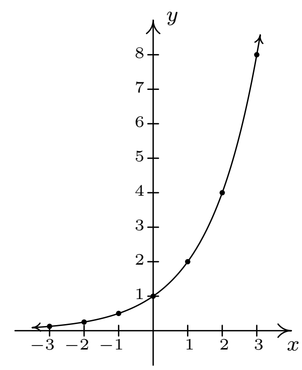
Chapter 1

Exponential and Logarithmic Functions

1.1 Introduction to Exponential and Logarithmic Functions

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most.[1] This section will introduce us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like x^2 or $x^{2/3}$, in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every College Algebra book we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect them in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	2^{-1}	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(2, 8)$



$y = f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$ and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the x-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of f is $(0, \infty)$. The graph of f passes the Horizontal Line Test which means f is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous[2] and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation.

[1] Take a class in Differential Equations and you'll see why.

[2] Recall that this means there are no holes or other kinds of breaks in the graph.

The number $\sqrt{3} = 1.73205\ldots$ is an irrational number[3] and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason[4] we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.[5]

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 1.1. A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called a **base b exponential function**.

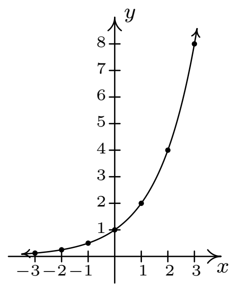
We leave it to the reader to verify[6] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Thinking back to Section **??**, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis. As such, we have

[3] You can actually prove this by considering the polynomial $p(x) = x^2 - 3$ and showing it has no rational zeros by applying Theorem **??**.

[4] This is where Calculus and continuity come into play.

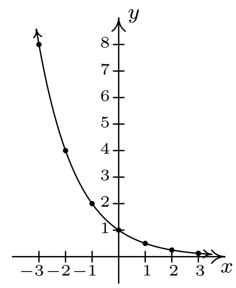
[5] Want more information? Look up "convergent sequences" on the Internet.

[6] Meaning, graph some more examples on your own.



$y = f(x) = 2^x$

reflect across y-axis $\longrightarrow$ multiply each x-coordinate by -1



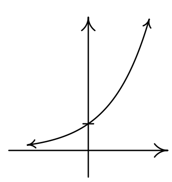
$y = g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g. We summarize these observations, and more, in the following theorem whose proof ultimately requires Calculus.

THEOREM 1.1. **Properties of Exponential Functions:** Suppose $f(x) = b^x$.

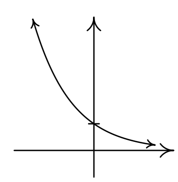
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[a]

- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

[a] Recall that this means the graph of f has no sharp turns or corners.

Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. We will more formally discuss the origins of this number in Section 1.5. For now, it is enough to know that since $e > 1$, $f(x) = e^x$ is an increasing exponential function. The following examples offer a glimpse as to the kind of real-world phenomena these functions can model.

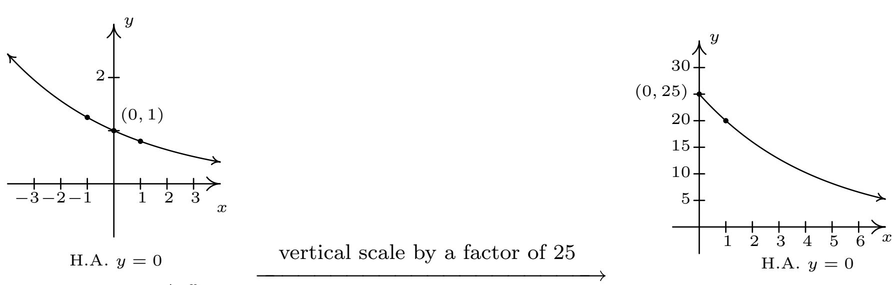
EXAMPLE 1.1.1. The value of a car can be modeled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

SOLUTION.

1. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since x represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of \$25,000. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was \$25,000.

2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the y-intercept $(0, 1)$ and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from f by 25, in other words, $V(x) = 25f(x)$. In accordance with Theorem **??**, this results in a vertical stretch by a factor of 25. We multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) and obtain the points $\left(-1, \frac{125}{4}\right)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us. We have the result below.



$y = f(x) = \left(\frac{4}{5}\right)^x$ $\qquad$ $y = V(x) = 25f(x)$, $x \geq 0$

3. We see from the graph of V that its horizontal asymptote is $y = 0$. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0.

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 1.5. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations from Section **??**.

Example 1.1.2. According to Netwon's Law of Cooling[7] the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

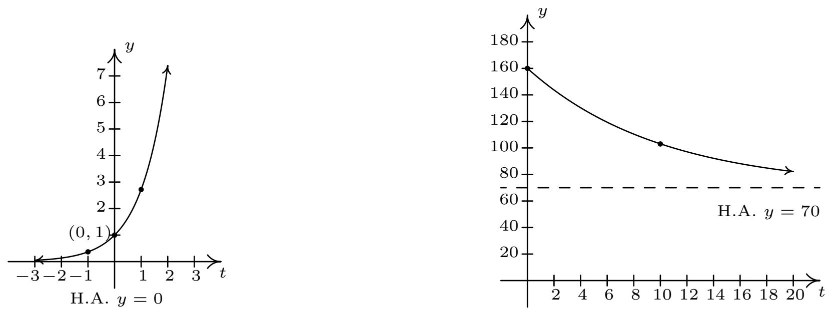
1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph.

Solution.

1. To find $T(0)$, we replace every occurrence of the independent variable t with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of f is an increasing exponential with y-intercept $(0, 1)$ and horizontal asymptote $y = 0$. The points $\left(-1, e^{-1}\right) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. Since the formula $T(t)$ looks rather complicated, we rewrite $T(t)$ in the form presented in Theorem **??** and use that result to track the changes to our three points and the horizontal asymptote. We have $T(t) = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Multiplication of the input to f, t, by -0.1 results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis. We divide each of the x values of our points by -0.1 (which amounts to multiplying them by -10) to obtain $\left(10, e^{-1}\right)$, $(0, 1)$, and $(-10, e)$. Since none of these changes affected the y values, the horizontal asymptote remains $y = 0$. Next, we see that the output from f is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the y-coordinates by 90 to obtain $\left(10, 90e^{-1}\right)$, $(0, 90)$, and $(-10, 90e)$. We also multiply the y value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the y-coordinates, which shifts the graph upwards to obtain $\left(10, 90e^{-1} + 70\right) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of f and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is below.

[7] We will discuss this in greater detail in Section 1.5.



$y = f(t) = e^t$ $\longrightarrow$ $y = T(t)$

3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically and to review some of the 'number sense' developed in Chapter **??**. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$, $e^{\text{very big }(-)} = \frac{1}{e^{\text{very big }(+)}} \approx \frac{1}{\text{very big }(+)} \approx$ very small $(+)$. The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70 +$ very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature. $\square$

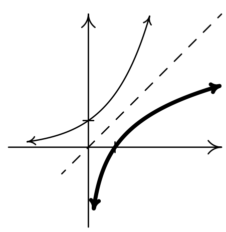
As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 1.2. The inverse of the exponential function $f(x) = b^x$ is called the **base b logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ The expression $\log_b(x)$ is read 'log base b of x.'

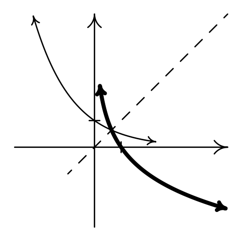
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 1.3. The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

Since logs are defined as the inverses of exponential functions, we can use Theorems **??** and **??** to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$ as shown below. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$
$\boldsymbol{y = \log_b(x)}$, $b > 1$



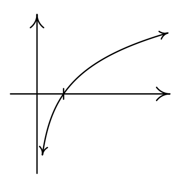
$y = b^x$, $0 < b < 1$
$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

THEOREM 1.2. **Properties of Logarithmic Functions:** Suppose $f(x) = \log_b(x)$.

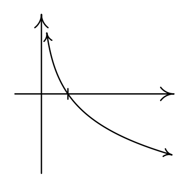
- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 1.2 is a consequence of Theorems **??** and **??**. However, it is worth the reader's time to understand Theorem 1.2 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 1.2 at this level instead of just merely memorizing the facts.

EXAMPLE 1.1.3. Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\frac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

SOLUTION.

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.
2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.
3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.
4. First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.
5. Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.
6. We can use Theorem 1.2 directly to simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.
7. We note that we cannot apply Theorem 1.2 directly to $117^{-\log_{117}(6)}$. (Why not?) We use a property of exponents to rewrite $117^{-\log_{117}(6)}$ as $\dfrac{1}{117^{\log_{117}(6)}}$. At this point, we can apply Theorem 1.2 to get $117^{\log_{117}(6)} = 6$ and thus $117^{-\log_{117}(6)} = \dfrac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$. It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$. □

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument[8] of the log must be strictly positive.

[8]See page **??** if you've forgotten what this term means.

Example 1.1.4. Find the domain of the following functions. Check your answers graphically using the calculator.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution.

1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph from the calculator below verifies this. Note that we could have graphed f using transformations. Taking a cue from Theorem **??**, we rewrite $f(x) = 2\log_{10}(-x+3) - 1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x+3) - 1$, Theorem **??** tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multplying by -1 gives us the vertical asymptote $x = 3$. Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.

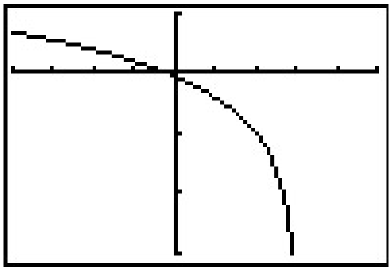
2. To find the domain of g, we set $\frac{x}{x-1} > 0$ and use a sign diagram to solve this inequality. We define $r(x) = \frac{x}{x-1}$ find its domain to be r is $(-\infty, 1) \cup (1, \infty)$. Setting $r(x) = 0$ gives $x = 0$.

   $(+)\ 0\ \ (-)\ \ ‽\ (+)$

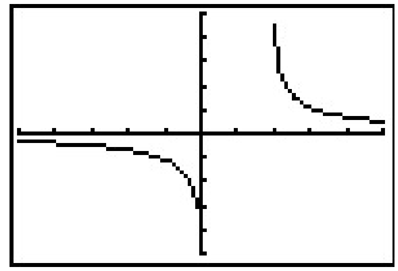
   $\longleftarrow\!\!+\!\!-\!\!-\!\!+\!\!\longrightarrow$

   $0 \qquad 1$

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of g. The graph of $y = g(x)$ confirms this. We can tell from the graph of g that it is not the result of Section **??** transformations being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behavior of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter **??** and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.



$y = f(x) = 2\log(3-x) - 1$



$y = g(x) = \ln\left(\dfrac{x}{x-1}\right)$

$\square$

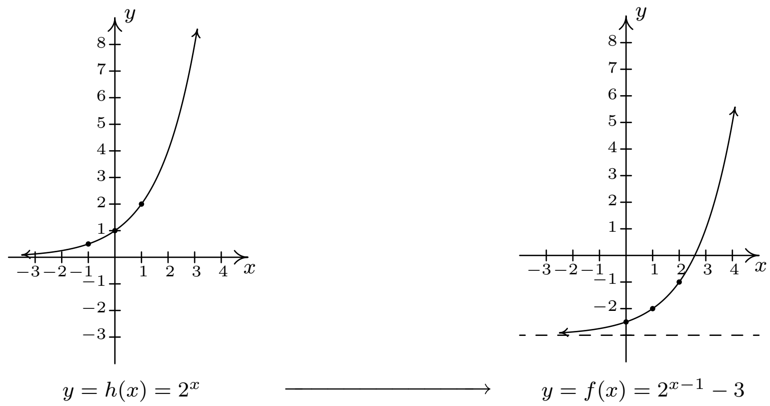
While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

Example 1.1.5. Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check the symmetry about the line $y = x$.

