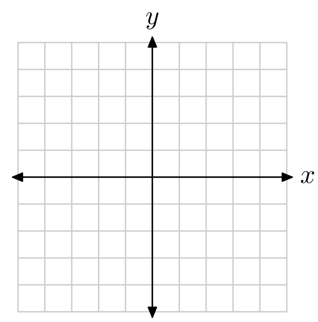
$y = e^{x-1}$

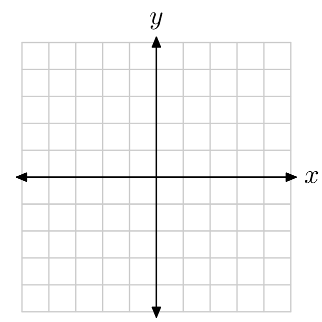



$y = e^{-x+1}$




**Note:** Again, always be sure to find and label your intercepts and asymptotes.

**Example.** For each of the following, determine if the given function is an exponential function, power function, or neither.

- $f(x) = e^x$
- $f(x) = 2^x$
- $f(x) = x^2$
- $f(x) = x^e$
- $f(x) = e^2$
- $f(x) = x^x$

# Lesson 24: Introduction to Logarithmic Functions

**Example.** Answer each of the following questions.

- To what exponent must you raise 2 in order to yield 8?

- To what exponent must you raise 3 in order to yield $\sqrt{3}$?

- To what exponent must you raise 10 in order to yield $\frac{1}{1000}$?

- To what exponent must you raise 4 in order to yield 32?

**Logarithms**:

Suppose $b > 0$ and $x$ are given.

Then the exponent to which $b$ must be raised in order to yield $x$ is $\underline{\log_b(x)}$, read "log base $b$ of $x$".

**Example.** Convert the following sentences into equations.

- 2 is the exponent to which 5 must be raised in order to yield 25.

- 4 is the exponent to which 2 must be raised in order to yield 16.

- $a$ is the exponent to which $b$ must be raised in order to yield 10.

- $x - 4$ is the exponent to which $b$ must be raised in order to yield $p$.

- $y$ is the exponent to which $b$ must be raised in order to yield $x$.

Explicitly, any logarithmic equation of the form $\underline{y = \log_b(x)}$ can be interpreted as

"$y$ is the exponent to which $b$ must be raised in order to yield $x$",

which is the same as the equation $\underline{x = b^y}$.

**Example.** Evaluate the following logarithms.

- $\log_2(4)$
- $\log_3(\frac{1}{3})$
- $\log_{10}(10)$
- $\log_4(0)$
- $\log_4(2)$
- $\log_{\frac{1}{3}}(\sqrt{27})$
- $\log_8(4)$
- $\log_2\left(\dfrac{1}{2}\right)$
- $\log_e\left(\dfrac{1}{e^2}\right)$
- $\log_{3479}\left(3479^8\right)$

## Errors to Avoid

- $\dfrac{\log_2 8}{\log_2 4}$
- $\dfrac{\log_2 16}{16}$

A **logarithmic function** is a function of the form $f(x) = \log_b(x)$,
where $b$ is the base and $b > 0$ and $b \neq 1$.

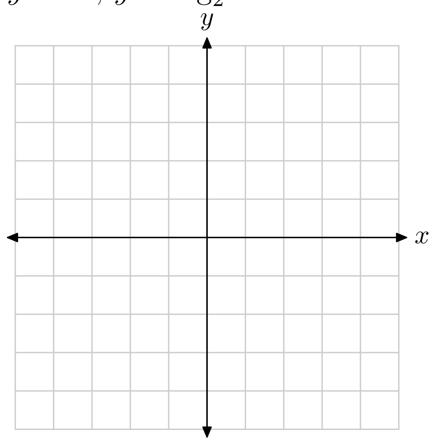
**Example.** Find the inverse of $f(x) = \log_2(x)$.

**Conclusion:**

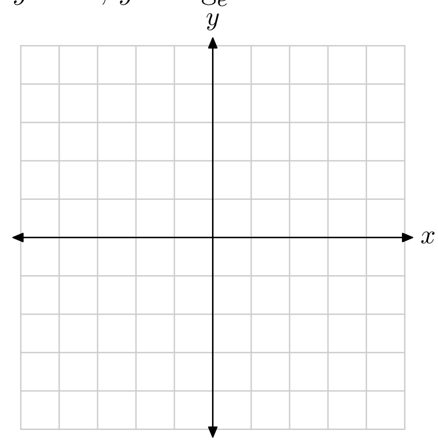
A logarithmic function is the **inverse** of an **exponential** function.

**Example.** Graph $y = \log_2 x$ and $y = \log_e x$ by graphing their inverses, then reflecting over $y = x$.

$y = 2^x$, $y = \log_2 x$




$y = e^x$, $y = \log_e x$




**Summary**

Exponential Graphs

- Domain $(-\infty, \infty)$
- Range $(0, \infty)$
- Intercepts $y$-intercept at $(0, 1)$
- Asymptotes horizontal at $y = 0$

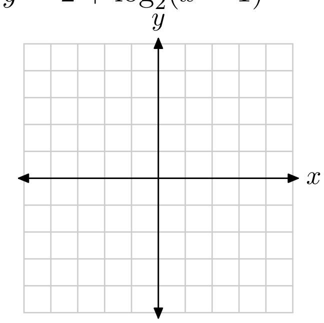
Logarithmic Graphs

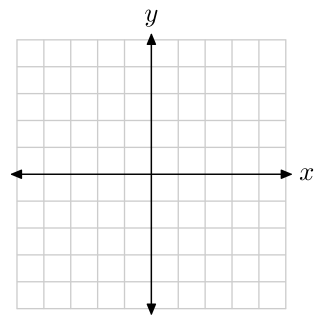
- Domain $(0, \infty)$
- Range $(-\infty, \infty)$
- Intercepts $x$-intercept at $(1, 0)$
- Asymptotes vertical at $x = 0$

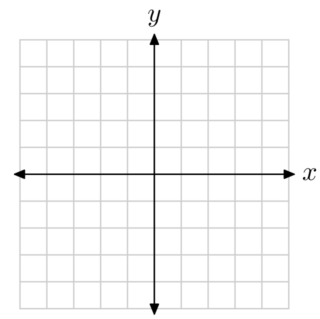
**Notation:** We use $\log_e$ and $\log_{10}$ frequently (we will soon see they are the only ones we need).
Therefore, when you see $\log x$ with no subscript, interpret that as $\log_{10} x$.
When you see $\ln x$, interpret that as $\log_e x$ (thank the French for that!)

