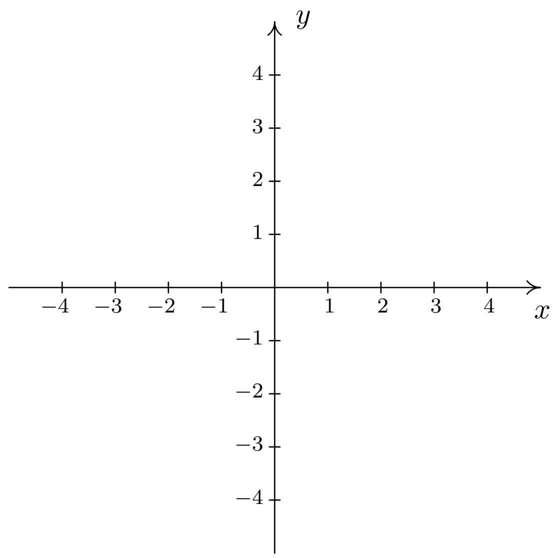
Chapter 1

Coordinates

1.1 The Cartesian Coordinate Plane

In order to visualize the pure excitement that is Algebra, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**.[1] Imagine two real number lines crossing at a right angle at 0 as below.



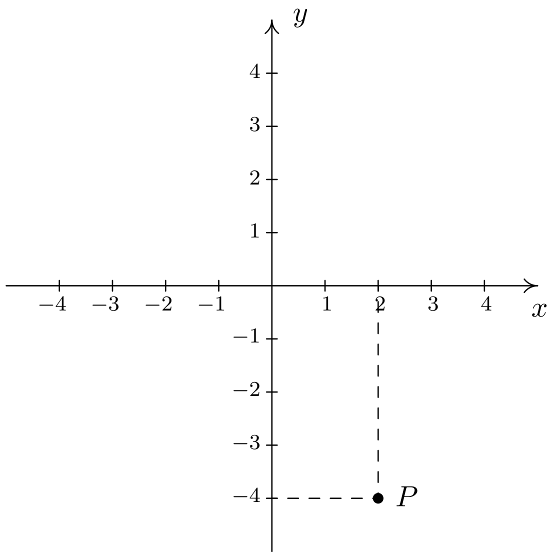
The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**.[2] As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the position of points off of the number lines as well as points on the lines themselves.

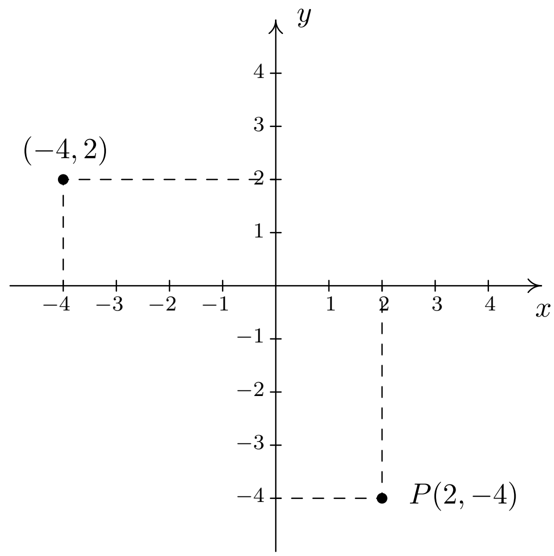
For example, consider the point P below on the left. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the x-axis to P and extending a horizontal line from the y-axis to P. We then describe the point P using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**.[3] Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point P. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach P from the origin by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important $-$ if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.

[1] So named in honor of René Descartes.

[2] The labels can vary depending on the context of application.

[3] Again, the names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the t-axis. The first number in the ordered pair would then be the t-coordinate.