Solution.

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x-1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0,1)$ and $(1,2)$ on the graph of g along with the horizontal asymptote $y = 0$ to track through the transformations. By Theorem **??** we first add 1 to the x-coordinates of the points on the graph of g (shifting g to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1,1)$ and $(2,2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the y-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1,-2)$ and $(2,-1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of g, we get the graph below. We see that the domain of f is the same as g, namely $(-\infty, \infty)$, but that the range of f is $(-3, \infty)$.



2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find f takes an input x and applies the steps

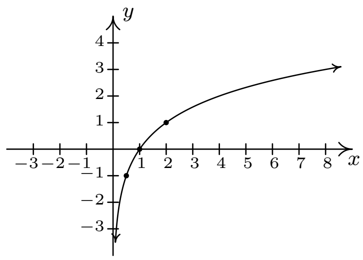
   (a) subtract 1
   (b) put as an exponent on 2
   (c) subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should

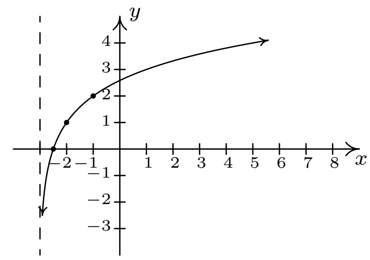
   (a) add 3
   (b) take the logarithm base 2
   (c) add 1

   In symbols, $f^{-1}(x) = \log_2(x + 3) + 1$.

3. To graph $f^{-1}(x) = \log_2(x + 3) + 1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations using Theorem **??**. Since $f^{-1}(x) = j(x + 3) + 1$, we first subtract 3 from each of the x values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, $(-2, 0)$ and $(-1, 1)$ with a vertical asymptote $x = -3$. Next, we add 1 to the y values on the graph and get $\left(-\frac{5}{2}, 0\right)$, $(-2, 1)$ and $(-1, 2)$. If you are experiencing *déjà vu*, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f.



$y = j(x) = \log_2(x)$ $\longrightarrow$ $y = f^{-1}(x) = \log_2(x+3)+1$



4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3)+1$ satisfy the composition requirement for inverses. For all real numbers x,

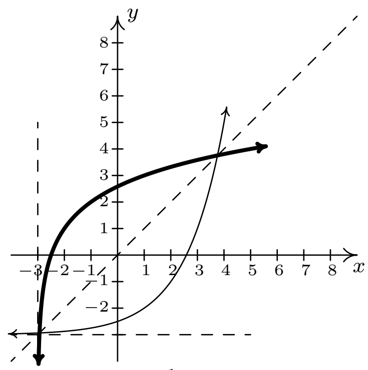
$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1} - 3\right) \\
&= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\
&= \log_2\left(2^{x-1}\right) + 1 \\
&= (x-1) + 1 && \text{Since } \log_2\left(2^u\right) = u \text{ for all real numbers } u \\
&= x \checkmark
\end{aligned}$$

For all real numbers $x > -3$, we have[9]

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3) + 1\right) \\
&= 2^{(\log_2(x+3)+1)-1} - 3 \\
&= 2^{\log_2(x+3)} - 3 \\
&= (x+3) - 3 && \text{Since } 2^{\log_2(u)} = u \text{ for all real numbers } u > 0 \\
&= x \checkmark
\end{aligned}$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$.

[9] Pay attention - can you spot in which step below we need $x > -3$?



$y = f(x) = 2^{x-1} - 3$

$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

$\square$

1.1.1 Exercises

1. Evaluate the expression.

   (a) $\log_3(27)$
   (b) $\log_6(216)$
   (c) $\log_2(32)$
   (d) $\log_6\left(\frac{1}{36}\right)$
   (e) $\log_8(4)$
   (f) $\log_{36}(216)$
   (g) $\log_{\frac{1}{5}}(625)$
   (h) $\log_{\frac{1}{6}}(216)$
   (i) $\log_{36}(36)$
   (j) $\log\left(\frac{1}{1000000}\right)$
   (k) $\log(0.01)$
   (l) $\ln\left(e^3\right)$
   (m) $\log_4(8)$
   (n) $\log_6(1)$
   (o) $\log_{13}\left(\sqrt{13}\right)$
   (p) $\log_{36}\left(\sqrt[4]{36}\right)$
   (q) $7^{\log_7(3)}$
   (r) $36^{\log_{36}(216)}$
   (s) $\log_{36}\left(36^{216}\right)$
   (t) $\ln\left(e^5\right)$
   (u) $\log\left(\sqrt[9]{10^{11}}\right)$
   (v) $\log\left(\sqrt[3]{10^5}\right)$
   (w) $\ln\left(\frac{1}{\sqrt{e}}\right)$
   (x) $\log_5\left(3^{\log_3(5)}\right)$
   (y) $\log\left(e^{\ln(100)}\right)$

2. Find the domain of the function.

   (a) $f(x) = \ln(x^2 + 1)$
   (b) $f(x) = \log_7(4x + 8)$
   (c) $f(x) = \ln(4x - 20)$
   (d) $f(x) = \log\left(x^2 + 9x + 18\right)$
   (e) $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$
   (f) $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$
   (g) $f(x) = \ln(7 - x) + \ln(x - 4)$
   (h) $f(x) = \ln(4x - 20) + \ln\left(x^2 + 9x + 18\right)$
   (i) $f(x) = \log\left(x^2 + x + 1\right)$
   (j) $f(x) = \sqrt[4]{\log_4(x)}$
   (k) $f(x) = \log_9(|x + 3| - 4)$
   (l) $f(x) = \ln(\sqrt{x-4} - 3)$
   (m) $f(x) = \dfrac{1}{3 - \log_5(x)}$
   (n) $f(x) = \dfrac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$
   (o) $f(x) = \ln(-2x^3 - x^2 + 13x - 6)$

3. For each function given below, find its inverse from the 'procedural perspective' discussed in Example 1.1.5 and graph the function and its inverse on the same set of axes.

   (a) $f(x) = 3^{x+2} - 4$
   (b) $f(x) = \log_4(x - 1)$
   (c) $f(x) = -2^{-x} + 1$
   (d) $f(x) = 5\log(x) - 2$

4. Show that $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0,\ b \neq 1$.

5. (Crazy bonus question) Without using your calculator, determine which is larger: e^π or π^e.

6. (The Logarithmic Scales) There are three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay close attention to the subtle differences.

(a) Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[10] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

i. Show that $M(0.001) = 0$.
ii. Compute $M(80,000)$.
iii. Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.
iv. Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

(b) While the decibel scale can be used in many disciplines,[11] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound.[12] The Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\frac{W}{m^2}$ is the threshold of human hearing.

i. Compute $L(10^{-6})$.
ii. Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?
iii. Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

[10] Rock-solid, perhaps?

[11] See this webpage for more information.

[12] As of the writing of this exercise, the Wikipedia page given here states that it may not meet the "general notability guideline" nor does it cite any references or sources. I find this odd because it is this very usage of the decibel scale which shows up in every College Algebra book I have read. Perhaps those other books have been wrong all along and we're just blindly following tradition.

(c) The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\mathrm{pH} = -\log[\mathrm{H}^+]$ where $[\mathrm{H}^+]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

i. The hydrogen ion concentration of pure water is $[\mathrm{H}^+] = 10^{-7}$. Find its pH.

ii. Find the pH of a solution with $[\mathrm{H}^+] = 6.3 \times 10^{-13}$.

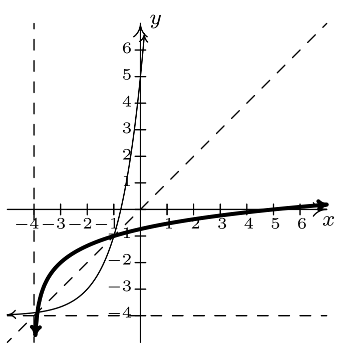
iii. The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

1.1.2 Answers

1. (a) $\log_3(27) = 3$
   (b) $\log_6(216) = 3$
   (c) $\log_2(32) = 5$
   (d) $\log_6\left(\frac{1}{36}\right) = -2$
   (e) $\log_8(4) = \frac{2}{3}$
   (f) $\log_{36}(216) = \frac{3}{2}$
   (g) $\log_{\frac{1}{5}}(625) = -4$
   (h) $\log_{\frac{1}{6}}(216) = -3$
   (i) $\log_{36}(36) = 1$
   (j) $\log \frac{1}{1000000} = -6$
   (k) $\log(0.01) = -2$
   (l) $\ln\left(e^3\right) = 3$
   (m) $\log_4(8) = \frac{3}{2}$
   (n) $\log_6(1) = 0$
   (o) $\log_{13}\left(\sqrt{13}\right) = \frac{1}{2}$
   (p) $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$
   (q) $7^{\log_7(3)} = 3$
   (r) $36^{\log_{36}(216)} = 216$
   (s) $\log_{36}\left(36^{216}\right) = 216$
   (t) $\ln(e^5) = 5$
   (u) $\log\left(\sqrt[9]{10^{11}}\right) = \frac{11}{9}$
   (v) $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$
   (w) $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$
   (x) $\log_5\left(3^{\log_3 5}\right) = 1$
   (y) $\log\left(e^{\ln(100)}\right) = 2$

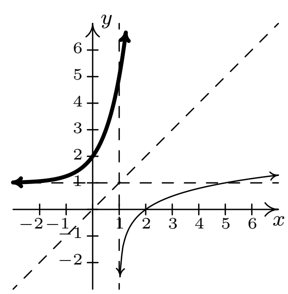
2. (a) $(-\infty, \infty)$
   (b) $(-2, \infty)$
   (c) $(5, \infty)$
   (d) $(-\infty, -6) \cup (-3, \infty)$
   (e) $(-2, -1) \cup (1, \infty)$
   (f) $(-6, -3) \cup (5, \infty)$
   (g) $(4, 7)$
   (h) $(5, \infty)$
   (i) $(-\infty, \infty)$
   (j) $[1, \infty)$
   (k) $(-\infty, -7) \cup (1, \infty)$
   (l) $(13, \infty)$
   (m) $(0, 125) \cup (125, \infty)$
   (n) No domain
   (o) $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

3. (a) $f(x) = 3^{x+2} - 4$
   $f^{-1}(x) = \log_3(x+4) - 2$



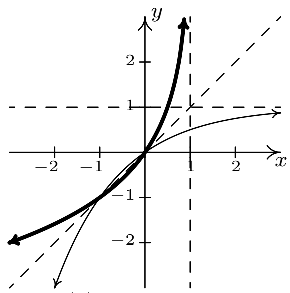
$y = f(x) = 3^{x+2} - 4$
$\boldsymbol{y = f^{-1}(x) = \log_3(x+4) - 2}$

   (b) $f(x) = \log_4(x-1)$
   $f^{-1}(x) = 4^x + 1$



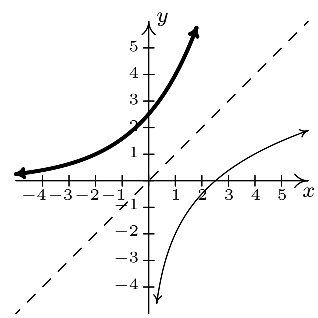
$y = f(x) = \log_4(x-1)$
$\boldsymbol{y = f^{-1}(x) = 4^x + 1}$

(c) $f(x) = -2^{-x} + 1$
$f^{-1}(x) = -\log_2(1-x)$



$y = f(x) = -2^{-x} + 1$
$\boldsymbol{y = f^{-1}(x) = -\log_2(1-x)}$

(d) $f(x) = 5\log(x) - 2$
$f^{-1}(x) = 10^{\frac{x+2}{5}}$



$y = f(x) = 5\log(x) - 2$
$\boldsymbol{y = f^{-1}(x) = 10^{\frac{x+2}{5}}}$

6. (a) i. $M(0.001) = \log\left(\dfrac{0.001}{0.001}\right) = \log(1) = 0.$

   ii. $M(80,000) = \log\left(\dfrac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9.$

   (b) i. $L(10^{-6}) = 60$ decibels.

   ii. $I = 10^{-.5} \approx 0.316$ watts per square meter.

   iii. Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

   (c) i. The pH of pure water is 7.

   ii. If $[\mathrm{H}^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.

   iii. $[\mathrm{H}^+] = 10^{-0.7} \approx .1995$ moles per liter.