**Example.** Graph the following translates.

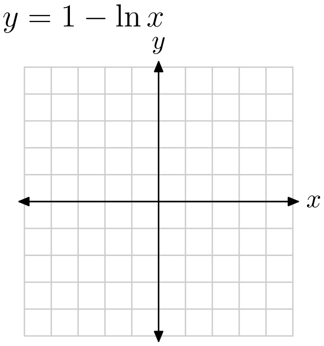
$y = 2 + \log_2(x - 1)$

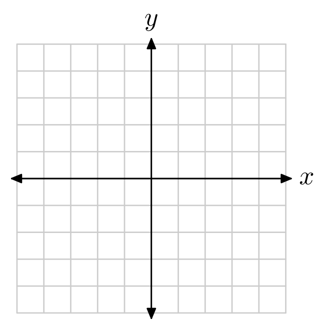


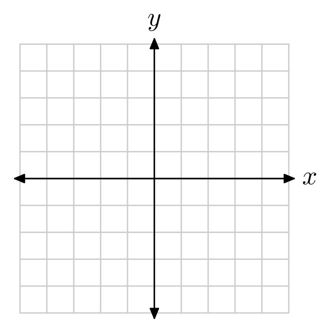



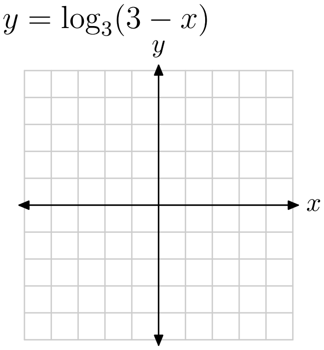

$y = 1 - \ln x$

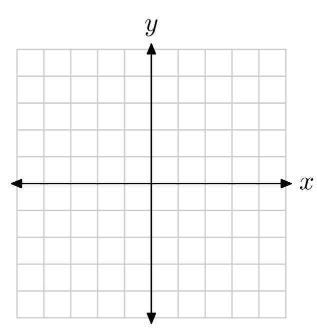


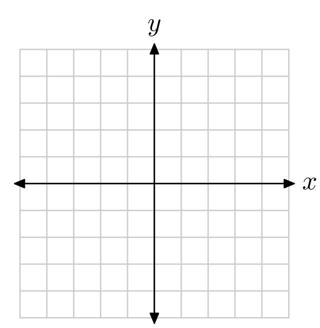




$y = \log_3(3 - x)$




**Note:** Remember to find and label all intercepts and asymptotes on your final graph.

## Domains for Logarithmic Functions

Recall that the domain for $\log_b(x)$ is all positive numbers. So when finding domains for functions with logarithms, you must make sure that no matter the input, you never try to take the log of 0 or a negative number.

To find permissible values in a logarithm, take its input and solve when it is **positive**.

**Example.** Find the domain for each of the following functions.

- $f(x) = \ln(x - 4)$
- $f(x) = \log_2(x - 4)$
- $f(x) = \log_4(1 - 2x)$
- $f(x) = \log_5(x^2 + 1)$

## Solving Equations

Recall that if $y = b^x$, then $\underline{x = log_b(y)}$ and vice versa.

**Example.** Solve for the appropriate variable.

- $10^x = 200$
- $e^{t-3} = 50$
- $\ln x = 5$
- $\log 3z = 2.5$

**Example.** The Richter scale is used to measure intensity of earthquakes. The equation used for finding the magnitude of an earthquake is

$$M = C + \log I,$$

where $M$ is the magnitude, $I$ is the largest output (intensity) on a seismograph, and $C$ is a constant (determined by the type of seismograph, distance from the earthquake, etc.).

Suppose two earthquakes have magnitudes of $M_1 = 4.4$ and $M_2 = 5.8$. By what factor is the second more intense than the first? In other words, if the intensity of the first earthquake is $I_1$ and the intensity of the second earthquake is $I_2$, what is $I_2/I_1$?

(Hint: Don't worry, in this problem you never need to know (nor can you solve for the value of) $C$.)

# Lesson 25: Properties of Logarithms

**Example.** Evaluate the logarithms in the following table:

| | | |
|---|---|---|
| $\log_2(4)$ | $\log_2(8)$ | $\log_2(4 \cdot 8)$ |
| $\log_3(3)$ | $\log_3(9)$ | $\log_3(3 \cdot 9)$ |
| $\log_4(16)$ | $\log_4(\frac{1}{4})$ | $\log_4(16 \cdot \frac{1}{4})$ |

**Property 1:** $\log_b(P) + \log_b(Q) = \underline{\log_b(P \cdot Q)}$.

**Why is this true?**

**Example.** Evaluate the following expressions by using Property 1 to rewrite, then compute.

- $\log_{10}(4) + \log_{10}(25)$
- $\log_8(32) + \log_8(\frac{1}{4})$
- $\log_2(x^2 + x)$

**Example.** Evaluate the logarithms in the following table:

| | | |
|---|---|---|
| $\log_2(32)$ | $\log_2(16)$ | $\log_2(\frac{32}{16})$ |
| $\log_5(25)$ | $\log_5(125)$ | $\log_5(\frac{25}{125})$ |
| $\log_3(\sqrt{3})$ | $\log_3(\sqrt[3]{3})$ | $\log_3(\frac{\sqrt{3}}{\sqrt[3]{3}})$ |

**Property 2:** $\log_b(P) - \log_b(Q) = \underline{\log_b\left(\dfrac{P}{Q}\right)}$.

**Why is this true?**

**Example.** Simplify the following expressions using Property 2.

- $\log_6(72) - \log_6(2)$
- $\log_8(28) - \log_8(\frac{7}{2})$
- $\log_7\left(\dfrac{1}{x}\right)$

**Example.** Use Property 1 $\{\log_b(PQ) = \log_b(P) + \log_b(Q)\}$ to rewrite the following logarithms.

- $\log_2(49)$

- $\log_4(x^3)$

- $\log_5(2^{10})$

**Property 3:** $\log_b(P^n) = \underline{n \cdot \log_b(P)}$.

**Why is this true?**

**Example.** Use Property 3 to rewrite each of the following:

- $\log_5(8)$

- $\log_b(x^{10})$

- $\frac{1}{2}\ln(x+1)$

**Example.** Suppose that $\log_b(2) = .33$, $\log_b(3) = .52$, and $\log_b(5) = .76$. Evaluate or write with simplied log arguments:

- $\log_b(10)$
- $\log_b(30)$
- $\log_b(25)$
- $\log_b(\sqrt{3})$
- $\log_b(2^{10})$
- $\log_b(\frac{1}{2})$
- $\log_b(.1)$
- $\log_b(.03)$
- $\log_b(.4)$