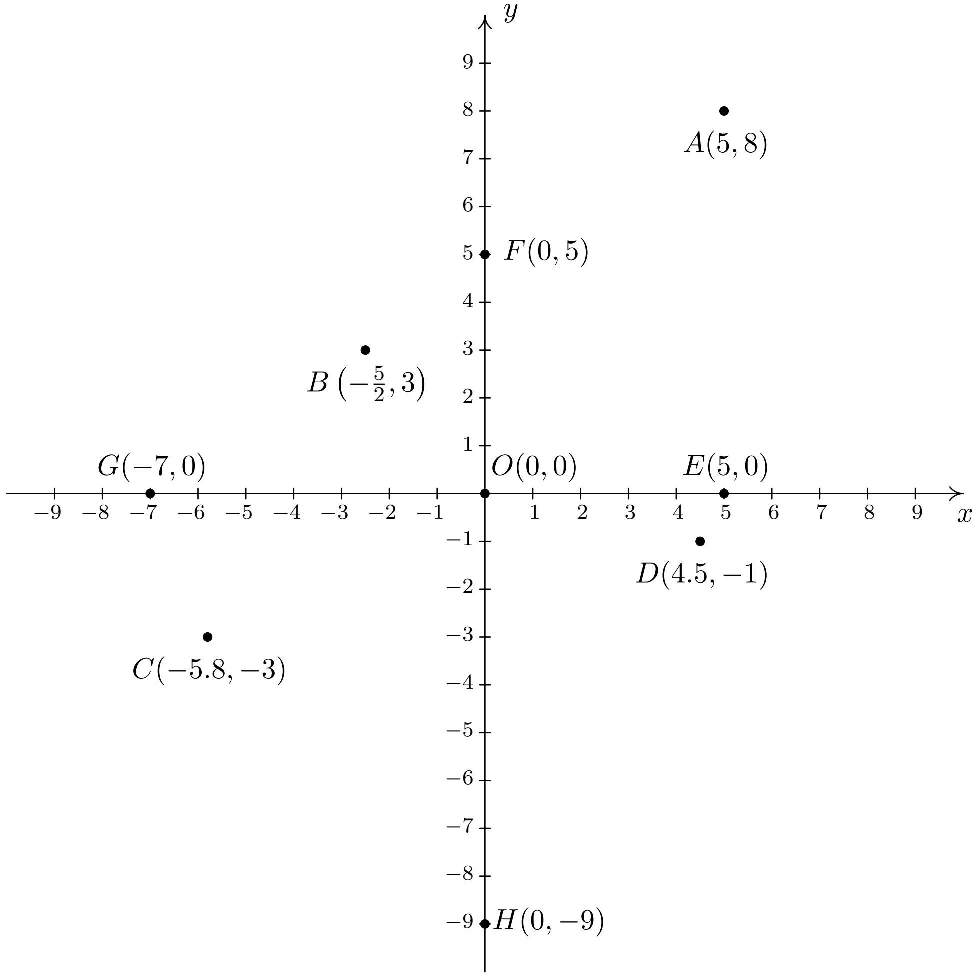




EXAMPLE 1.1.1. Plot the following points: $A(5,8)$, $B\left(-\frac{5}{2},3\right)$, $C(-5.8,-3)$, $D(4.5,-1)$, $E(5,0)$, $F(0,5)$, $G(-7,0)$, $H(0,-9)$, $O(0,0)$.[4]

SOLUTION. To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.

[4] The letter O is almost always reserved for the origin.



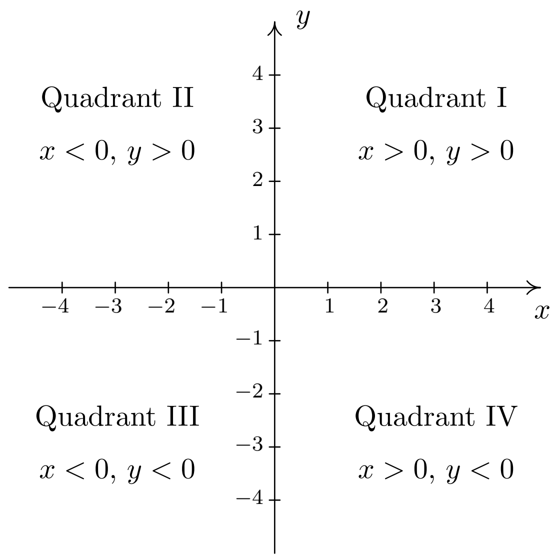
$\square$

When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

The axes divide the plane into four regions called **quadrants**. They are labeled with Roman numerals and proceed counterclockwise around the plane:



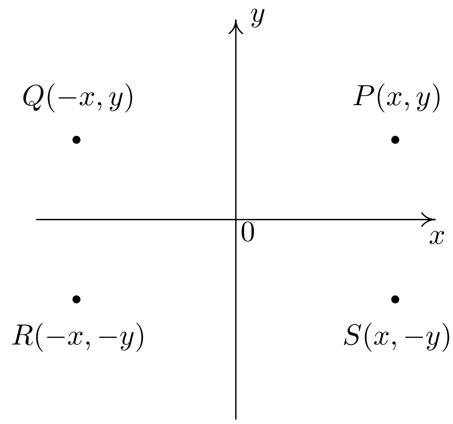
For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III, and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to the point as lying on the positive or negative x-axis (if $y = 0$) or on the positive or negative y-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive y-axis whereas $(-117, 0)$ lies on the negative x-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of mathematics is **symmetry**.There are many types of symmetry in mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 1.1. Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the $\boldsymbol{x}$-axis** if $a = c$ and $b = -d$
- **symmetric about the $\boldsymbol{y}$-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

Schematically,



In the above figure, P and S are symmetric about the x-axis, as are Q and R; P and Q are symmetric about the y-axis, as are R and S; and P and R are symmetric about the origin, as are Q and S.

Example 1.1.2. Let P be the point $(-2, 3)$. Find the points which are symmetric to P about the:

1. x-axis
2. y-axis
3. origin

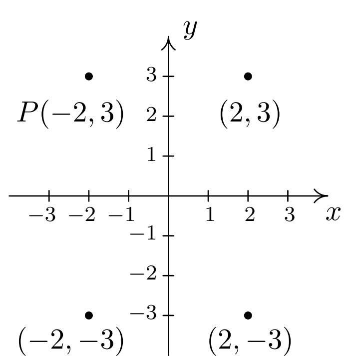
Check your answer by graphing.