1.2 Properties of Logarithms

In Section 1.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge rule in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 1.2 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 1.3. **(Inverse Properties of Exponential and Log Functions)** Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out in more detail what it means for exponential and logarithmic functions to be one-to-one.

Theorem 1.4. **(One-to-one Properties of Exponential and Log Functions)** Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one. In other words:

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base b is restricted to $b > 0$, $b \neq 1$. An added benefit of this restriction is that it eliminates the pathologies discussed in Section **??** when, for example, we simplified $\left(x^{2/3}\right)^{3/2}$ and obtained $|x|$ instead of what we had expected from the arithmetic in the exponents, $x^1 = x$.

THEOREM 1.5. **(Algebraic Properties of Exponential Functions)** Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u+w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u-w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 1.5 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

THEOREM 1.6. **(Algebraic Properties of Logarithm Functions)** Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 1.6 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of b^x, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly. From a purely functional approach, we can see the properties in Theorem 1.6 as an example of how inverse functions interchange the roles of inputs in outputs. For instance, the Product Rule for exponential functions given in Theorem 1.5, $f(u+w) = f(u)f(w)$, says that adding inputs results in multiplying outputs. Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely what the Product Rule for Logarithmic functions states in Theorem 1.6: $g(uw) = g(u)+g(w)$. The reader is encouraged to view the remaining properties listed in Theorem 1.6 similarly. The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 1.6 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log. While it is the opposite process, which we will practice later, that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

EXAMPLE 1.2.1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\frac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\frac{3}{ex}\right)^2$
4. $\log\sqrt[3]{\frac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2-4\right)$

SOLUTION.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u=8$ and $w=x$ and simplify.

$$\begin{aligned} \log_2\left(\frac{8}{x}\right) &= \log_2(8)-\log_2(x) && \text{Quotient Rule} \\ &= 3-\log_2(x) && \text{Since } 2^3=8 \\ &= -\log_2(x)+3 \end{aligned}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product. In order to use the Product Rule, the *entire* quantity inside the logarithm must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u=10$ and $w=x^2$. Once we get the x^2 by itself inside the log, we may apply the Power Rule with $u=x$ and $w=2$ and simplify.

$$\begin{aligned} \log_{0.1}\left(10x^2\right) &= \log_{0.1}(10)+\log_{0.1}\left(x^2\right) && \text{Product Rule} \\ &= \log_{0.1}(10)+2\log_{0.1}(x) && \text{Power Rule} \\ &= -1+2\log_{0.1}(x) && \text{Since } (0.1)^{-1}=10 \\ &= 2\log_{0.1}(x)-1 \end{aligned}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u=\frac{3}{ex}$ and $w=2$. Next, we see the Quotient Rule is applicable, with $u=3$ and $w=ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3)-\ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3)-\ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u=e$ and

$w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule} \\
&= 2\left[\ln(3) - \ln(ex)\right] && \text{Quotient Rule} \\
&= 2\ln(3) - 2\ln(ex) && \\
&= 2\ln(3) - 2\left[\ln(e) + \ln(x)\right] && \text{Product Rule} \\
&= 2\ln(3) - 2\ln(e) - 2\ln(x) && \\
&= 2\ln(3) - 2 - 2\ln(x) && \text{Since } e^1 = e \\
&= -2\ln(x) + 2\ln(3) - 2 &&
\end{aligned}$$

4. In Theorem 1.6, there is no mention of how to deal with radicals. However, thinking back to Definition **??**, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule[1], and we keep in mind that the common log is log base 10.

$$\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3} && \\
&= \tfrac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule} \\
&= \tfrac{1}{3}\left[\log\left(100x^2\right) - \log\left(yz^5\right)\right] && \text{Quotient Rule} \\
&= \tfrac{1}{3}\log\left(100x^2\right) - \tfrac{1}{3}\log\left(yz^5\right) && \\
&= \tfrac{1}{3}\left[\log(100) + \log\left(x^2\right)\right] - \tfrac{1}{3}\left[\log(y) + \log\left(z^5\right)\right] && \text{Product Rule} \\
&= \tfrac{1}{3}\log(100) + \tfrac{1}{3}\log\left(x^2\right) - \tfrac{1}{3}\log(y) - \tfrac{1}{3}\log\left(z^5\right) && \\
&= \tfrac{1}{3}\log(100) + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Power Rule} \\
&= \tfrac{2}{3} + \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) && \text{Since } 10^2 = 100 \\
&= \tfrac{2}{3}\log(x) - \tfrac{1}{3}\log(y) - \tfrac{5}{3}\log(z) + \tfrac{2}{3} &&
\end{aligned}$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2 - 4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2 - 4 = (x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule.

[1] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

$$\begin{aligned} \log_{117}\left(x^2-4\right) &= \log_{117}\left[(x+2)(x-2)\right] && \text{Factor} \\ &= \log_{117}(x+2)+\log_{117}(x-2) && \text{Product Rule} \end{aligned}$$

□

A couple of remarks about Example 1.2.1 are in order. First, while not explicitly stated in the above example, a general rule of thumb to determine which log property to apply first to a complicated problem is 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. In a multi-step problem, this rule can give the required guidance on which log property to apply at each step. The reader is encouraged to look through the solutions to Example 1.2.1 to see this rule in action. Second, while we were instructed to assume when necessary that all quantities represented positive real numbers, the authors would be committing a sin of omission if we failed to point out that, for instance, the functions $f(x) = \log_{117}\left(x^2-4\right)$ and $g(x) = \log_{117}(x+2)+\log_{117}(x-2)$ have different domains, and, hence, are different functions. We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$. In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so. One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten[2] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 1.2.2. Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1)-\log_3(x+1)$
2. $\log(x)+2\log(y)-\log(z)$
3. $4\log_2(x)+3$
4. $-\ln(x)-\frac{1}{2}$

Solution. Whereas in Example 1.2.1 we read the properties in Theorem 1.6 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1)-\log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.

2. In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x)+2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it

[2]The authors relish the irony involved in writing what follows.

all together, we have

$$\begin{aligned}
\log(x) + 2\log(y) - \log(z) &= \log(x) + \log\left(y^2\right) - \log(z) && \text{Power Rule} \\
&= \log\left(xy^2\right) - \log(z) && \text{Product Rule} \\
&= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule}
\end{aligned}$$

3. We can certainly get started rewriting $4\log_2(x) + 3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 1.3, we know $3 = \log_2\left(2^3\right)$, so we get

$$\begin{aligned}
4\log_2(x) + 3 &= \log_2\left(x^4\right) + 3 && \text{Power Rule} \\
&= \log_2\left(x^4\right) + \log_2\left(2^3\right) && \text{Since } 3 = \log_2\left(2^3\right) \\
&= \log_2\left(x^4\right) + \log_2(8) \\
&= \log_2\left(8x^4\right) && \text{Product Rule}
\end{aligned}$$

4. To get started with $-\ln(x) - \frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x) = \ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 1.3 gives us $\frac{1}{2} = \ln\left(e^{1/2}\right) = \ln\left(\sqrt{e}\right)$. We have

$$\begin{aligned}
-\ln(x) - \tfrac{1}{2} &= (-1)\ln(x) - \tfrac{1}{2} \\
&= \ln\left(x^{-1}\right) - \tfrac{1}{2} && \text{Power Rule} \\
&= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2} = \ln\left(e^{1/2}\right) \\
&= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\
&= \ln\left(\frac{1}{x\sqrt{e}}\right)
\end{aligned}$$

□

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1,\infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty,-1)\cup(1,\infty)$. We will need to keep this in mind when we solve equations involving logarithms in Section 1.4 - it is precisely for this reason we will have to check for extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 1.7. **(Change of Base)** Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

The proofs of the Change of Base formulas are a result of the other properties studied in this section. If we start with $b^{x \log_b(a)}$ and use the Power Rule in the exponent to rewrite $x\log_b(a)$ as $\log_b(a^x)$ and then apply one of the Inverse Properties in Theorem 1.3, we get

$$b^{x\log_b(a)} = b^{\log_b(a^x)} = a^x,$$

as required. To verify the logarithmic form of the property, we also use the Power Rule and an Inverse Property. We note that

$$\log_a(x)\cdot\log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x),$$

and we get the result by dividing through by $\log_b(a)$. Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$. While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.[3] What Theorem 1.7 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base - be it 10, e, 42, or 117. Your Calculus teacher will have more to say about this when the time comes.

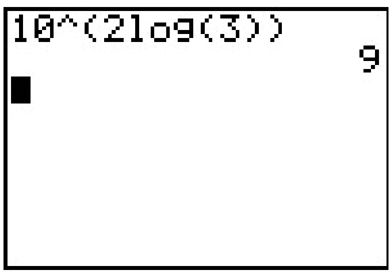
Example 1.2.3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a calculator, as appropriate.

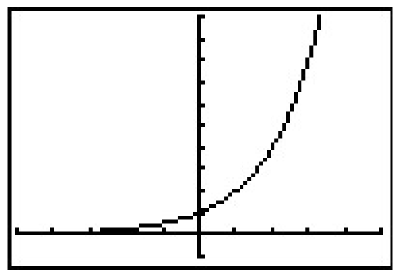
[3]The authors feel so strongly about showing students that every property of logarithms comes from and corresponds to a property of exponents that we have broken tradition with the vast majority of other authors in this field. This isn't the first time this happened, and it certainly won't be the last.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

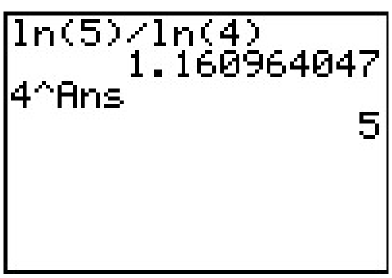
1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.
2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ and $g(x) = e^{x\ln(2)}$. Their graphs are indistinguishable which provides evidence that they are the same function.

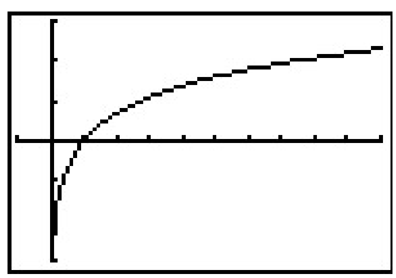




$y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The calculator confirms this.[4]
4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.





$y = f(x) = \ln(x)$ and $y = g(x) = \frac{\log(x)}{\log(e)}$

[4]Which means if it is lying to us about the first answer it gave us, at least it is being consistent.