**Example.** Rewrite each of the following as a single logarithm ( rewrite as $\log_b$(something...) ):

- $\ln x + \ln 3 - \ln(x - 2)$

- $2\ln x - 3\ln(x + 1)$

- $\dfrac{1}{2}\ln(x + 4) + 2\ln(2x - 3) - 4\ln x$

**Example.** Rewrite each of the following with simplified log arguments.

- $\log_7\left(\dfrac{x(x+2)}{x^2+1}\right)$

- $\log_2\sqrt{\dfrac{(x+1)^5}{(x-3)^3}}$

- $\log\left(\dfrac{36\sqrt{x}}{\sqrt[3]{x+1}}\right)$

# Lesson 26: Compositions of Exponential and Logarithmic Functions

**Example.** Evaluate the quantities in the table below:

| | |
|---|---|
| $\log_2(8)$ | $2^{\log_2(8)}$ |
| $\log_3(\frac{1}{3})$ | $3^{\log_3(\frac{1}{3})}$ |
| $\log_5(\sqrt{5})$ | $5^{\log_5(\sqrt{5})}$ |

**Property 4:** $b^{\log_b(P)} = \underline{P}$.

**Why is this true?**

**Example.** Evaluate the following logarithms using property 4:

- $2^{\log_2 3}$
- $5^{\log_5 7}$
- $e^{\ln 6}$
- $e^{2 \ln 6}$

## The Change-of-Base Formula

**Example.** Suppose that $\ln(2) = .69$ and $\ln(7) = 1.95$. Evaluate $\log_2(7)$.

**Example.** Fill in the table as directed on each step.

| | |
|---|---|
| Suppose $y = \log_b(x)$. | Given. |
| Then | Rewrite this equation without logarithms. |
| Then | Apply ln to both sides of the equation. |
| Then | Use Property 3 to move the exponent. |
| Then | Solve for $y$. |

> **The Change-of-Base Theorem:** $\log_b(x) = \dfrac{\log_c(x)}{\log_c(b)}$.

**Example.** Using that $\ln(2) = .69$, $\ln(3) = 1.1$, $\ln(5) = 1.61$, and $\ln(7) = 1.95$, evaluate the following logarithms:

- $\log_2 5$
- $\log_3 8$
- $\log_5 14$

> Note: For this course, you will be expected to rewrite all logarithms in terms of ln and log (base 10). These are the only log functions that WeBWorK accepts, so everything typed into WeBWorK must be rewritten using this theorem.

**Example.** Let $f(x) = e^{2x}$ and $g(x) = \ln(x + 2)$.

- Find $f(g(x))$ and $g(f(x))$.

**Example.** Let $f(x) = \ln(x) + 3$ and $g(x) = e^{4x}$.

- Find $f(g(x))$ and $g(f(x))$.

**Example.** Determine whether $f(x) = \ln(x+2)$ and $g(x) = e^x - 2$ are inverses.

**Example.** Find the inverse functions for each of the following functions.

- $f(x) = 1 + 2 \cdot e^{x+2}$

- $f(x) = 4\ln(7x - 3)$

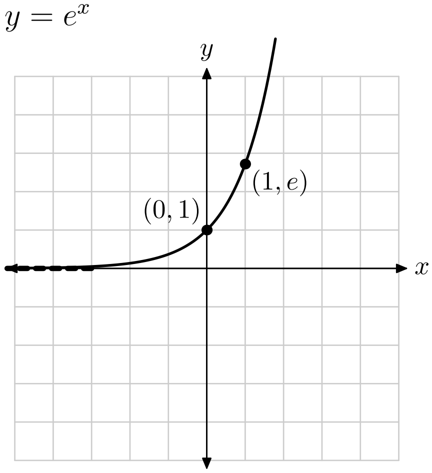
# Lesson 27: Exponential Equations and Inequalities

**Example.** Fill in the chart below using previous lectures:

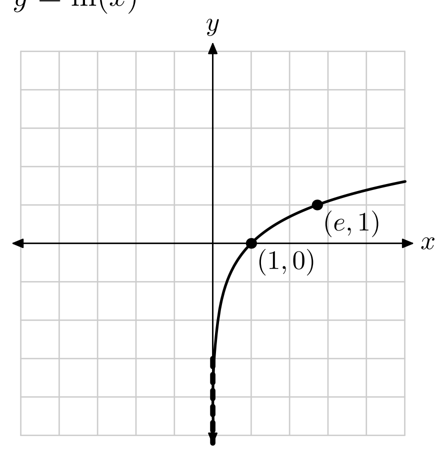
- $\log_b b =$ ______
- $\log_b 1 =$ ______
- $\log_b(P \cdot Q) =$ ____________________________________
- $\log_b \left(\dfrac{P}{Q}\right) =$ ____________________________________
- $\log_b P^n =$ ________________________
- $b^{\log_b P} =$ ______

Recall the graphs of $y = e^x$ and $y = \ln(x)$:

$y = e^x$




$y = \ln(x)$




Note: Both of these graphs are **increasing**.

**Properties of Inequalities**

The following are results of the behavior of the above graphs:

- If $a < b$, then $e^a < e^b$.
- If $e^a < e^b$, then $a < b$.
- If $a < b$, and $a > 0$, then $\ln(a) < \ln(b)$.
- If $\ln(a) < \ln(b)$, then $a < b$.

**Example.** Solve for the appropriate variable.

- $2^x = 3$

- $5^{2t-3} = 10$

- $10^{2x} - 3 \cdot 10^x - 4 = 0$

**Example.** Solve for the appropriate variable.

- $6^{z+2} = 10$

- $2^x = 3^{x-3}$

- $2^y \cdot 3^{-y} \cdot 5^{2-3y} = 10^{y+2}$

**Example.** Solve the following inequalities:

- $6^{x-3} < 2$

- $10^x < 2^{x+1}$

**Example.** Solve more inequalities:

- $12^{x^2} \geq 12^{3x+4}$

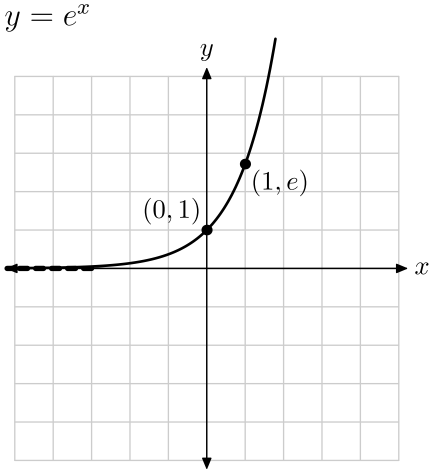
# Lesson 28: Logarithmic Equations and Inequalities

**Example.** Fill in the chart below using previous lectures:

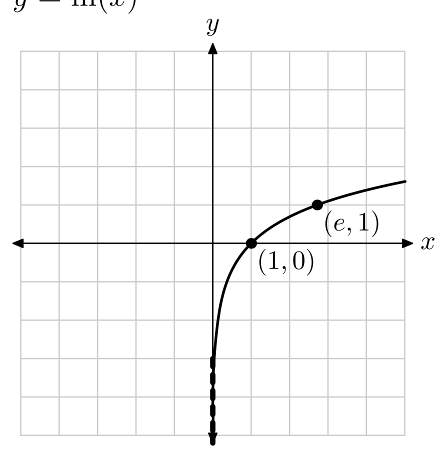
- $\log_b b =$ ______
- $\log_b 1 =$ ______
- $\log_b(P \cdot Q) =$ ____________________________________
- $\log_b \left(\dfrac{P}{Q}\right) =$ ____________________________________
- $\log_b P^n =$ ________________________
- $b^{\log_b P} =$ ______

Recall the graphs of $y = e^x$ and $y = \ln(x)$:

$y = e^x$




$y = \ln(x)$




Note: Both of these graphs are **increasing**.