Solution. The figure after Definition 1.1 gives us a good way to think about finding symmetric points in terms of taking the opposites of the x- and/or y-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the x-axis, we replace the y-coordinate with its opposite to get $(-2, -3)$.

2. To find the point symmetric about the y-axis, we replace the x-coordinate with its opposite to get $(2, 3)$.

3. To find the point symmetric about the origin, we replace the x- **and** y-coordinates with their opposites to get $(2, -3)$.



□

One way to visualize the processes in the previous example is with the concept of **reflections**. If we start with our point $(-2,3)$ and pretend the x-axis is a mirror, then the reflection of $(-2,3)$ across the x-axis would lie at $(-2,-3)$. If we pretend the y-axis is a mirror, the reflection of $(-2,3)$ across that axis would be $(2,3)$. If we reflect across the x-axis and then the y-axis, we would go from $(-2,3)$ to $(-2,-3)$ then to $(2,-3)$, and so we would end up at the point symmetric to $(-2,3)$ about the origin. We summarize and generalize this process below.

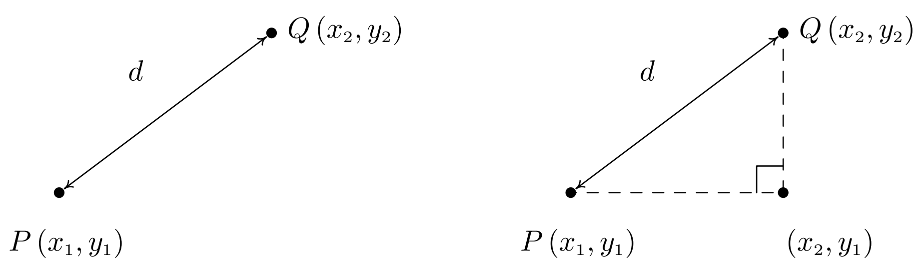
Reflections

To reflect a point (x,y) about the:

- x-axis, replace y with $-y$.
- y-axis, replace x with $-x$.
- origin, replace x with $-x$ and y with $-y$.

1.1.1 Distance in the Plane

Another important concept in geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P\left(x_1,y_1\right)$ and $Q\left(x_2,y_2\right)$, in the plane. By the **distance** d between P and Q, we mean the length of the line segment joining P with Q. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation below on the left.



With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_2-x_1|$ and $|y_2-y_1|$ so the Pythagorean Theorem gives us

$$|x_2-x_1|^2+|y_2-y_1|^2=d^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

EQUATION 1.1. **The Distance Formula:** The distance d between the points $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$ is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader to verify Equation 1.1 for these cases.

EXAMPLE 1.1.3. Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

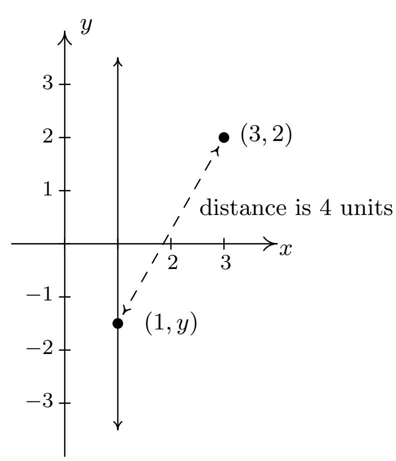
SOLUTION.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \end{aligned}$$

So, the distance is $3\sqrt{5}$. □

EXAMPLE 1.1.4. Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

SOLUTION. We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$. Visually,

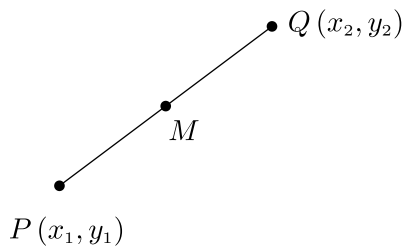


We require that the distance from $(3,2)$ to $(1,y)$ be 4. The Distance Formula, Equation 1.1, yields

$$\begin{array}{rclr}
d & = & \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} & \\
4 & = & \sqrt{(1-3)^2+(y-2)^2} & \\
4 & = & \sqrt{4+(y-2)^2} & \\
4^2 & = & \left(\sqrt{4+(y-2)^2}\right)^2 & \text{squaring both sides} \\
16 & = & 4+(y-2)^2 & \\
12 & = & (y-2)^2 & \\
(y-2)^2 & = & 12 & \\
y-2 & = & \pm\sqrt{12} & \text{extracting the square root} \\
y-2 & = & \pm 2\sqrt{3} & \\
y & = & 2\pm 2\sqrt{3} &
\end{array}$$

We obtain two answers: $(1, 2+2\sqrt{3})$ and $(1, 2-2\sqrt{3})$. The reader is encouraged to think about why there are two answers. □

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P\,(x_1,y_1)$ and $Q\,(x_2,y_2)$, the **midpoint**, M, of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.