1.2.1 Exercises

1. Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

   (a) $\ln(x^3y^2)$

   (b) $\log_2\left(\dfrac{128}{x^2+4}\right)$

   (c) $\log_5\left(\dfrac{z}{25}\right)^3$

   (d) $\log(1.23\times 10^{37})$

   (e) $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$

   (f) $\log_5\left(x^2-25\right)$

   (g) $\log_{\sqrt{2}}\left(4x^3\right)$

   (h) $\log_{\frac{1}{3}}(9x(y^3-8))$

   (i) $\log\left(1000x^3y^5\right)$

   (j) $\log_3\left(\dfrac{x^2}{81y^4}\right)$

   (k) $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$

   (l) $\log_6\left(\dfrac{216}{x^3y}\right)^4$

   (m) $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

2. Use the properties of logarithms to write the following as a single logarithm.

   (a) $4\ln(x)+2\ln(y)$

   (b) $3-\log(x)$

   (c) $\log_2(x)+\log_2(y)-\log_2(z)$

   (d) $\log_3(x)-2\log_3(y)$

   (e) $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$

   (f) $2\ln(x)-3\ln(y)-4\ln(z)$

   (g) $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$

   (h) $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$

   (i) $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

   (j) $\log_2(x)+\log_4(x-1)$

   (k) $\log_5(x)-3$

   (l) $\log_7(x)+\log_7(x-3)-2$

   (m) $\ln(x)+\frac{1}{2}$

3. Use an appropriate change of base formula to convert the following expressions to ones with the indicated base.

   (a) 7^{x-1} to base e

   (b) $\log_3(x+2)$ to base 10

   (c) $\left(\dfrac{2}{3}\right)^x$ to base e

   (d) $\log(x^2+1)$ to base e

4. Use the appropriate change of base formula to approximate the following logarithms.

   (a) $\log_3(12)$

   (b) $\log_5(80)$

   (c) $\log_6(72)$

   (d) $\log_4\left(\dfrac{1}{10}\right)$

   (e) $\log_{\frac{3}{5}}(1000)$

   (f) $\log_{\frac{2}{3}}(50)$

5. Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.

6. Prove the Quotient Rule and Power Rule for Logarithms.

7. Give numerical examples to show that, in general,

   (a) $\log_b(x + y) \neq \log_b(x) + \log_b(y)$

   (b) $\log_b(x - y) \neq \log_b(x) - \log_b(y)$

   (c) $\log_b\left(\frac{x}{y}\right) \neq \frac{\log_b(x)}{\log_b(y)}$

8. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction
$$HA \rightleftharpoons H^+ + A^-.$$
The acid disassociation constant, K_a, is given by
$$K_\alpha = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$
where the square brackets denote the concentrations just as they did in Exercise 6c in Section 1.1. The symbol pK_a is defined similarly to pH in that p$K_a = -\log(K_a)$. Using the definition of pH from Exercise 6c and the properties of logarithms, derive the Henderson-Hasselbalch Equation which states
$$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

9. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

10. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

1.2.2 Answers

1. (a) $3\ln(x) + 2\ln(y)$
   (b) $7 - \log_2(x^2 + 4)$
   (c) $3\log_5(z) - 6$
   (d) $\log(1.23) + 37$
   (e) $\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$
   (f) $\log_5(x - 5) + \log_5(x + 5)$
   (g) $3\log_{\sqrt{2}}(x) + 4$
   (h) $-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y - 2) + \log_{\frac{1}{3}}(y^2 + 2y + 4)$
   (i) $3 + 3\log(x) + 5\log(y)$
   (j) $2\log_3(x) - 4 - 4\log_3(y)$
   (k) $\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$
   (l) $12 - 12\log_6(x) - 4\log_6(y)$
   (m) $\frac{1}{3}\ln(x) - \ln(10) - \frac{1}{2}\ln(y) - \frac{1}{2}\ln(z)$

2. (a) $\ln(x^4y^2)$
   (b) $\log\left(\dfrac{1000}{x}\right)$
   (c) $\log_2\left(\dfrac{xy}{z}\right)$
   (d) $\log_3\left(\dfrac{x}{y^2}\right)$
   (e) $\log_3\left(\dfrac{\sqrt{x}}{y^2z}\right)$
   (f) $\ln\left(\dfrac{x^2}{y^3z^4}\right)$
   (g) $\log\left(\dfrac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
   (h) $\ln\left(\sqrt[3]{\dfrac{z}{xy}}\right)$
   (i) $\log_2\left(\dfrac{x}{x-1}\right)$
   (j) $\log_2\left(x\sqrt{x-1}\right)$
   (k) $\log_5\left(\dfrac{x}{125}\right)$
   (l) $\log_7\left(\dfrac{x(x-3)}{49}\right)$
   (m) $\ln\left(x\sqrt{e}\right)$

3. (a) $7^{x-1} = e^{(x-1)\ln(7)}$
   (b) $\log_3(x+2) = \dfrac{\log(x+2)}{\log(3)}$
   (c) $\left(\dfrac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$
   (d) $\log(x^2+1) = \dfrac{\ln(x^2+1)}{\ln(10)}$

4. (a) $\log_3(12) \approx 2.26186$
   (b) $\log_5(80) \approx 2.72271$
   (c) $\log_6(72) \approx 2.38685$
   (d) $\log_4\left(\dfrac{1}{10}\right) \approx -1.66096$
   (e) $\log_{\frac{3}{5}}(1000) \approx -13.52273$
   (f) $\log_{\frac{2}{3}}(50) \approx -9.64824$

1.3 Exponential Equations and Inequalities

In this section we will develop techniques for solving equations involving exponential functions. Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 1.4, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution. Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 1.3 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$. This makes sense because, after all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 1.7, to give us something more calculator friendly,[1] say $\log_2(129) = \frac{\ln(129)}{\ln(2)}$. Another way to arrive at this answer is as follows

$$\begin{array}{rcll} 2^x & = & 129 & \\ \ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\ x\ln(2) & = & \ln(129) & \text{Power Rule} \\ x & = & \dfrac{\ln(129)}{\ln(2)} & \end{array}$$

'Taking the natural log' of both sides is akin to squaring both sides: since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[2] Also note that we treat $\ln(2)$ as any other non-zero real number and divide it through[3] to isolate the variable x. We summarize below the two common ways to solve exponential equations, motivated by our examples.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

Example 1.3.1. Solve the following equations. Check your answer graphically using a calculator.

[1] You can use natural logs or common logs. We choose natural logs. (In Calculus, you'll learn these are the most 'mathy' of the logarithms.)

[2] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 1.4.

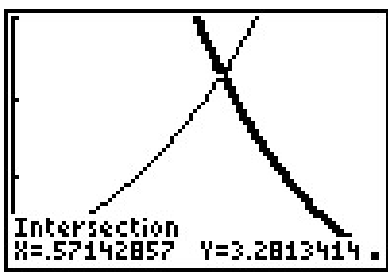
[3] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

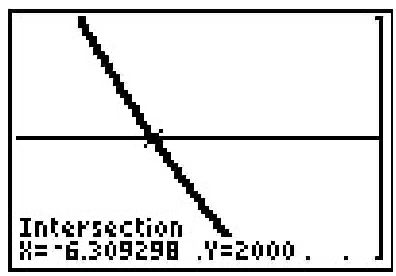
Solution.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$. Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$. To check graphically, we set $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.5714$.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$. Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$. Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ to get $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$. On the calculator, we graph $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ and find that they intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.3093$.



$y = f(x) = 2^{3x}$ and
$\boldsymbol{y = g(x) = 16^{1-x}}$



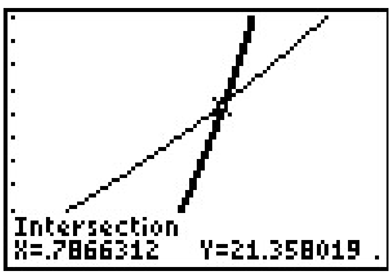
$y = f(x) = 2000$ and
$\boldsymbol{y = g(x) = 1000 \cdot 3^{-0.1x}}$

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$. Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$. The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants. The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

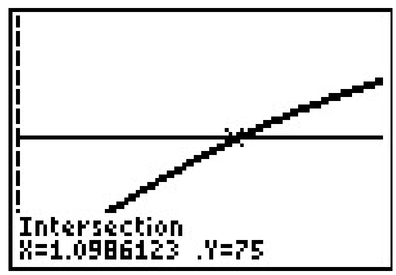
$$\begin{array}{rcll}
(x+2)\ln(3) & = & 2x\ln(7) & \\
x\ln(3) + 2\ln(3) & = & 2x\ln(7) & \\
2\ln(3) & = & 2x\ln(7) - x\ln(3) & \\
2\ln(3) & = & x(2\ln(7) - \ln(3)) & \text{Factor.} \\
x & = & \frac{2\ln(3)}{2\ln(7) - \ln(3)} &
\end{array}$$

Graphing $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ on the calculator, we see that these two graphs intersect at $x = \frac{2\ln(3)}{2\ln(7) - \ln(3)} \approx 0.7866$.

4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential. To that end, we clear denominators and get $75\left(1 + 3e^{-2t}\right) = 100$. From this we get $75 + 225e^{-2t} = 100$, which leads to $225e^{-2t} = 25$, and finally, $e^{-2t} = \frac{1}{9}$. Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. We can also use the Power Rule to write $\ln\left(\frac{1}{9}\right) = -\ln(9)$. Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?) The calculator confirms the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$ intersect at $x = \ln(3) \approx 1.099$.



$y = f(x) = 9 \cdot 3^x$ and
$\boldsymbol{y = g(x) = 7^{2x}}$

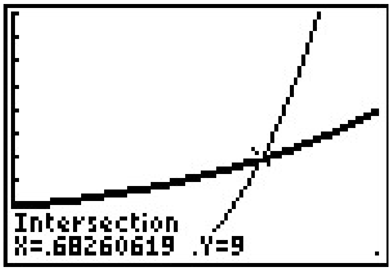


$y = f(x) = 75$ and
$\boldsymbol{y = g(x) = \frac{100}{1+3e^{-2x}}}$

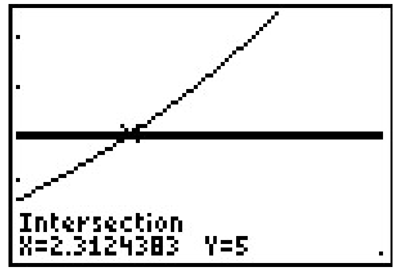
5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$. Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs. If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'. Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$. Since $5^x = -2$ has no real solution, (Why not?) we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln(5^x) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$. When we graph $f(x) = 25^x$ and $g(x) = 5^x + 6$, we see that they intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.6826$.

6. At first, it's unclear how to proceed with $\frac{e^x - e^{-x}}{2} = 5$, besides clearing the denominator to obtain $e^x - e^{-x} = 10$. Of course, if we rewrite $e^{-x} = \frac{1}{e^x}$, we see we have another denominator lurking in the problem: $e^x - \frac{1}{e^x} = 10$. Clearing this denominator gives us $e^{2x} - 1 = 10e^x$, and once again, we have an equation with three terms where the exponent on one term is exactly twice that of another - a 'quadratic in disguise.' If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$, we obtain

by the quadratic formula $u = 5 \pm \sqrt{26}$. From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$, but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$. If we graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$, we see that the graphs intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$



$y = f(x) = 25^x$ and
$\boldsymbol{y = g(x) = 5^x + 6}$



$y = f(x) = \frac{e^x - e^{-x}}{2}$ and
$\boldsymbol{y = g(x) = 5}$

□

The authors would be remiss not to mention that Example 1.3.1 still holds great educational value. Much can be learned about logarithms and exponentials by verifying the solutions obtained in Example 1.3.1 analytically. For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and obtain

$$\begin{array}{rcll}
2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\
2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\
2000 & \stackrel{\checkmark}{=} & 2000 &
\end{array}$$

The other solutions can be verified by using a combination of log and inverse properties. Some fall out quite quickly, while others are more involved. We leave them to the reader.