**Properties of Inequalities**

The following are results of the behavior of the above graphs:

- If $a < b$, then $e^a < e^b$.
- If $e^a < e^b$, then $a < b$.
- If $a < b$, and $a > 0$, then $\ln(a) < \ln(b)$.
- If $\ln(a) < \ln(b)$, then $a < b$.

**Example.** Solve for $x$:

- $\log_3 [\log_2 x] = 1$

- $\log_3 [\log_2 [\log_5 [\ln x]]] = 0$

**Example.** Solve for $x$: (Make sure to check your answers!)

- $\ln(x^2) = (\ln(x))^2$

- $[\ln x]^2 - 3\ln x + 2 = 0$

**Example.** Solve for $x$: (Make sure to check your answers!)

- $\log_4(x^2 + 6x) = 2$

- $\log x + \log(x + 3) = 1$

**Example.** Solve for $x$: (Check your answers!)

- $\log_2(x) + \log_2(x-2) = 3$

- $2\log(x) = 1 + \log(x - 1.6)$

**Example.** Solve more inequalities:

- $\log_2(x-3) < 4$

- $\ln x \geq \ln(2x+4)$

**Example.** Solve more inequalities (Make sure to check your answers!)

- $\log_2 x + \log_2(x-3) < 2$

# Lesson 29: Applications - Compound Interest

**Example.** Suppose we invest \$250 in a savings account which accrues 6% interest annually.

- How much interest is made after 1 year?

- How much money is in the account after 1 year?

- How much money is in the account after 2 years?

With this in mind, fill in the following table, expressing how much money would be present after $t$ years:

| $t$ (years) | Amount in Account |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 10 | |
| 40 | |

If an account accrues interest $r$ (as a decimal) every year and had initial principal $\$P$, the amount in the account after $t$ years is $\underline{A = P(1+r)^t}$.

**Example.** Use the formula $A = P(1 + r)^t$ to solve each of the following problems.

- If you invest \$500 at 7% interest compounded annually, how much money would you have after 3 years?

- If you invest \$500 for 3 years accruing compound interest annually, and end up with \$800 in your account, what was the interest rate?

- If you end up with \$1000 dollars after investing for 4 years at 8% compounded annually, what was the initial investment?

- If you invest \$300 at 8% and yield \$700, for how long did you invest?

- If you invest \$700 at 5% for 3 years, how much interest do you earn?

## Compounding more than once a year

If a bank account accrues 12% interest, but the bank compounds **monthly** rather than yearly, instead of getting 12% applied each year, you would get $\frac{12\%}{12} = 1\%$ applied each month.

In general, if a bank account accrues $r$ interest, compounded $n$ times per year (call this new time interval a **period**), it is done by accruing $\frac{r}{n}$ interest every period. This amount is called the **periodic interest rate**.

**Example.** An account gets 12% interest compounded quarterly.

- What is the periodic interest rate?

If you invest \$200, how much is in the account after

- one quarter?

- four quarters (a year)?

- 10 years (40 quarters)?

## Compound Interest Formula

An account with initial amount $\$P$ which gets interest $r$ (as a decimal) compounded $n$ times per year will yield $A = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$ dollars after $t$ years (or $n \cdot t$ periods).

## Compounding Continuously

**Example.** Suppose you would like to invest \$1000. Banks X, Y, and Z each will give you 8% interest, but bank X compounds annually, bank Y compounds monthly, and bank Z compounds daily. How much money would be in your account after one year if you were to invest with each bank?

Your get-rich-quick scheme. If your bank compounds interest more often per year, the return is greater. So what if you could find a bank that would compound your interest infinitely many times in a year?

A Fun Fact from Calculus

As $n$ gets really large, we have that

$$P\left(1+\frac{r}{n}\right)^{n\cdot t} \to Pe^{rt},$$

where $e = 2.71828\ldots$, $P$ is the principal, and $r$ is the interest rate.

So if you compound more often, there is a cap on how much your account will accrue. Epic Fail.

## Continuous Compounding Interest

If a bank offers interest compounded continuously (interest is compounded at every moment) the formula giving the amount in the account when investing \$$P$ after $t$ years at interest $r$ would be

$$A = P \cdot e^{r\cdot t}$$

**Example.** If you invest \$150 at 6% compounded continuously, how much money would be in your account after 4 years?

**Example.** If you invest money in an account which compounds continuously and you wish for your money to double in 8 years, what must the interest rate be? What if we wanted the money to double in 10 years? 20 years?

**Example.** If one invests at 8% compounded continuously, how long will it take the investment to triple?

**Effective interest rate**

**Example.** Suppose you invest \$200 in a bank account which gives 6% interest compounded continuously.

- How much would you have after a year?

- What annual interest rate would make you the same amount of money? (We call this interest rate the **effective interest rate** for the account)

**Example.** What is the effective interest rate of investing at 6.1% compounded monthly? Is investing at 6.1% compounded monthly better than investing at 6% compounded continuously?

# Lesson 30: Applications - Population Growth and Decay

**Example.** The population of Gondor is current 50,000 people. We expect the population will double every 25 years.

- How many people will be in Gondor 75 years from now?

- How many people were in Gondor 25 years ago?

**Example.** The half-life of a substance is the time required for the amount of a substance to halven. We currently have 800 grams of Fermium-255, which has a half-life of 20 hours.

- How much Fermium will be remaining in 40 hours?

- How much Fermium was present 20 hours ago?

This type of growth or decay is called **exponential**.

A population (or amount of substance) which has exponential growth or decay is modeled by the function $\underline{N(t) = N_0 \cdot e^{k \cdot t}}$, where

- $N(t)$ is the population (or amount of substance) after $t$ time units have passed,
- $N_0$ is the initial population (or amount of substance),
- $t$ is the amount of time elapsed, and
- $k$ is called the **growth constant**.

**Example.** Find the model for the growth of the population of Gondor, with time being measured in years from when the population was 50,000.