If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

EQUATION 1.2. **The Midpoint Formula:** The midpoint M of the line segment connecting $P\,(x_1,y_1)$ and $Q\,(x_2,y_2)$ is:

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

If we let d denote the distance between P and Q, we leave it as an exercise to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Equation 1.2 gives the coordinates of the midpoint.

Example 1.1.5. Find the midpoint of the line segment connecting $P(-2,3)$ and $Q(1,-3)$.

Solution.

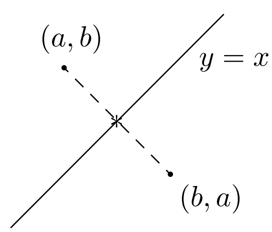
$$\begin{aligned} M &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) \\ &= \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$. □

An interesting application[5] of the midpoint formula follows.

Example 1.1.6. Prove that the points (a,b) and (b,a) are symmetric about the line $y=x$.

Solution. By 'symmetric about the line $y=x$', we mean that if a mirror were placed along the line $y=x$, the points (a,b) and (b,a) would be mirror images of one another. (You should compare and contrast this with the other types of symmetry presented back in Definition 1.1.) Schematically,



From the figure, we see that this problem amounts to showing that the midpoint of the line segment connecting (a,b) and (b,a) lies on the line $y=x$. Applying Equation 1.2 yields

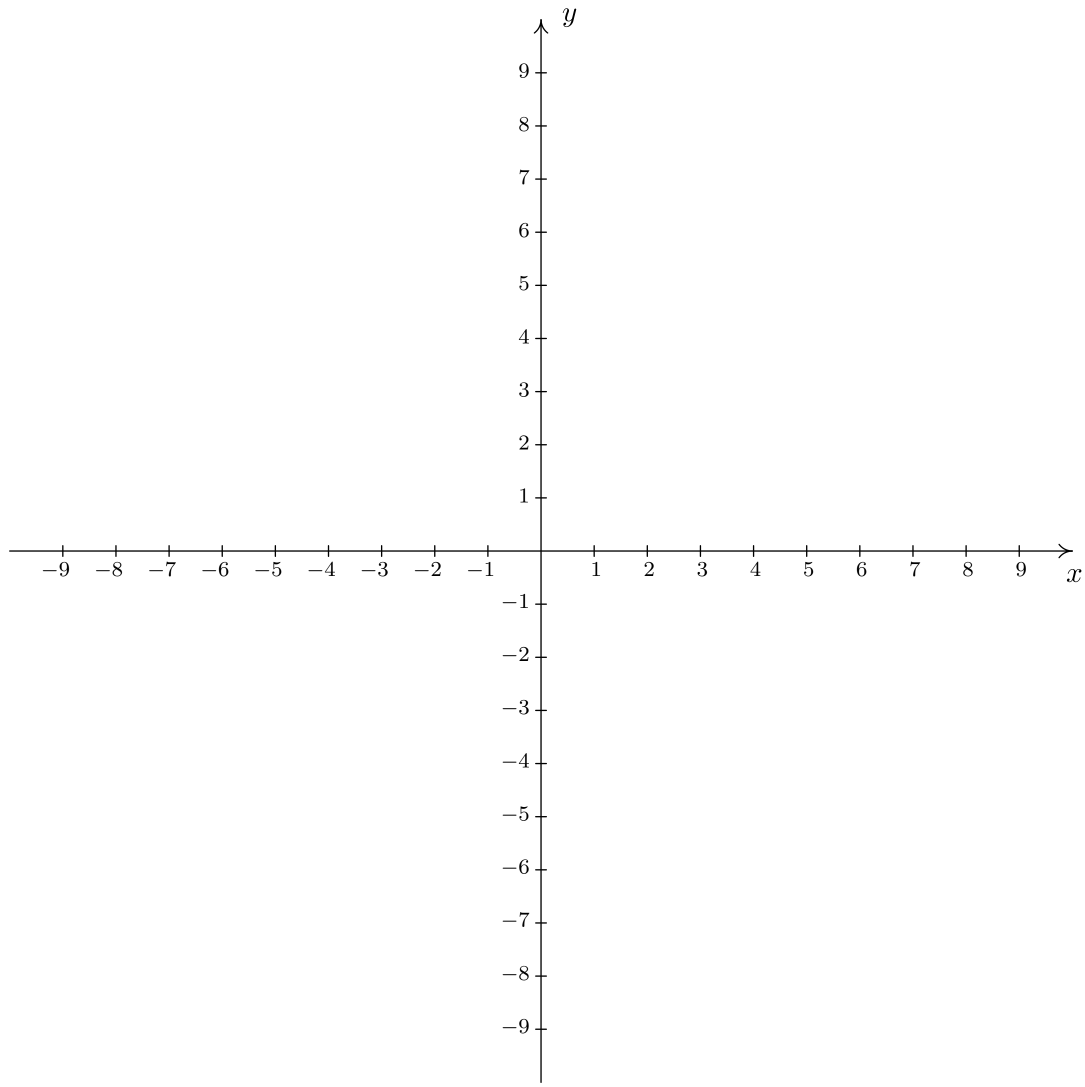
$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y=x$, as required. □

[5]This is a key concept in the development of inverse functions. See Section ??