Since exponential functions are continuous on their domains, the Intermediate Value Theorem **??** applies. As with the algebraic functions in Section **??**, this allows us to solve inequalities using sign diagrams as demonstrated below.

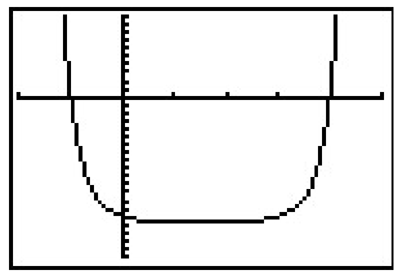
EXAMPLE 1.3.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $xe^{2x} < 4x$

SOLUTION.

1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$. The domain of r is all real numbers, so in order to construct our sign diagram, we seed to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by the one-to-one property of exponential functions, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$, is on or above the x-axis.

$$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & \vert & \text{———} & \vert & \longrightarrow \\ & -1 & & 4 & \end{array}$$



$y = r(x) = 2^{x^2-3x} - 16$

2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

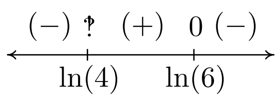
$$\begin{aligned} \frac{e^x}{e^x-4} &\leq 3 \\ \frac{e^x}{e^x-4} - 3 &\leq 0 \\ \frac{e^x}{e^x-4} - \frac{3\left(e^x-4\right)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12-2e^x}{e^x-4} &\leq 0 \end{aligned}$$

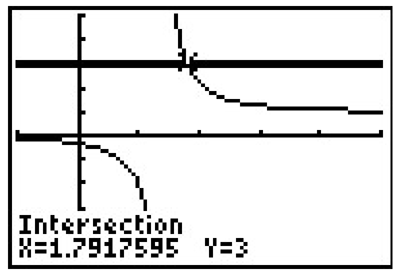
We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. While the prospect of determining the sign of $r(\ln(3))$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r(\ln(3)) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

[4] This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

We determine the signs of $r(\ln(5))$ and $r(\ln(7))$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$. Using the calculator, we see the graph of $f(x) = \frac{e^x}{e^x-4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x = \ln(6) \approx 1.792$.





$y = f(x) = \frac{e^x}{e^x-4}$

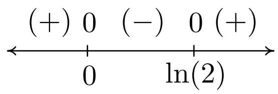
$\boldsymbol{y = g(x) = 3}$

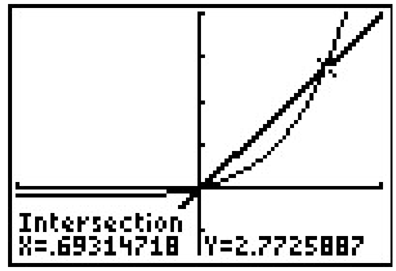
3. As before, we start solving $xe^{2x} < 4x$ by getting 0 on one side of the inequality, $xe^{2x} - 4x < 0$. We set $r(x) = xe^{2x} - 4x$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers. Setting $r(x) = 0$ produces $xe^{2x} - 4x = 0$. With x both in and out of the exponent, this could cause some difficulty. However, before panic sets in, we factor out the x to obtain $x\left(e^{2x} - 4\right) = 0$ which gives $x = 0$ or $e^{2x} - 4 = 0$. To solve the latter, we isolate the exponential and take logs to get $2x = \ln(4)$, or $x = \frac{\ln(4)}{2} = \ln(2)$. (Can you explain the last equality using properties of logs?) As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we have $r(\ln\left(\frac{1}{2}\right)) = \ln\left(\frac{1}{2}\right)e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right)$. Applying the Power Rule to the log in the exponent, we obtain $\ln\left(\frac{1}{2}\right)e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right)e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right)$. Using the inverse properties of logs, this reduces to $\frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right)$. Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$. Continuing in this manner, we find $r(x) < 0$ on $(0, \ln(2))$. The calculator confirms that the graph of $f(x) = xe^{2x}$ is below the graph of $g(x) = 4x$ on this intervals.[7]

[5] We could, of course, use the calculator, but what fun would that be?

[6] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

[7] Note: $\ln(2) \approx 0.693$.



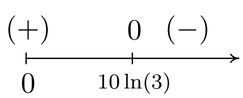


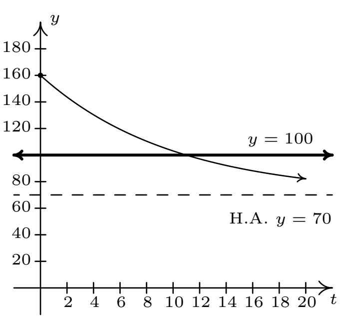
$y = f(x) = xe^{2x}$ and
$\boldsymbol{y = g(x) = 4x}$

□

EXAMPLE 1.3.3. Recall from Example 1.1.2 that the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

SOLUTION. We need to find when $T(t) > 100$, or in other words, we need to solve the inequality $70 + 90e^{-0.1t} > 100$. Getting 0 on one side of the inequality, we have $90e^{-0.1t} - 30 > 0$, and we set $r(t) = 90e^{-0.1t} - 30$. The domain of r is artificially restricted due to the context of the problem to $[0, \infty)$, so we proceed to find the zeros of r. Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which, after a quick application of the Power Rule leaves us with $t = 10\ln(3)$. If we wish to avoid using the calculator to choose test values, we note that since $1 < 3$, $0 = \ln(1) < \ln(3)$ so that $10\ln(3) > 0$. So we choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $10\ln(3) < 10\ln(4)$, so the latter is our choice of a test value for the interval $(10\ln(3), \infty)$. Our sign diagram is below, and next to it is our graph of $t = T(t)$ from Example 1.1.2 with the horizontal line $y = 100$.





$y = T(t)$

In order to interpret what this means in the context of the real world, we need a reasonable approximation of the number $10\ln(3) \approx 10.986$. This means it takes approximately 11 minutes for the coffee to cool to 100°F. Until then, the coffee is warmer than that.[8] □

[8]Critics may point out that since we needed to use the calculator to interpret our answer anyway, why not use it earlier to simplify the computations? It is a fair question which we answer unfairly: it's our book.

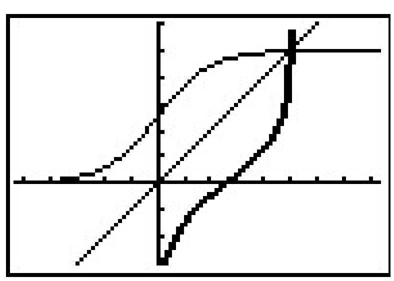
We close this section by finding the inverse of a function which is a composition of a rational function with an exponential function.

EXAMPLE 1.3.4. The function $f(x) = \dfrac{5e^x}{e^x+1}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

SOLUTION. We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{aligned}
y &= \frac{5e^x}{e^x+1} \\
x &= \frac{5e^y}{e^y+1} && \text{Switch } x \text{ and } y \\
x\left(e^y+1\right) &= 5e^y \\
xe^y + x &= 5e^y \\
x &= 5e^y - xe^y \\
x &= e^y(5-x) \\
e^y &= \frac{x}{5-x} \\
\ln\left(e^y\right) &= \ln\left(\frac{x}{5-x}\right) \\
y &= \ln\left(\frac{x}{5-x}\right)
\end{aligned}$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1}\circ f\right)(x) = x$ for all x in the domain of f and that $\left(f\circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this to the reader. To verify our solution graphically, we graph $y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$ on the same set of axes and observe the symmetry about the line $y = x$. Note the domain of f is the range of g and vice-versa.



$y = f(x) = \frac{5e^x}{e^x+1}$ and $\boldsymbol{y = g(x) = \ln\left(\frac{x}{5-x}\right)}$

$\square$

1.3.1 Exercises

1. Solve the following equations analytically.

   (a) $3^{(x-1)} = 27$
   (b) $3^{(x-1)} = 29$
   (c) $3^{(x-1)} = 2^x$
   (d) $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
   (e) $8^x = \frac{1}{128}$
   (f) $3^{7x} = 81^{4-2x}$
   (g) $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
   (h) $7e^{2x} = 28e^{-6x}$
   (i) $7^{3+7x} = 3^{4-2x}$
   (j) $\left(1 + \frac{0.06}{12}\right)^{12t} = 3$
   (k) $e^{-5730k} = \frac{1}{2}$
   (l) $2^{(x^3-x)} = 1$
   (m) $e^{2x} = 2e^x$
   (n) $70 + 90e^{-0.1t} = 75$
   (o) $\frac{150}{1+29e^{-0.8t}} = 75$
   (p) $25\left(\frac{4}{5}\right)^x = 10$

2. Solve the following inequalities analytically.

   (a) $e^x > 53$
   (b) $1000\left(1 + \frac{0.06}{12}\right)^{12t} \geq 3000$
   (c) $2^{(x^3-x)} < 1$
   (d) $25\left(\frac{4}{5}\right)^x \geq 10$
   (e) $\frac{150}{1+29e^{-0.8t}} \leq 130$
   (f) $70 + 90e^{-0.1t} \leq 75$

3. Use your calculator to help you solve the following equations and inequalities.

   (a) $e^x < x^3 - x$
   (b) $2^x = x^2$
   (c) $e^{\sqrt{x}} = x + 1$
   (d) $e^{-x} - xe^{-x} \geq 0$
   (e) $3^{(x-1)} < 2^x$
   (f) $e^x = \ln(x) + 5$

4. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram.

5. Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

6. In Example 1.3.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.

   (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
   (b) Find the range of f by finding the domain of f^{-1}.
   (c) Let $g(x) = \dfrac{5x}{x+1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. [9]

7. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

[9] We know this is true in general by Exercise **??** in Section **??**, but it's nice to see a specific example of the property.

1.3.2 Answers

1. (a) $x = 4$
   (b) $x = \dfrac{\ln(29) + \ln(3)}{\ln(3)}$
   (c) $x = \dfrac{\ln(3)}{\ln(3) - \ln(2)}$
   (d) $x = \dfrac{\ln(3) + 5\ln\left(\frac{1}{2}\right)}{\ln(3) - \ln\left(\frac{1}{2}\right)}$
   (e) $x = -\frac{7}{3}$
   (f) $x = \frac{16}{15}$
   (g) $x = -\frac{2}{11}$
   (h) $x = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
   (i) $x = \dfrac{4\ln(3) - 3\ln(7)}{7\ln(7) + 2\ln(3)}$
   (j) $t = \dfrac{\ln(3)}{12\ln(1.005)}$
   (k) $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730}$
   (l) $x = -1,\ 0,\ 1$
   (m) $x = \ln(2)$
   (n) $t = 10\ln(18)$
   (o) $t = \dfrac{\ln\left(\frac{1}{29}\right)}{-0.8}$
   (p) $x = \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}$

2. (a) $(\ln(53), \infty)$
   (b) $\left[\dfrac{\ln(3)}{12\ln(1.005)}, \infty\right)$
   (c) $(-\infty, -1) \cup (0, 1)$
   (d) $\left(-\infty, \dfrac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right]$
   (e) $\left(-\infty, \dfrac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, -\frac{5}{4}\ln\left(\frac{2}{377}\right)\right]$
   (f) $\left[\dfrac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$

3. (a) $(2.3217, 4.3717)$
   (b) $x \approx -0.76666$, $x = 2$, $x = 4$
   (c) $x = 0$
   (d) $(-\infty, 1]$
   (e) $(-\infty, 2.7095)$
   (f) $x \approx 0.01866$, $x \approx 1.7115$

4. $x > \frac{1}{3}(\ln(6) + 1)$

5. $f^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$. Both f and f^{-1} have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

1.4 Logarithmic Equations and Inequalities

In Section 1.3 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from. For example, suppose we wish to solve $\log_2(x) = \log_2(5)$. Theorem 1.4 tells us that the *only* solution to this equation is $x = 5$. Now suppose we wish to solve $\log_2(x) = 3$. If we want to use Theorem 1.4, we need to rewrite 3 as a logarithm base 2. We can use Theorem 1.3 to do just that: $3 = \log_2\left(2^3\right) = \log_2(8)$. Our equation then becomes $\log_2(x) = \log_2(8)$ so that $x = 8$. However, we could have arrived at the same answer, in fewer steps, by using Theorem 1.3 to rewrite the equation $\log_2(x) = 3$ as $2^3 = x$, or $x = 8$. We summarize the two common ways to solve log equations below.