- What will the population be 75 years later?

- What will the population be 30 years later?

- When will the population reach 80,000 people?

**Example.** Suppose a population of bacteria in a Jack O' Lantern $t$ days after Halloween can be modeled by

$$N(t) = 100e^{\frac{\ln 2}{5}t}$$

- How many bacteria are present on Halloween?

- How many are present 10 days after Halloween?

- How long does it take the bacteria to double its initial amount?

**Example.** Suppose the population in Burkina Faso can be modeled exponentially. The 1996 census gave the population to be about 9.9 million people. The 2005 estimate is 13.2 million people.

- Find a population model where $t$ is the number of years since 1996 by identifying $N_0$ and the growth constant $k$.

- When would you expect the population to reach 15 million?

**Example.** Suppose the popualation of American Bald Eagles in Wisconsin follows an expontential model. There were 358 eagles in 1990 and 1065 in 2006.

- Find the model for the population $t$ years after 1990 by finding $N_0$ and growth constant $k$.

- How long after 1990 did it take the population to double? How long will it take to quadruple?

**Example.** Radioactive decay can always be modelled by an exponential model. The half-life of platinum-186 is 2 hours.

- Find the decay constant $k$ and the model $N(t)$, where $t$ is in hours.

- If we start with 100g, how much is left after 5 hours?

- Suppose we again start with 100 grams. When will we have only 10g left?

**Example.** We have 100g of a radioactive substance and after 5 days, we have 60g.

- Find the model for the decay of this substance.

- Find the half-life.

- How many half-lives must pass for the amount to be reduced by a factor of 100?

## Lesson 31: Systems of Linear Equations

A point $\underline{(x_0, y_0)}$ is a **solution** to $ax + by = c$ if $\underline{ax_0 + by_0 = c}$.

**Example.** Check if $(2, 3)$ and $(1, 1)$ are solutions for the following **linear equations**:

- $x - 3y = -7$

- $2x - y = 1$

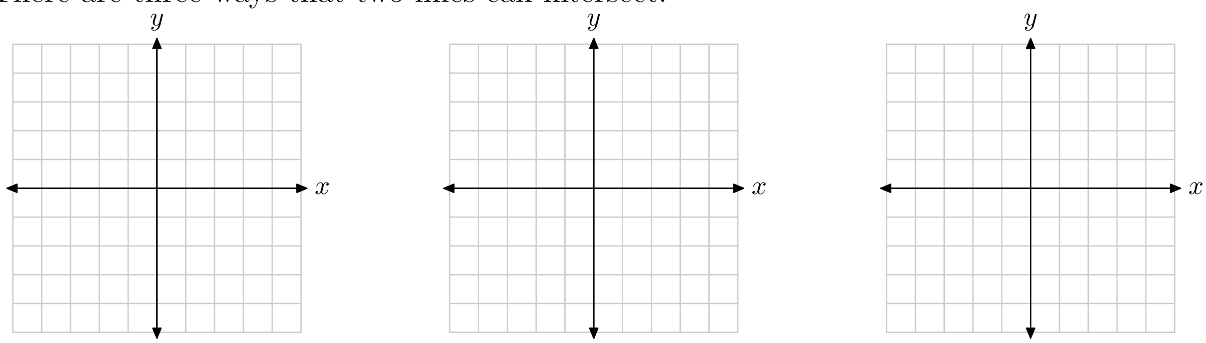
- $3x + y = 4$

A collection of two or more equations is called a <u>**system**</u> of equations. A point which is a solution to each equation in a given system is called a **solution to the system of equations**.

**What are the possible solutions to a system of two linear equations?**

Solutions to a system of linear equations are points where their lines **intersect**.

There are three ways that two lines can intersect:




One method to solve a system of equations for its solutions is called **substitution**.

Using one equation, first **solve for one variable**,

then **replace the variable in all other equations**.

**Example.** Solve the system using substitution.

$$\begin{aligned} x - 3y &= -7 \\ 2x - y &= 1 \end{aligned}$$

Another method for solving systems of equations is the **addition-subtraction** method.

In this method, we eliminate a variable by possibly multiplying our equations by numbers, then combining the left sides together and the right sides together.

**Example.** Solve using addition-subtraction (also known as linear combinations):

$$\begin{array}{rcrcr} 2x & - & y & = & 1 \\ 3x & + & y & = & 4 \end{array}$$

**Example.** Solve using linear combinations:

$$\begin{array}{rcrcr} 3x & - & 4y & = & 10 \\ 7x & + & 5y & = & 9 \end{array}$$

**Example.** Solve using linear combinations:

$$\begin{array}{rcrcr} 2x & - & 3y & = & 3 \\ -4x & + & 6y & = & 6 \end{array}$$

**Example.** Solve using linear combinations:

$$\begin{array}{rcrcr} 4x & + & 6y & = & 2 \\ 6x & + & 9y & = & 3 \end{array}$$

**Example.** Suppose a chemist has 10% and 15% acid solutions in stock. (If a solution is a 15% acid solution, that means that 15% of the solution is acid, and the remaining 85% is water) How much of each should the chemist mix if 100 mL of a 12% solution is desired?

**Example.** Find the equation for a parabola $y = ax^2 + bx - 2$ which goes through $(1, 0)$ and $(2, 6)$.

# Lesson 32: Systems of Nonlinear Equations

**Example.** Solve the following system of equations:

$$\begin{aligned} x^2 + y &= 4 \\ 2x - y &= 20 \end{aligned}$$

**Example.** Solve the following system of equations:

$$\begin{aligned} 2x - y &= -2 \\ x^2 + y^2 &= 25 \end{aligned}$$

**Example.** Solve the following system of equations:

$$\begin{aligned} xy &= -2 \\ y &= 3x + 7 \end{aligned}$$

**Example.** Solve the following system of equations:

$$\begin{aligned} y &= \sqrt{x-3} \\ y &= x-5 \end{aligned}$$

**Example.** Solve the following system of equations:

$$\begin{aligned} y+3 &= (x^2+1)^2 \\ 2(x^2+1)+y &= 0 \end{aligned}$$

**Example.** If a right triangle has an area of 6 and a hypotenuse of length 5, what are the lengths of the legs of the triangle?

# Lesson 33: Sequences

A **sequence** is an **ordered list**.

**Example.** Determine which of the following are sequences. Is there a pattern?

- $2, 4, 8, \frac{2}{3}, -\frac{1}{2}$
- $1, -1, i, \ldots$
- $\{1, 2, 3, 4, 5\}$
- $1, 3, 5, 7, 9, \ldots$
- $1, 4, 1, 5, 9, 2, 6, \ldots$
- $2, 3, 5, 7, 11, \ldots$
- $1, 2, 4, 8, 16, \ldots$
- $2, 1, 0, 0, 0, \ldots$
- $1, 3, 6, 10, 15, \ldots$

**Notation.**

Let $a$ denote a sequence. To indicate specific terms of a sequence, we use **subscripts**.