1.1.2 Exercises

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above

   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

3. For each of the following pairs of points, find the distance d between them and find the midpoint M of the line segment connecting them.

   (a) $(1, 2)$, $(-3, 5)$
   (b) $(3, -10)$, $(-1, 2)$
   (c) $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$
   (d) $\left(-\frac{2}{3}, \frac{3}{2}\right)$, $\left(\frac{7}{3}, 2\right)$
   (e) $\left(\frac{24}{5}, \frac{6}{5}\right)$, $\left(-\frac{11}{5}, -\frac{19}{5}\right)$.
   (f) $\left(\sqrt{2}, \sqrt{3}\right)$, $\left(-\sqrt{8}, -\sqrt{12}\right)$
   (g) $\left(2\sqrt{45}, \sqrt{12}\right)$, $\left(\sqrt{20}, \sqrt{27}\right)$.
   (h) $(0, 0)$, (x, y)

4. Find all of the points of the form $(x, -1)$ which are 4 units from the point $(3, 2)$.

5. Find all of the points on the y-axis which are 5 units from the point $(-5, 3)$.

6. Find all of the points on the x-axis which are 2 units from the point $(-1, 1)$.

7. Find all of the points of the form $(x, -x)$ which are 1 unit from the origin.

8. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

9. Verify the Distance Formula 1.1 for the cases when:

   (a) The points are arranged vertically. (Hint: Use $P(a, y_1)$ and $Q(a, y_2)$.)
   (b) The points are arranged horizontally. (Hint: Use $P(x_1, b)$ and $Q(x_2, b)$.)
   (c) The points are actually the same point. (You shouldn't need a hint for this one.)

10. Verify the Midpoint Formula by showing the distance between $P(x_1, y_1)$ and M and the distance between M and $Q(x_2, y_2)$ are both half of the distance between P and Q.

11. Show that the points A, B and C below are the vertices of a right triangle.

    (a) $A(-3, 2)$, $B(-6, 4)$, and $C(1, 8)$
    (b) $A(-3, 1)$, $B(4, 0)$ and $C(0, -3)$

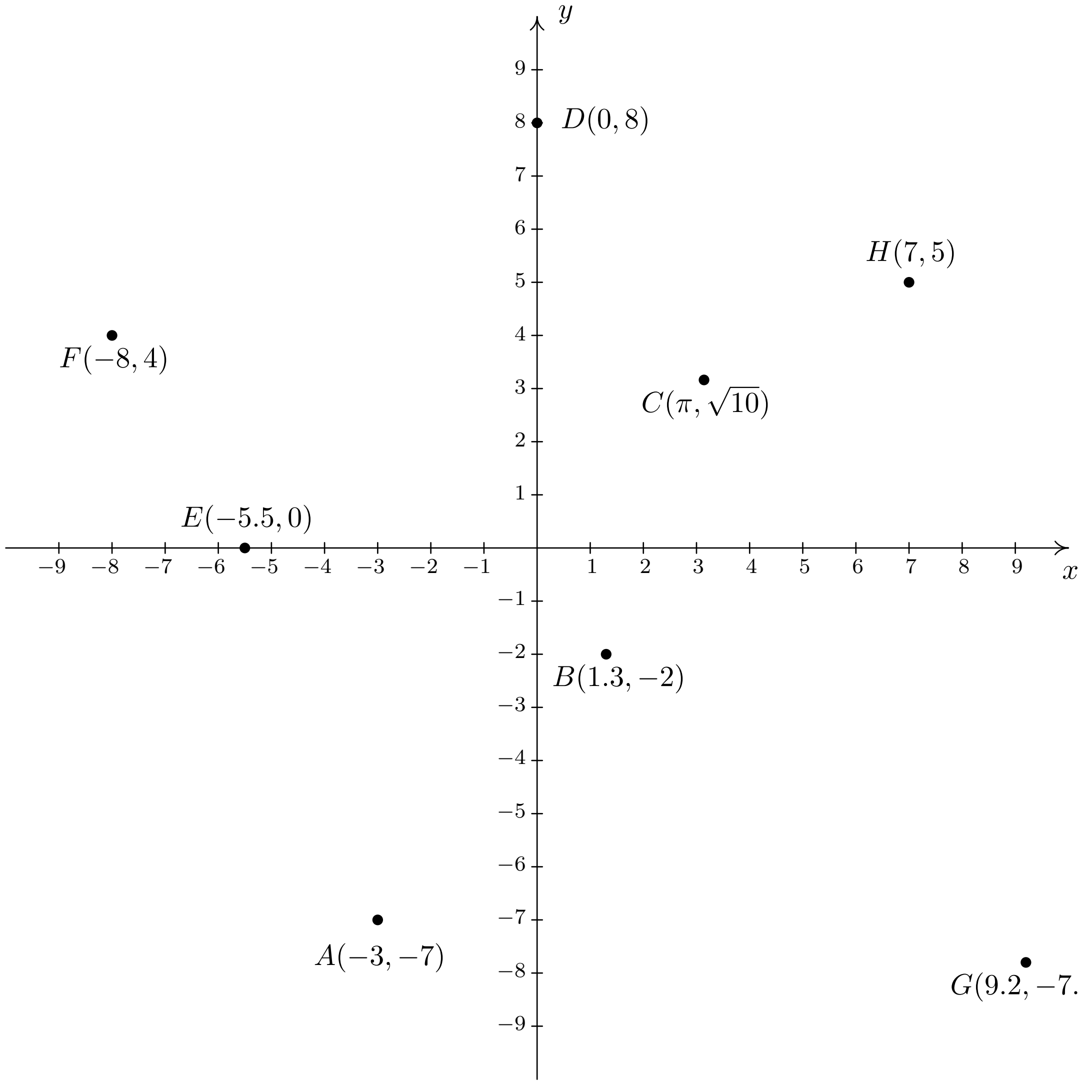
12. Find a point $D(x, y)$ such that the points $A(-3, 1)$, $B(4, 0)$, $C(0, -3)$ and D are the corners of a square. Justify your answer.