Steps for Solving an Equation involving Logarithmic Fuctions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate the arguments of the log functions.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 1.4.1. Solve the following equations. Check your solutions graphically using a calculator.

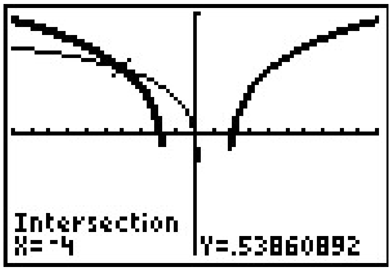
1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(x-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2x) = 1 - \log_7(3-x)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(x+1) = 2\log_2(x)$

Solution.

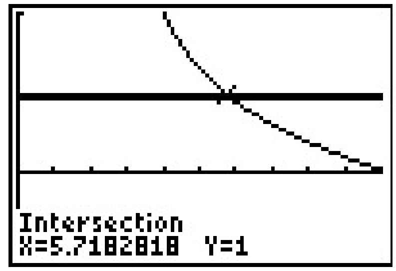
1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate what's inside the logs to get $1 - 3x = x^2 - 3$. Solving $x^2 + 3x - 4 = 0$ gives $x = -4$ and $x = 1$. To check these answers using the calculator, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$ and we see they intersect only at $x = -4$. To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x = 1$ is not in the domain of the original equation, and is not a solution.

[1] They do, however, represent the same **family** of complex numbers. We stop ourselves at this point and refer the reader to a good course in Complex Variables.

2. Our first objective in solving $2-\ln(x-3)=1$ is to isolate the logarithm. We get $\ln(x-3)=1$, which, as an exponential equation, is $e^1=x-3$. We get our solution $x=e+3$. On the calculator, we see the graph of $f(x)=2-\ln(x-3)$ intersects the graph of $g(x)=1$ at $x=e+3\approx 5.718$.

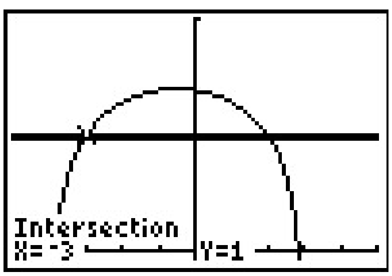


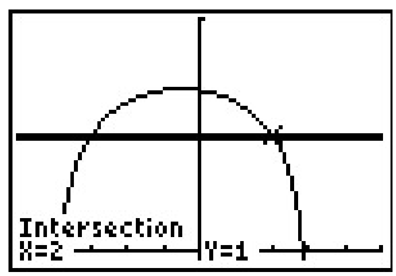
$y=f(x)=\log_{117}(1-3x)$ and
$\boldsymbol{y=g(x)=\log_{117}\left(x^2-3\right)}$



$y=f(x)=2-\ln(x-3)$ and
$\boldsymbol{y=g(x)=1}$

3. We can start solving $\log_6(x+4)+\log_6(3-x)=1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right]=1$. Rewriting this as an exponential equation, we get $6^1=(x+4)(3-x)$. This reduces to $x^2+x-6=0$, which gives $x=-3$ and $x=2$. Graphing $y=f(x)=\frac{\ln(x+4)}{\ln(6)}+\frac{\ln(3-x)}{\ln(6)}$ and $y=g(x)=1$, we see they intersect twice, at $x=-3$ and $x=2$.



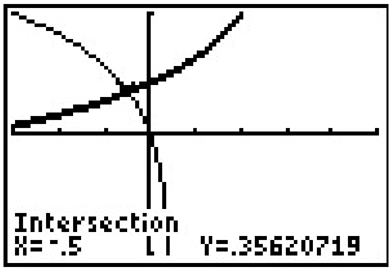


$y=f(x)=\log_6(x+4)+\log_6(3-x)$ and $\boldsymbol{y=g(x)=1}$

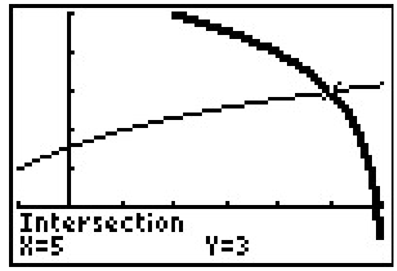
4. Taking a cue from the previous problem, we begin solving $\log_7(1-2x)=1-\log_7(3-x)$ by first collecting the logarithms on the same side, $\log_7(1-2x)+\log_7(3-x)=1$, and then using the Product Rule to get $\log_7[(1-2x)(3-x)]=1$. Rewriting this as an exponential equation gives $7^1=(1-2x)(3-x)$ which gives the quadratic equation $2x^2-7x-4=0$. Solving, we find $x=-\frac{1}{2}$ and $x=4$. Graphing, we find $y=f(x)=\frac{\ln(1-2x)}{\ln(7)}$ and $y=g(x)=1-\frac{\ln(3-x)}{\ln(7)}$ intersect only at $x=-\frac{1}{2}$. Checking $x=4$ in the original equation produces $\log_7(-7)=1-\log_7(-1)$, which is a clear domain violation.

5. Starting with $\log_2(x+3)=\log_2(6-x)+3$, we gather the logarithms to one side and get $\log_2(x+3)-\log_2(6-x)=3$, and then use the Quotient Rule to obtain $\log_2\left(\frac{x+3}{6-x}\right)=3$.

Rewriting this as an exponential equation gives $2^3 = \frac{x+3}{6-x}$. This reduces to the linear equation $8(6-x) = x+3$, which gives us $x = 5$. When we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$, we find they intersect at $x = 5$.

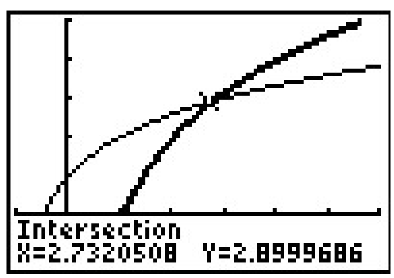


$y = f(x) = \log_7(1-2x)$ and
$\boldsymbol{y = g(x) = 1 - \log_7(3-x)}$



$y = f(x) = \log_2(x+3)$ and
$\boldsymbol{y = g(x) = \log_2(6-x) + 3}$

6. Starting with $1 + 2\log_4(x+1) = 2\log_2(x)$, we gather the logs to one side to get the equation $1 = 2\log_2(x) - 2\log_4(x+1)$. Before we can combine the logarithms, however, we need a common base. Since 4 is a power of 2, we use change of base to convert $\log_4(x+1) = \frac{\log_2(x+1)}{\log_2(4)} = \frac{1}{2}\log_2(x+1)$. Hence, our original equation becomes $1 = 2\log_2(x) - 2\left(\frac{1}{2}\log_2(x+1)\right)$ or $1 = 2\log_2(x) - \log_2(x+1)$. Using the Power and Quotient Rules, we obtain $1 = \log_2\left(\frac{x^2}{x+1}\right)$. Rewriting this in exponential form, we get $\frac{x^2}{x+1} = 2$ or $x^2 - 2x - 2 = 0$. Using the quadratic formula, we get $x = 1 \pm \sqrt{3}$. Graphing $f(x) = 1 + \frac{2\ln(x+1)}{\ln(4)}$ and $g(x) = \frac{2\ln(x)}{\ln(2)}$, we see the graphs intersect only at $x = 1+\sqrt{3} \approx 2.732$. The solution $x = 1 - \sqrt{3} < 0$, which means if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined.



$y = f(x) = 1 + 2\log_4(x+1)$ and $\boldsymbol{y = g(x) = 2\log_2(x)}$

□

If nothing else, Example 1.4.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot - any supposed solution which causes a negative number

[2] Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

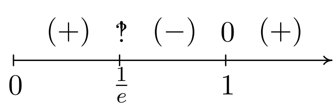
inside a logarithm needs to be discarded. As with the equations in Example 1.3.1, much can be learned from checking all of the answers in Example 1.4.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

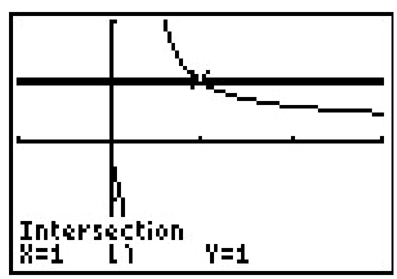
Example 1.4.2. Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1} \leq 1$ 2. $(\log_2(x))^2 < 2\log_2(x) + 3$ 3. $x\log(x+1) \geq x$

Solution.

1. We start solving $\frac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1} - 1 \leq 0$. Getting a common denominator yields $\frac{1}{\ln(x)+1} - \frac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\frac{\ln(x)}{\ln(x)+1} \geq 0$. We define $r(x) = \frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x)+1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of r is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$. To find the zeros of r, we set $r(x) = \frac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$. In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$. To determine the sign of $r\left(\frac{1}{e^2}\right)$, we use the fact that $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$, and find $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2$, which is $(+)$. The rest of the test values are determined similarly. From our sign diagram, we find the solution to be $\left(0, \frac{1}{e}\right) \cup [1, \infty)$. Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see the the graph of f is below the graph of g on the solution intervals, and that the graphs intersect at $x = 1$.

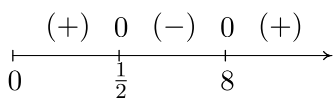


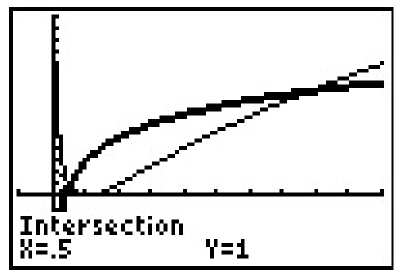


$y = f(x) = \frac{1}{\ln(x)+1}$ and $\boldsymbol{y = g(x) = 1}$

2. Moving all of the nonzero terms of $(\log_2(x))^2 < 2\log_2(x) + 3$ to one side of the inequality, we have $(\log_2(x))^2 - 2\log_2(x) - 3 < 0$. Defining $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3 = 0$ which results in a 'quadratic in disguise.' We set $u = \log_2(x)$ so our equation becomes $u^2 - 2u - 3 = 0$ which gives us $u = -1$ and $u = 3$. Since

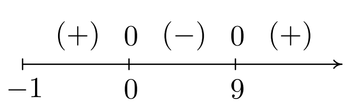
$u = \log_2(x)$, we get $\log_2(x) = -1$, which gives us $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which yields $x = 2^3 = 8$. We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$, and from our sign diagram, we see $r(x) < 0$ on $\left(\frac{1}{2}, 8\right)$. Geometrically, we see the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on the solution interval.