The first term of a sequence would be $a_1$, the second would be $a_2$, and so on.

**Example.** Consider the sequence $a: \ 2, \frac{1}{2}, -\frac{3}{4}, 10, e, \imath, 12, 16, \ldots$
Evaluate the following expressions.

- $a_3$
- $a_1 + a_4$
- $a_5/a_6$

## Explicit Formulas for Sequences

**Definition.**

A sequence $a$ has an **explicit formula** if there exists a **function** $f$ such that $\underline{a_n = f(n)}$.

**Example.** Calculate the first four terms $(a_1, a_2, a_3, a_4)$ of the following sequences:

- $a_n = 3n + 2$

- $a_n = 20 \cdot \left(-\frac{1}{2}\right)^n$

- $a_n = n!$ (the product of the integers from 1 up to $n$)

- $a_n = \frac{1}{2} \cdot n \cdot (n+1)$

## Recursive Formulas for Sequences

A **recursively defined** sequence is a sequence where

- the first term(s) of the sequence are given, and
- the subsequent terms are computed using the **previous terms**.

**Example.** Calculate the first four terms $(b_1, b_2, b_3, b_4)$ of the following sequences:

- $\begin{cases} b_1 & = & 5 \\ b_n & = & b_{n-1} + 3, \text{ for } n \geq 2 \end{cases}$

- $\begin{cases} b_1 & = & -10 \\ b_n & = & -\frac{1}{2} b_{n-1}, \text{ for } n \geq 2 \end{cases}$

- $\begin{cases} b_1 & = & 1 \\ b_n & = & n \cdot b_{n-1}, \text{ for } n \geq 2 \end{cases}$

- $\begin{cases} b_1 & = & 1 \\ b_n & = & b_{n-1} + n, \text{ for } n \geq 2 \end{cases}$

- $\begin{cases} b_1 & = & 1 \\ b_2 & = & 1 \\ b_n & = & b_{n-1} + b_{n-2}, \text{ for } n \geq 3 \end{cases}$

## Sigma Notation

**Sigma notation** is a short-hand method for **adding things**.

The sum $a_1 + a_2 + \cdots + a_n$ is expressed as $\displaystyle\sum_{k=1}^{n} a_k$.

The variable $k$ is called the **index of summation**.

## Procedure for Evaluation

**Example.** Consider the sequence $a\colon\ = 2, 10, i, -7, 12$

- Evaluate $\displaystyle\sum_{i=1}^{4} a_i$

- Evaluate $\displaystyle\sum_{j=2}^{5} a_{j-1}$

- Evaluate $\displaystyle\sum_{k=3}^{3} a_k$

- Evaluate $\displaystyle\sum_{p=1}^{4} a_p \cdot a_{p+1}$

**Example.** Evaluate the following summations:

- $\displaystyle\sum_{i=1}^{5} i$

- $\displaystyle\sum_{j=1}^{5} j^2$

- $\displaystyle\sum_{k=1}^{5} 2$
  (notice the terms are not dependent on $k$; they are constant.)

- $\displaystyle\sum_{p=1}^{5} j$
  (notice the terms are not dependent on $p$; they are constant.)

- $\displaystyle\sum_{q=-1}^{3} q^2 - 1$

- $\displaystyle\sum_{r=1}^{6} \frac{1}{r} - \frac{1}{r+1}$

**Example.** Express the following sums using sigma ($\sum$) notation.

For uniformity, have your summation look like $\displaystyle\sum_{k=1}^{\square} \square$

- $2 + 3 + 4 + \cdots + 100$

- $1^2 + 2^2 + \cdots 10^2$

- $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^{25}}$

- $x + x^2 + x^3 + x^4 + x^5$

- $\log_{10}(2 \cdot 4 \cdot 6 \cdot 8 \cdots 20)$

- $1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots$

# Lesson 34: Series

## Geometric Sequences

**Definition.**
A **geometric sequence** is a sequence of the form $a$, $a \cdot r$, $a \cdot r^2$, ...

The value of $r$ is called the ________________________ .

**Example.** Which of the following sequences appear to be geometric?

- $1, 2, 3, 4, \dots$
- $21, 15, 10, 6, \dots$
- $-3, 2, -\frac{4}{3}, \frac{8}{9}, \dots$
- $9, -6, 4, -3, \dots$
- $18, 12, 8, \frac{16}{3}, \dots$
- $1, 2, 4, 8, \dots$

**Example.** In a certain geometric sequence, the $a_1 = 4$ and $a_3 = 8$. Find all possible $r$, and possible values for $a_2$.

### Formulae for Geometric Sequences

Again, the general form for a geometric sequence is $a, a \cdot r, a \cdot r^2, \ldots$

The recursive definition for a geometric sequence is

$$\begin{cases} a_1 & = \\ & \\ a_n & = \end{cases}$$

The explicit formula for the $n$th term of the geometric sequence is $a_n =$ __________________ .

**Example.** Find a formula for the $n$th term of the following geometric sequences:

- $1, \frac{1}{2}, \frac{1}{4}, \ldots$

- $2, 3, \frac{9}{2}, \frac{27}{4}, \ldots$

- $18, -12, 8, -\frac{16}{3}, \ldots$

- $2, -6, 18, -54, 162, \ldots$

### Finite Geometric Series

The end goal is to determine a formula for adding up terms in a geometric sequence (i.e. evaluating a geometric series). To accomplish this, we must first complete some fun polynomial arithmetic:

**Example.** Expand the following products.

- $(1-x)(1+x) =$

- $(1-x)(1+x+x^2) =$

- $(1-x)(1+x+x^2+x^3) =$

- $(1-x)(1+x+x^2+x^3+x^4) =$

- $(1-x)(1+x+\cdots+x^8+x^9) =$

So if $n \geq 1$, $(1-x)(1+x+\cdots+x^{n-2}+x^{n-1}) =$

So $\quad 1+x+\cdots+x^{n-2}+x^{n-1} =$ ____________________________ .
This makes $S_n$, the sum of the first $n$ terms in a geometric sequence, be

$$\begin{aligned} S_n &= a+ar+ar^2+\cdots ar^{n-1} \\ &= \\ &= \end{aligned}$$

So the sum of the first $n$ terms would be $S_n =$ ________________________________ .