13. The world is not flat.[6] Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

[6] There are those who disagree with this statement. Look them up on the Internet some time when you're bored.

1.1.3 Answers

1. The required points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$, and $H(7,5)$ are plotted in the Cartesian Coordinate Plane below.



2. (a) The point $A(-3,-7)$ is
   - in Quadrant III
   - symmetric about x-axis with $(-3,7)$
   - symmetric about y-axis with $(3,-7)$
   - symmetric about origin with $(3,7)$

   (b) The point $B(1.3,-2)$ is
   - in Quadrant IV
   - symmetric about x-axis with $(1.3,2)$
   - symmetric about y-axis with $(-1.3,-2)$
   - symmetric about origin with $(-1.3,2)$

   (c) The point $C(\pi,\sqrt{10})$ is
   - in Quadrant I
   - symmetric about x-axis with $(\pi,-\sqrt{10})$
   - symmetric about y-axis with $(-\pi,\sqrt{10})$

- symmetric about origin with $(-\pi, -\sqrt{10})$

(d) The point $D(0, 8)$ is
- on the positive y-axis
- symmetric about x-axis with $(0, -8)$
- symmetric about y-axis with $(0, 8)$
- symmetric about origin with $(0, -8)$

(e) The point $E(-5.5, 0)$ is
- on the negative x-axis
- symmetric about x-axis with $(-5.5, 0)$
- symmetric about y-axis with $(5.5, 0)$
- symmetric about origin with $(5.5, 0)$

(f) The point $F(-8, 4)$ is
- in Quadrant II
- symmetric about x-axis with $(-8, -4)$
- symmetric about y-axis with $(8, 4)$
- symmetric about origin with $(8, -4)$

(g) The point $G(9.2, -7.8)$ is
- in Quadrant IV
- symmetric about x-axis with $(9.2, 7.8)$
- symmetric about y-axis with $(-9.2, -7.8)$
- symmetric about origin with $(-9.2, 7.8)$

(h) The point $H(7, 5)$ is
- in Quadrant I
- symmetric about x-axis with $(7, -5)$
- symmetric about y-axis with $(-7, 5)$
- symmetric about origin with $(-7, -5)$

3. (a) $d = 5$, $M = \left(-1, \frac{7}{2}\right)$

(b) $d = 4\sqrt{10}$, $M = (1, -4)$

(c) $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$

(d) $d = \frac{\sqrt{37}}{2}$, $M = \left(\frac{5}{6}, \frac{7}{4}\right)$

(e) $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$.

(f) $d = 3\sqrt{5}$, $M = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\right)$

(g) $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$.

(h) $d = \sqrt{x^2 + y^2}$, $M = \left(\frac{x}{2}, \frac{y}{2}\right)$

4. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$

5. $(0, 3)$

6. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$

7. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

8. $(-3, -4)$, 5 miles, $(4, -4)$

10. (a) The distance from A to B is $\sqrt{13}$, the distance from A to C is $\sqrt{52}$, and the distance from B to C is $\sqrt{65}$. Since

$$\left(\sqrt{13}\right)^2 + \left(\sqrt{52}\right)^2 = \left(\sqrt{65}\right)^2,$$