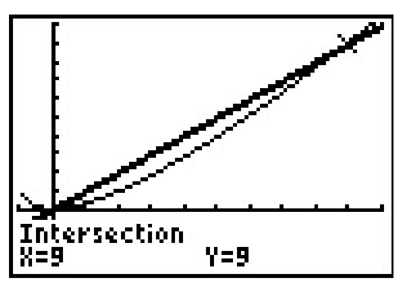




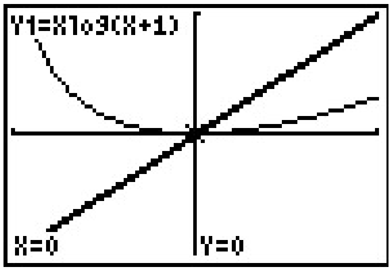
$y = f(x) = (\log_2(x))^2$ and $\boldsymbol{y = g(x) = 2\log_2(x) + 3}$

3. We begin to solve $x\log(x+1) \geq x$ by subtracting x from both sides to get $x\log(x+1) - x \geq 0$. We define $r(x) = x\log(x+1) - x$ and due to the presence of the logarithm, we require $x+1 > 0$, or $x > -1$. To find the zeros of r, we set $r(x) = x\log(x+1) - x = 0$. Factoring, we get $x\left(\log(x+1) - 1\right) = 0$, which gives $x = 0$ or $\log(x+1) - 1 = 0$. The latter gives $\log(x+1) = 1$, or $x + 1 = 10^1$, which admits $x = 9$. We select test values x so that $x + 1$ is a power of 10, and we obtain $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$. Our sign diagram gives the solution to be $(-1, 0] \cup [9, \infty)$. The calculator indicates the graph of $y = f(x) = x\log(x+1)$ is above $y = g(x) = x$ on the solution intervals, and the graphs intersect at $x = 0$ and $x = 9$.

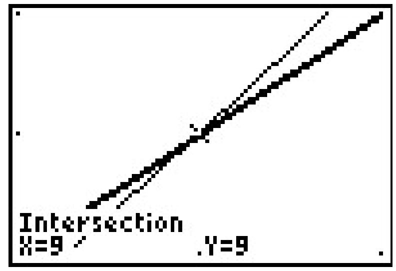




$y = f(x) = x\log(x+1)$ and $\boldsymbol{y = g(x) = x}$



Near $x = 0$



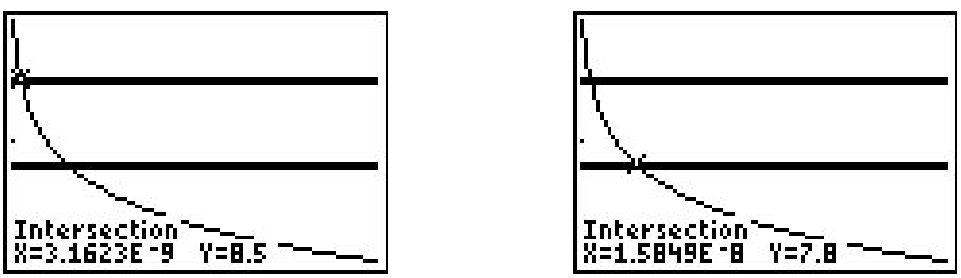
Near $x = 9$

$\square$

Our next example revisits the concept of pH as first introduced in the exercises in Section 1.1.

Example 1.4.3. In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration.

Solution. Recall from Exercise 6c in Section 1.1 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-7.8 \geq \log[\text{H}^+] \geq -8.5$. To solve this compound inequality we solve $-7.8 \geq \log[\text{H}^+]$ and $\log[\text{H}^+] \geq -8.5$ and take the intersection of the solution sets.[3] The former inequality yields $0 < [\text{H}^+] \leq 10^{-7.8}$ and the latter yields $[\text{H}^+] \geq 10^{-8.5}$. Taking the intersection gives us our final answer $10^{-8.5} \leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) After carefully adjusting the viewing window on the graphing calculator we see that the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.16 \times 10^{-9}, 1.58 \times 10^{-8}]$.



The graphs of $y = f(x) = -\log(x)$, $\boldsymbol{y = 7.8}$ and $\boldsymbol{y = 8.5}$

$\square$

We close this section by finding an inverse of a one-to-one function which involves logarithms.

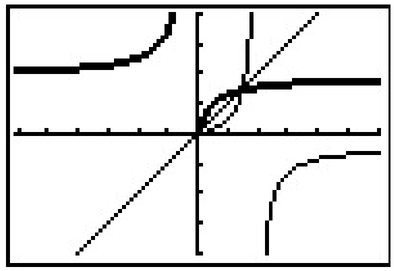
Example 1.4.4. The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

[3] Refer to page **??** for a discussion of what this means.

SOLUTION. We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{aligned}
y &= f(x) \\
y &= \frac{\log(x)}{1-\log(x)} \\
x &= \frac{\log(y)}{1-\log(y)} && \text{Interchange } x \text{ and } y. \\
x\,(1-\log(y)) &= \log(y) \\
x - x\log(y) &= \log(y) \\
x &= x\log(y) + \log(y) \\
x &= (x+1)\log(y) \\
\frac{x}{x+1} &= \log(y) \\
y &= 10^{\frac{x}{x+1}} && \text{Rewrite as an exponential equation.}
\end{aligned}$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same viewing window yields



$y = f(x) = \dfrac{\log(x)}{1 - \log(x)}$ and $\boldsymbol{y = g(x) = 10^{\frac{x}{x+1}}}$

□

1.4.1 Exercises

1. Solve the following equations analytically.

   (a) $\log_{\frac{1}{2}} x = -3$
   (b) $\ln(x^2) = (\ln(x))^2$
   (c) $\log_3(x-4) + \log_3(x+4) = 2$
   (d) $\log_5(2x+1) + \log_5(x+2) = 1$
   (e) $\log_2(x^3) = \log_2(x)$
   (f) $\log_{169}(3x+7) - \log_{169}(5x-9) = \frac{1}{2}$
   (g) $\log\left(\frac{x}{10^{-3}}\right) = 4.7$
   (h) $-\log(x) = 5.4$
   (i) $10\log\left(\frac{x}{10^{-12}}\right) = 150$
   (j) $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$
   (k) $\log_{125}\left(\frac{3x-2}{2x+3}\right) = \frac{1}{3}$
   (l) $\ln(x+1) - \ln(x) = 3$
   (m) $\ln(\ln(x)) = 3$
   (n) $2\log_7(x) = \log_7(2) + \log_7(x+12)$
   (o) $\log(x) - \log(2) = \log(x+8) - \log(x+2)$

2. Solve the following inequalities analytically.

   (a) $x\ln(x) - x > 0$
   (b) $5.6 \leq \log\left(\frac{x}{10^{-3}}\right) \leq 7.1$
   (c) $10\log\left(\frac{x}{10^{-12}}\right) \geq 90$
   (d) $2.3 < -\log(x) < 5.4$
   (e) $\frac{1-\ln(x)}{x^2} < 0$
   (f) $\ln(x^2) \leq (\ln(x))^2$

3. Use your calculator to help you solve the following equations and inequalities.

   (a) $\ln(x) = e^{-x}$
   (b) $\ln(x^2+1) \geq 5$
   (c) $\ln(x) = \sqrt[4]{x}$
   (d) $\ln(-2x^3 - x^2 + 13x - 6) < 0$

4. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x+1) < 3$ without a sign diagram. Also, compare this exercise to question 4 in Section 1.3.

5. Solve $\ln(3-y) - \ln(y) = 2x + \ln(5)$ for y.

6. In Example 1.4.4 we found the inverse of $f(x) = \frac{\log(x)}{1-\log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

   (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
   (b) Find the range of f by finding the domain of f^{-1}.
   (c) Let $g(x) = \frac{x}{1-x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise **??** in Section **??**, but it's nice to see a specific example of the property.)

7. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.

8. Explain the equation in Exercise 1g and the inequality in Exercise 2b above in terms of the Richter scale for earthquake magnitude. (See Exercise 6a in Section 1.1.)

9. Explain the equation in Exercise 1i and the inequality in Exercise 2c above in terms of sound intensity level as measured in decibels. (See Exercise 6b in Section 1.1.)

10. Explain the equation in Exercise 1h and the inequality in Exercise 2d above in terms of the pH of a solution. (See Exercise 6c in Section 1.1.)

11. With the help of your classmates, solve the inequality $\sqrt[n]{x} > \ln(x)$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = \ln(x)$ grows versus any principal n^{th} root function?

1.4.2 Answers

1. (a) $x = 8$
   (b) $x = 1$, $x = e^2$
   (c) $x = 5$
   (d) $x = \frac{1}{2}$
   (e) $x = 1$
   (f) $x = 2$
   (g) $x = 10^{1.7}$
   (h) $x = 10^{-5.4}$
   (i) $x = 10^3$
   (j) $x = 81$
   (k) $x = -\frac{17}{7}$
   (l) $x = \frac{1}{e^3-1}$
   (m) $x = e^{e^3}$
   (n) $x = 6$
   (o) $x = 4$

2. (a) (e, ∞)
   (b) $\left[10^{2.6}, 10^{4.1}\right]$
   (c) $\left[10^{-3}, \infty\right)$
   (d) $\left(10^{-5.4}, 10^{-2.3}\right)$
   (e) (e, ∞)
   (f) $(0, 1] \cup [e^2, \infty)$

3. (a) $x \approx 1.3098$
   (b) $(-\infty, -12.1414) \cup (12.1414, \infty)$
   (c) $x \approx 4.177$, $x \approx 5503.665$
   (d) $(-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$

4. $-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$

5. $y = \dfrac{3}{5e^{2x} + 1}$

7. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. (The reason for this rewriting will be explained much later in the text.) The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

1.5 Applications of Exponential and Logarithmic Functions

As we mentioned in Section 1.1, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, the calculator is often used to express our answers as decimal approximations.

1.5.1 Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have \$100 to invest at your local bank and they are offering a whopping 5 % annual percentage interest rate. This means that after one year, the bank will pay **you** 5% of that \$100, or $\$100(0.05) = \$5$ in interest, so you now have \$105.[1] This is in accordance with the formula for **simple interest** which you have undoubtedly run across at some point in your mathematical upbringing.

Equation 1.1. **Simple Interest** The amount of interest I accrued at an annual rate r on an investment[a] P after t years is

$$I = Prt$$

The amount A in the account after t years is given by

$$A = P + I = P + Prt = P(1 + rt)$$

[a]Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[2] Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial \$100 yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest. You take your \$102.50 off to the competitor and find out that those restrictions which *may* apply actually do apply to you, and you return to your bank which happily accepts your \$102.50 for the remaining six months of the year. To your surprise and delight, at the end of the year your statement reads \$105.06, not \$105 as you had expected.[3] Where did those extra six cents come from? For the first six months of the year, interest was earned on the original principal of \$100, but for the second six months, interest was earned on \$102.50, that is, you earned interest on your interest. This is the basic concept behind **compound interest**. In the previous discussion, we would say that the interest was compounded twice, or semiannually.[4] If more money can be earned by earning interest

[1] How generous of them!
[2] Some restrictions may apply.
[3] Actually, the final balance should be \$105.0625.
[4] Using this convention, simple interest after one year is the same as compounding the interest only once.

on interest already earned, a natural question to ask is what happens if the interest is compounded more often, say 4 times a year, which is every three months, or 'quarterly.' In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1 + rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the \$101.25 for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is \$103.79, and, at last, we obtain \$105.08. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound. In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let A_k denote the amount in the account after the k^{th} compounding. Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$. Since we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

EQUATION 1.2. **Compounded Interest:** If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount A in the account after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 1.2 becomes $A = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A = 100(1.0125)^{4t}$. This equation defines the amount A as an exponential function of time t, $A(t)$. To check this against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

EXAMPLE 1.5.1. Suppose \$2000 is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change[5] of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

SOLUTION.