**Example.** Find the sum of the following geometric series:

- $\displaystyle\sum_{k=1}^{10} 2^{k-1}$

- $\displaystyle\sum_{j=2}^{6} \left(-\frac{2}{5}\right)^{j}$

- $6 - 18 + 54 - 162 + \cdots 4374$

- $1 + \dfrac{1}{3} + \dfrac{1}{9} + \cdots + \dfrac{1}{729}$

# Lesson 35: Applications of Series

### Lecture 39: Application of Series

### The Compound Interest Formula

Recall that if one invests $P$ dollars at an interest rate $r$ compounded $n$ times per year, the amount yielded after $t$ years is __________________________ .

**Example.** A bank account accrues interest at 6% compounded monthly.

- If you make a deposit of \$100, how much is in the account at that time? (Don't think hard)

- If you make a deposit of \$100 now and another \$100 deposit one month from now, how much is in the account at that time?

- If you make a three deposits of \$100 each month, how much is in the account immediately after the third deposit?

**Example.** (Continued)

- Suppose you make a deposit of \$100 every month for the next 15 years. How much is in your account after the last deposit?

**Annuities**

If one invests $P$ dollars $n$ times per year for $t$ years earning at an interest rate $r$, the total amount after the last investment is ______________________________ .

**Example.** You would like to buy a \$400,000 house in 8 years, so you start an annuity at 7% compounded quarterly. What should your investments be every month to achieve this?

**Example.** You deposit \$1500 each week into an account which earns 4% compounded weekly.

- How much will be in the account after 2 years worth of deposits?

- How much will be in the account after 4 years worth of deposits?

- How many years would be needed to amass \$1,000,000?

## Present Value

**Example.** Suppose you invested some money at 6% and after five years, your investment is worth \$5000. How much was your initial investment?

**Example.** You owe a debt of \$7,000 that is to be paid in one lump sum nine years from now. Suppose the going interest rate in a savings account is 3.5%. If you could, how much money should you put away now in a savings account so that nine years from now, you will have exactly the amount of money to pay off this debt?

**Example.** You intend to buy a Nintendo Wii in 7 months for \$150. You can currently open a savings account at 2% compounded monthly. How much should you deposit now so you will have money for the Wii in 7 months?

## Introduction to Amortization

**Example.** Suppose the going interest rate is 6% compounded monthly.

- If you need to pay the bank \$250 at the end of one month, but would like to settle the debt today, how much should you pay them?

- If you need to pay the bank \$250 at the end of one month AND at the end of the second month, how much could you pay today to settle this debt?

- If you need to pay the bank \$250 at the end of each of the next three months, how much could you pay today to settle this debt?

**Example (continued)**

▸ If you need to pay the bank \$250 at the end of each month for the next 60 months, how much could you pay today to settle this debt?

**Payment Schedules**

1 If one pays $P$ dollars each period for a length of $t$ years (or $nt$ periods), then the current value of the payments with respect to an interest rate $r$ would be ________________________________ .

**Example.** Suppose you would like to buy a car that is presently worth \$20000. The bank is willing to forward you the money and allow you to purchase it, and have you pay them back with 60 monthly payments with an interest rate of 3%. How much would your payments be?

The process of determining a payment schedule for paying off loans is called _______________ .

**Example.** What would the payment plans be if you financed a \$15000 car at 4% amortized over

- 3 years, paying monthly?

- 6 years, paying monthly?

- If you could only pay up to \$300 per month, what is the minimum number of years it would take to pay off the car?

**Example.** When purchasing a \$250,000 house, you put down \$25,000 at signing and take out a mortgage (mortgage means an amortized payment plan) at 2.5%.

- What will your payments be each month if you take out a 20-year mortgage?

- What would the payments be on a 30-year mortgage?

- If you can pay \$1000 a month, what is the shortest mortgage (in years) you can take?

**Example.** Suppose you are sending your kid to college, and the cost of room and board is $25,000 per semester. How much should be deposited in an account accruing 3% compounded semi-annually so that your kid can immediately withdrawl $25,000 for their first semester, and do so at the beginning of each of the remaining 7 semesters?

**Example.** If you were to set up an annuity at 5% compounded monthly, how much should have been invested every month for the last 18 years in order to allow your child to have $25,000 each semester for room and board?

# Lesson 36: Induction

# Lesson 37: The Binomial Theorem

**Factorials**
When $n$ is a non-negative integer, define $n!$ as

By convention, we define

**Example.** Simplify

- $1!$
- $3!$
- $5!$
- $7!$
- $\dfrac{10!}{9!}$
- $\dfrac{10!}{8!}$
- $\dfrac{10!}{8!2!}$
- $\dfrac{6!}{3!3!}$
- $\dfrac{n!}{(n-1)!}$
- $\dfrac{n!}{(n-2)!2!}$
- $\dfrac{(n+2)!}{(n-1)!}$
- $\dfrac{(n+1)!}{(n-2)!3!}$

## Binomial Coefficients

If $n$ and $k$ are positive integers, then

$$\binom{n}{k} =$$

**Example.** Simplify

- $\binom{10}{2}$
- $\binom{10}{8}$
- $\binom{8}{3}$
- $\binom{8}{5}$
- $\binom{7}{1}$
- $\binom{7}{0}$
- $\binom{6}{3}$
- $\binom{6}{6}$
- $\binom{n}{2}$
- $\binom{n}{n-1}$

## Binomials

A is a sum or difference of .

**Example.** Which is a binomial?

$$x + y, \quad xy, \quad x^2 + y^2, \quad xyzw + x^2yz, \quad 2xy - y^2$$

What happens when we raise a binomial to a power?

$$(a + b)^0 =$$

$$(a + b)^1 =$$

$$(a + b)^2 =$$

$$(a + b)^3 =$$

$$(a + b)^4 =$$

$$(a + b)^8 =$$

## Pascal's Triangle

____

____ ____

____ ____ ____

____ ____ ____ ____

____ ____ ____ ____ ____

____ ____ ____ ____ ____ ____

____ ____ ____ ____ ____ ____ ____

____ ____ ____ ____ ____ ____ ____ ____

The of Pascal's Triangle give the

**Example.** Find

$$(a+b)^6 =$$

$$(a+b)^7 =$$

$$(a+b)^8 =$$

**Example.** Expand.

- $(x + y)^4$
- $(x - y)^3$
- $(2x + y)^4$
- $(\frac{1}{2}x - \frac{1}{3}y)^3$
- $(x^2 + y)^5$
- $(2x^2 - 3y^3)^3$

**The Binomial Theorem** - The binomial expansion of $(a+b)^n$ is

**Example.** Find

- the $a^2b^7$ term in $(a+b)^9$
- the $a^2b^7$ term in $(a-b)^9$
- the $x^2y^4$ term in $(2x-y)^6$
- the $x^2y^4$ term in $(x^2+\frac{1}{2}y)^5$
- the 4th term of $(a+b)^9$
- the 3rd term of $(x-y)^{10}$
- the 50th term of $(x^2-\frac{1}{2}y)^{52}$

# Lesson 38: Complex Numbers

## Lecture 34: The Fundamental Theorem of Algebra - Part 1

**Example.** Find the solutions to the following equations:

- $x^2 - 1 = 0$
- $x^2 + 1 = 0$

### The Imaginary Number

Define $i$ as the imaginary number such that $\underline{i^2 = -1}$.