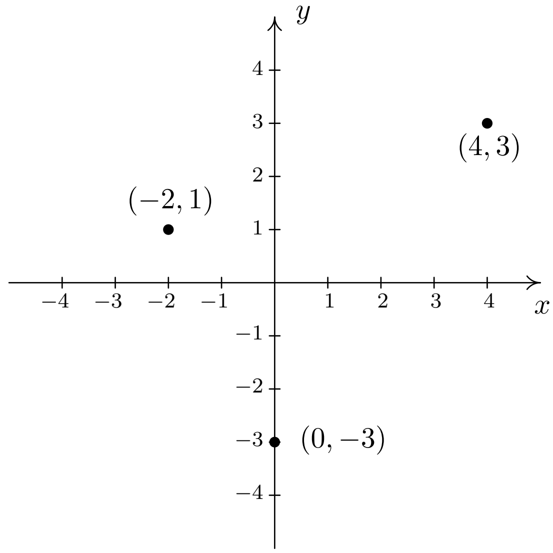
we are guaranteed by the converse of the Pythagorean Theorem that the triangle is right.

1.2 Relations

We now turn our attention to sets of points in the plane.

Definition 1.2. A **relation** is a set of points in the plane.

Throughout this text we will see many different ways to describe relations. In this section we will focus our attention on describing relations graphically, by means of the list (or roster) method and algebraically. Depending on the situation, one method may be easier or more convenient to use than another. Consider the set of points below



These three points constitute a relation. Let us call this relation R. Above, we have a **graphical** description of R. Although it is quite pleasing to the eye, it isn't the most portable way to describe R. The **list** (or **roster**) method of describing R simply lists all of the points which belong to R. Hence, we write: $R = \{(-2, 1), (4, 3), (0, -3)\}$.[1] The roster method can be extended to describe infinitely many points, as the next example illustrates.

Example 1.2.1. Graph the following relations.

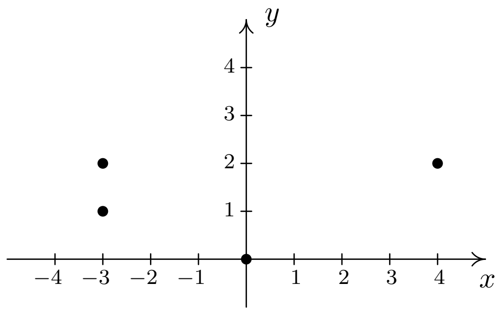
1. $A = \{(0, 0), (-3, 1), (4, 2), (-3, 2)\}$
2. $HLS_1 = \{(x, 3) : -2 \leq x \leq 4\}$
3. $HLS_2 = \{(x, 3) : -2 \leq x < 4\}$
4. $V = \{(3, y) : y \text{ is a real number}\}$

Solution.

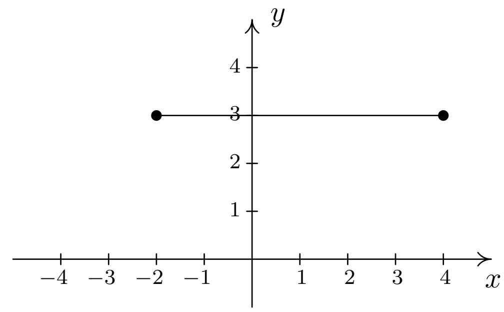
1. To graph A, we simply plot all of the points which belong to A, as shown below on the left.
2. Don't let the notation in this part fool you. The name of this relation is HLS_1, just like the name of the relation in part 1 was R. The letters and numbers are just part of its name, just

[1]We use 'set braces' {} to indicate that the points in the list all belong to the same set, in this case, R.

like the numbers and letters of the phrase 'King George III' were part of George's name. The next hurdle to overcome is the description of HLS_1 itself − a variable and some seemingly extraneous punctuation have found their way into our nice little roster notation! The way to make sense of the construction $\{(x, 3) : -2 \leq x \leq 4\}$ is to verbalize the set braces $\{\}$ as 'the set of' and the colon $:$ as 'such that'. In words, $\{(x, 3) : -2 \leq x \leq 4\}$ is: 'the set of points $(x, 3)$ such that $-2 \leq x \leq 4$.' The purpose of the variable x in this case is to describe infinitely many points. All of these points have the same y-coordinate, 3, but the x-coordinate is allowed to vary between -2 and 4, inclusive. Some of the points which belong to HLS_1 include some friendly points like: $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$, $(2, 3)$, $(3, 3)$, and $(4, 3)$. However, HLS_1 also contains the points $(0.829, 3)$, $\left(-\frac{5}{6}, 3\right)$, $(\sqrt{\pi}, 3)$, and so on. It is impossible to list all of these points, which is why the variable x is used. Plotting several friendly representative points should convince you that HLS_1 describes the horizontal line segment from the point $(-2, 3)$ up to and including the point $(4, 3)$.