[5] See Definition **??** in Section **??**.

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (monthly) into Equation 1.2 yields $A = 2000\left(1 + \frac{0.07125}{12}\right)^{12t}$. Using function notation, we get $A(t) = 2000(1.0059375)^{12t}$.

2. Since t represents the length of the investment, we substitute $t = 5$ into $A(t)$ to find $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately \$2852.92.

3. Our initial investment is \$2000, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$. Taking natural logs as in Section 1.3, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute $\frac{A(5)-A(4)}{5-4} \approx 195.63$. Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$. This means that the value of the investment is increasing at a rate of approximately \$195.63 per year between the end of the fourth and fifth years, while that rate jumps to \$1648.21 per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[6] □

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[7] Consider an investment of \$1 invested at 100% interest for 1 year compounded n times a year. Equation 1.2 tells us that the amount of money in the account after 1 year is $A = \left(1 + \frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing. We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so

[6] In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

[7] Once you've had a semester of Calculus, you'll be able to fully appreciate this very lame pun.

there is less time to build up additional interest. With Calculus, we can show[8] that as $n \to \infty$, $A = \left(1 + \frac{1}{n}\right)^n \to e$, where e is the natural base first presented in Section 1.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

THEOREM 1.8. If you invest \$1 at 100% interest compounded continuously, then you will have \$$e$ at the end of one year.

Using this definition of e and a little Calculus, we can take Equation 1.2 and produce a formula for continuously compounded interest.

EQUATION 1.3. **Continuously Compounded Interest:** If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount A in the account after t years is

$$A = Pe^{rt}$$

If we take the scenario of Example 1.5.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about \$24,035.28, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about \$24,213.18 - a difference of less than 1%.

Equations 1.2 and 1.3 both use exponential functions to describe the growth of an investment. Curiously enough, the same principles which govern compound interest are also used to model short term growth of populations. In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time.[9] In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Its solution is stated below.

[8] Or define, depending on your point of view.

[9] The average rate of change of a function over an interval was first introduced in Section **??**. *Instantaneous* rates of change are the business of Calculus, as is mentioned on Page **??**.

EQUATION 1.4. **Uninhibited Growth:** If a population increases according to The Law of Uninhibited Growth, the number of organisms N at time t is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)$$

It is worth taking some time to compare Equations 1.3 and 1.4. In Equation 1.3, we use P to denote the initial investment; in Equation 1.4, we use N_0 to denote the initial population. In Equation 1.3, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 1.4 corresponds to a growth rate as well. While Equations 1.3 and 1.4 look entirely different, they both represent the same mathematical concept.

EXAMPLE 1.5.2. In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, N, in thousands, after t days.

SOLUTION. We begin with $N(t) = N_0 e^{kt}$. Since N is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$. In order to complete the formula, we need to determine the growth rate k. We know that after one week, the number of cells has grown to five million. Since t measures days and the units of N are in thousands, this translates mathematically to $N(7) = 5000$. We get the equation $12e^{7k} = 5000$ which gives $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$. Hence, $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$. Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells. $\square$

Whereas Equations 1.3 and 1.4 model the growth of quantities, we can use equations like them to describe the decline of quantities. One example we've seen already is Example 1.1.1 in Section 1.1. There, the value of a car declined from its purchase price of \$25,000 to nothing at all. Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays. This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 1.4 with the exception that the rate constant k is negative.

EQUATION 1.5. **Radioactive Decay** The amount of a radioactive element A at time t is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } A(t) \text{ at time } t) = k\, A(t)$$

EXAMPLE 1.5.3. Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 1.5, and that the half-life[10] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

SOLUTION. Since we start with 5 grams initially, Equation 1.5 gives $A(t) = 5e^{kt}$. Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Hence, $A(8) = 2.5$ which means $5e^{8k} = 2.5$. Solving, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily. Hence, $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$. $\square$

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 1.1.2 of Section 1.1. In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where t was measured in minutes. In this situation, we know the physical limit of the temperature of the coffee is room temperature,[11] and the differential equation which gives rise to our formula for $T(t)$ takes this into account. Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature <u>gap</u> exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly. Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

[10] The time it takes for half of the substance to decay.

[11] The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

Equation 1.6. **Newton's Law of Cooling (Warming):** The temperature T of an object at time t is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } T(t) \text{ at time } t) = k\ (T(t) - T_a)$$

[a]That is, the temperature of the surroundings.

If we re-examine the situation in Example 1.1.2 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 1.6, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given. The rate constant $k = 0.1$ indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings. Note in Equation 1.6 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 1.5.4. A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast T as a function of its time in the oven, t, in hours.
2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution.

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming tells us $T(t) = 350 + (40 - 350)e^{-kt}$, or $T(t) = 350 - 310e^{-kt}$. To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is

$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. To determine when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. It takes roughly 3 hours and 15 minutes to cook the roast completely. □

If we had taken the time to graph $y = T(t)$ in Example 1.5.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion by applying a bit of 'number sense'. As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.[12] The function T is sometimes called a **limited** growth model, since the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate. The **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Equation 1.7. **Logistic Growth:** If a population behaves according to the assumptions of logistic growth, the number of organisms N at time t is given by the equation

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population[a], C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)\,(L - N(t))$$

[a] That is, as $t \to \infty$, $N(t) \to L$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[13]

Example 1.5.5. The number of people N, in hundreds, at a local community college who have heard the rumor 'Carl is afraid of Virginia Woolf' can be modeled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

where $t \geq 0$ is the number of days after April 1, 2009.

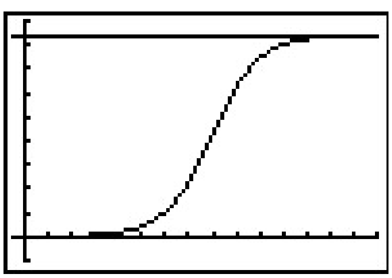
[12] at which point it would be more toast than roast.

[13] Which can be just as damaging as diseases.

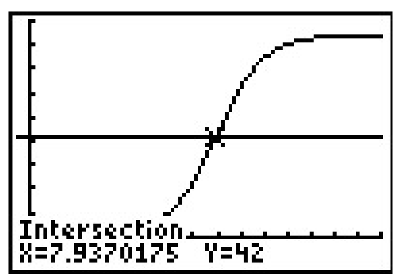
1. Find and interpret $N(0)$.
2. Find and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?
4. Check your answers to 2 and 3 using your calculator.

Solution.

1. We find $N(0) = \frac{84}{1+2799e^0} = \frac{84}{2800} = \frac{3}{100}$. Since $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2009, we may conclude that on that day, 3 people have heard the rumor.[14]
2. We could simply note that $N(t)$ is written in the form of Equation 1.7, and identify $L = 84$. However, to see why the answer is 84, we proceed analytically. Since the domain of N is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[15] as $t \to \infty$, have $1997e^{-t} \to 0^+$ and so $N(t) \approx \frac{84}{1+\text{very small } (+)} \approx 84$. Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.
3. To find how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$. It takes around 8 days until 4200 people have heard the rumor.
4. We graph $y = N(x)$ using the calculator and see that the line $y = 84$ is the horizontal asymptote of the graph, confirming our answer to part 2, and the graph intersects the line $y = 42$ at $x = \ln(2799) \approx 7.937$, which confirms our answer to part 3.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 84$



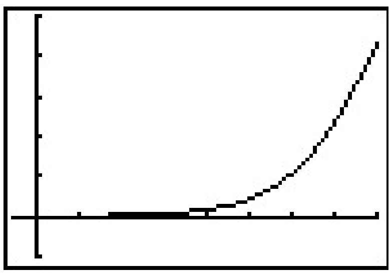
$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 42$

□

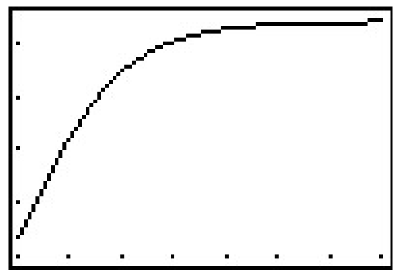
[14] Or, more likely, three people started the rumor. I'd wager Jeff, Jamie, and Jason started it. So much for telling your best friends something in confidence!

[15] See, for example, Example 1.1.2.

If we take the time to analyze the graph of $y = N(x)$ above, we can see graphically how logistic growth combines features of uninhibited and limited growth. The curve seems to rise steeply, then at some point, begins to level off. The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. At this point, even though the function is still increasing, the rate at which it does so begins to decline. It turns out the point of diminishing returns always occurs at half the limiting population. (In our case, when $y = 42$.) While these concepts are more precisely quantified using Calculus, below are two views of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth; the latter like limited growth.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $0 \leq x \leq 8$



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $8 \leq x \leq 16$

1.5.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 6a, 6b and 6c of Section 1.1, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 1.5.6. The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[16] password is comprised of letters and numbers only, find the associated information entropy.

2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution.

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.

[16]That is, upper and lower case letters are treated as different characters.

2. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[17] □

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 6c in Section 1.1.

Example 1.5.7. Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[18] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution. We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr. □

[17] Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.

[18] Derived from the Henderson-Hasselbalch Equation. See Exercise 8 in Section 1.2. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

1.5.3 Exercises

1. On May, 31, 2009, the Annual Percentage Rate listed at my bank for regular savings accounts was 0.25% compounded monthly. Use Equation 1.2 to answer the following.

   (a) If $P = 2000$ what is $A(8)$?

   (b) Solve the equation $A(t) = 4000$ for t.

   (c) What principal P should be invested so that the account balance is \$2000 is three years?

2. My bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

   (a) If $P = 2000$ what is $A(8)$?

   (b) Solve the equation $A(t) = 4000$ for t.

   (c) What principal P should be invested so that the account balance is \$2000 is three years?

   (d) The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

3. Use Equation 1.2 to show that the time it takes for an investment to double in value does not depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[19] in financial mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

4. Use Equation 1.5 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

5. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

   (a) Use Equation 1.5 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

   (b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

   (c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

   (d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

[19] Awesome pun!

6. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 1.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

7. In Example 1.1.1 in Section 1.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use the properties of logs and/or exponents to rewrite the model in the form $V(t) = 25e^{kt}$.

8. A pork roast was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F.[20] Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 1.6),

   (a) Express the temperature T as a function of time t.

   (b) Find the time at which the roast would have dropped to 140°F had it not been carved and eaten.

[20] This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

1.5.4 Answers

1. (a) $A(8) = 2000\left(1 + \frac{0.0025}{12}\right)^{12 \cdot 8} \approx \$2040.40$

   (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0025}{12}\right)} \approx 277.29$ years

   (c) $P = \dfrac{2000}{\left(1 + \frac{0.0025}{12}\right)^{36}} \approx \$1985.06$

2. (a) $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12 \cdot 8} \approx \$2394.03$

   (b) $t = \dfrac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years

   (c) $P = \dfrac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$

   (d) $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

5. (a) $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$

   (b) $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains

   (c) $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old

6. $A(t) = 2.3e^{-0.0138629t}$

7. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$

8. (a) $T(t) = 75 + 105e^{-0.005005t}$

   (b) The roast would have cooled to 140°F in about 95 minutes.