We call a **complex number** a number of the form $\underline{a + b \cdot i}$, where $a$ and $b$ are real numbers.

The value of $a$ is called the **real component** of $a + b \cdot i$.

The value of $b$ is called the **imaginary component** of $a + b \cdot i$.

### Property:

If $a + b \cdot i = c + d \cdot i$, then $a = c$ and $b = d$.

In other words, for two complex numbers to equal, their <u>components</u> must be equal.

**Example.** Find $r$ and $s$ so that $3 - 2i = \log_2 r + \dfrac{r}{s} i$.

## Properties of Complex Numbers

- Addition and subtraction:

  When adding or subtracting complex numbers, treat $i$ as a variable and combine like terms.

  - $(3+2i)+(4-i)=$
  - $(5+7i)-(2-3i)=$

- Multiplication:

  When multiplying complex numbers, treat $i$ as a variable and multiply out the binomials. Then, whenever you have $i^2$ in your result, replace it with $-1$.

  - $(1-i)\cdot(2+5i)=$

**Definition**:

The **complex conjugate** of $z=a+b\cdot i$ (denoted as $\overline{z}$) is $a-b\cdot i$.

**Example.** For each given $z$, find $\overline{z}$ and $z\cdot\overline{z}$.

- $z=3+4i$
- $z=2-i$
- $z=2i$
- $z=4$

**Property:** For $n$ positive, the radical $\sqrt{-n}$ is equivalent to $\sqrt{-1}\sqrt{n}=i\sqrt{n}$.

**Example.** Simplify the following radicals.

- $\sqrt{-4}$
- $\sqrt{-4\cdot-9}$
- $\sqrt{-4}\cdot\sqrt{-9}$

Note: You must rewrite the radical before doing any arithmetic.

## Division of Complex Numbers

When a fraction contains a complex number $a + bi$ in its denominator, we should multiply both numerator and denominator by its conjugate $a - bi$.

**Example.** Perform the indicated operations.

- $\dfrac{1 + 3i}{i}$
- $\dfrac{1 + 3i}{2 - i}$
- $\dfrac{5 - 2i}{3 + 4i}$

## Larger powers of $i$

**Example.** Compute

- $i^3$
- $i^4$
- $i^5$

$i$ to any power will always reduce to either $i, -1, -i,$ or 1. Do this by dividing the power by 4.

**Example.** Simplify these powers of $i$:

- $i^8$
- $i^{100}$
- $i^{2007}$
- $i^{483}$
- $i^{381}$
- $i^{502}$
- $i^{-1}$
- $i^{-2}$
- $i^{-3}$
- $i^{-10}$
- $i^{-15}$
- $i^{-207}$

**Example.** Solve the following quadratic equations:

- $x^2 + 4 = 0.$

- $x^2 - 6x + 13 = 0.$

- $x^2 + 2x + 9 = 0.$

# Lesson 39: The Fundamental Theorem of Algebra

## Multiplicities

**Example.** Find the roots for the following functions.

- $f(x) = x(x-1)$
- $f(x) = x^2(x-1)^3$
- $f(x) = (x-2)^4(x+1)^5$

**Definition.** We say $r$ is a root of $p(x)$ of **multiplicity** $m$ if __________________ is a factor of $p(x)$.

In other words, we can divide $p(x)$ by ____________ a total of ______ times.

**Example.** Find the roots and multiplicities for the following functions.

- $f(x) = x^3$
- $f(x) = (2x-1)^2(3x+1)^3$
- $f(x) = x(x-1)(x-2)^2(x-3)^3$
- $f(x) = x^4 - 2x^2 + 1$

**Example.** Find all roots of the following functions.

- $f(x) = x^3 - 6x^2 + 11x - 6$, given that 3 is a root.

- $f(x)x^3 - 5x^2 + 2x + 8$, given that 4 is a root.

**Example.** Find all roots of the following functions.

- $f(x) = x^4 - 4x^3 - 3x^2 + 10x + 8$, given that $-1$ is a root of multiplicity 2.

- $f(x) = x^3 - 9x - 10$, given that $-2$ is a root.

## The Fundamental Theorem of Algebra

**Theorem.** Every polynomial equation of the form $a_n x^2 + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ has at least one root within the complex number system.

What does this mean to us? It means that we can ____________________ any polynomial!

**The Linear Factors Theorem**
If $f(x)$ is a polynomial of degree $n$, then it can always be factored as

$f(x) =$ ____________________________________________________

where _______ is a constant, and

where ____________________ are the ______________ , and may not be ______________ .

**Example.** Write the following polynomials in the form

$$a(x - r_1)(x - r_2) \cdots (x - r_n) :$$

- $x^2 - 2x + 10$

- $3x^2 - 7x + 2$

**Example.** Write the following polynomials in the form $a(x - r_1)(x - r_2) \cdots (x - r_n)$ :

- $f(x) = x^3 - 5x^2 + 16x - 30$, given 3 is a root.

- $f(x) = x^4 + 5x^2 + 4$

- $x^3 - 7x + 6$, given that 1 is a root.

**Fact from High-Powered Algebra**
If all coefficients of a polynomial are ___________________

and a value which is ___________________ is a root, then so must its ___________________ .

**Example.** Write the following polynomials in the form

$$a(x - r_1)(x - r_2) \cdots (x - r_n):$$

- $x^4 + x^3 - 5x^2 - 3x + 6$, given that 1 and $-\sqrt{3}$ are roots.

- $f(x) = x^3 - 5x^2 + 17x - 13$, given that $2 + 3i$ is a root.

**Observation**.
Every polynomial of degree $n$ has ______ roots, when you account for ____________ .

**Example.** Find a minimal degree polynomial which meets the given requirements.

- leading coefficient of 1,

| Root | w/ Multiplicity |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |

- leading coefficient of 1,

| Root | w/ Multiplicity |
|---|---|
| 1 | 3 |
| 2 | 2 |

- Integer coefficients,

| Root | w/ Multiplicity |
|---|---|
| 0 | 1 |
| $\frac{1}{2}$ | 1 |
| $-\frac{2}{3}$ | 1 |

- goes through $(0, 1)$

| Root | w/ Multiplicity |
|---|---|
| 1 | 1 |
| 3 | 1 |
| $-2$ | 1 |

- leading coefficient of 1,

| Root | w/ Multiplicity |
|---|---|
| 0 | 1 |
| $i$ | 1 |
| $-i$ | 1 |