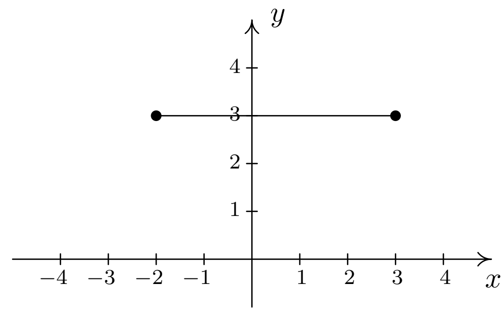


The graph of A

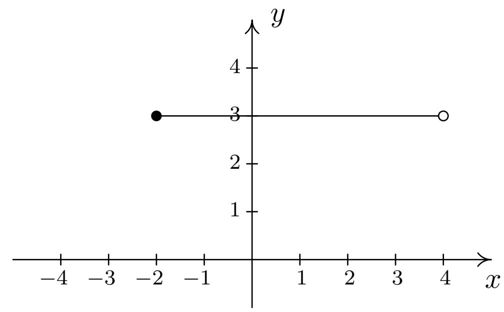


The graph of HLS_1

3. HLS_2 is hauntingly similar to HLS_1. In fact, the only difference between the two is that instead of '$-2 \leq x \leq 4$' we have '$-2 \leq x < 4$'. This means that we still get a horizontal line segment which includes $(-2, 3)$ and extends to $(4, 3)$, **but does not include** $(4, 3)$ because of the strict inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2, 3)$ end at $(3, 3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1, 3)$, $(3.5, 3)$, $(3.9, 3)$, $(3.99, 3)$, and so forth. There is no real number that comes 'immediately before' 4, and so to describe the set of points we want, we draw the horizontal line segment starting at $(-2, 3)$ and draw an 'open circle' at $(4, 3)$ as depicted below on the right.

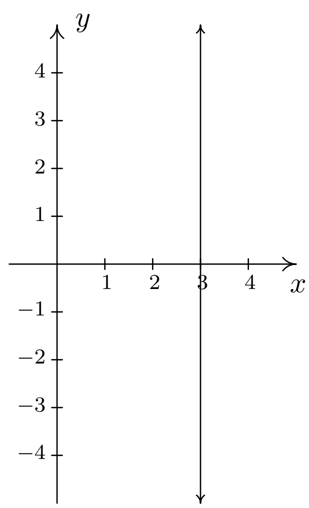


This is NOT the correct graph of HLS_2

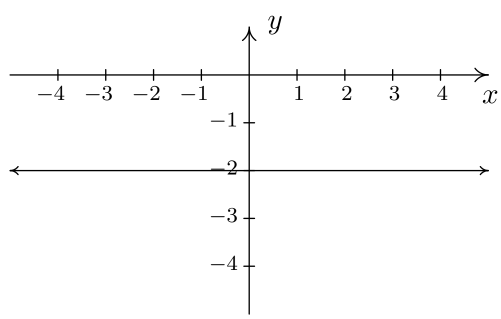


The graph of HLS_2

4. Our last example, V, describes the set of points $(3, y)$ such that y is a real number. All of these points have an x-coordinate of 3, but the y-coordinate is free to be whatever it wants to be, without restriction. Plotting a few 'friendly' points of V should convince you that all the points of V lie on a vertical line which crosses the x-axis at $x = 3$. Since there is no restriction on the y-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of V is below on the left.



The graph of V



The graph of $y = -2$

□

The relation V in the previous example leads us to our final way to describe relations: **algebraically**. We can simply describe the points in V as those points which satisfy the equation $x = 3$. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for x and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose x-coordinate is 3. Similarly, the equation $y = -2$ in this context corresponds to all points in the plane whose y-coordinate is -2. Since there are no restrictions on the x-coordinate listed, we would graph the relation $y = -2$ as the horizontal line above on the right. In general, we have the following.

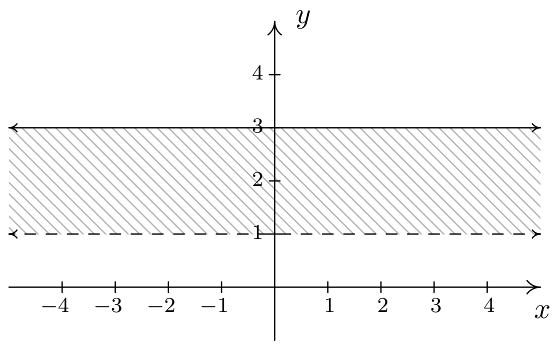
Equations of Vertical and Horizontal Lines

- The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
- The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

In the next section, and in many more after that, we shall explore the graphs of equations in great detail.[2] For now, we shall use our final example to illustrate how relations can be used to describe entire regions in the plane.

Example 1.2.2. Graph the relation: $R = \{(x, y) : 1 < y \leq 3\}$

Solution. The relation R consists of those points whose y-coordinate only is restricted between 1 and 3 excluding 1, but including 3. The x-coordinate is free to be whatever we like. After plotting some[3] friendly elements of R, it should become clear that R consists of the region between the horizontal lines $y = 1$ and $y = 3$. Since R requires that the y-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to R. Graphically,



The graph of R

□

1.2.1 Exercises

1. Graph the following relations.

   (a) $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$
   (b) $\{(-2, 2),\ (-2, -1),\ (3, 5),\ (3, -4)\}$
   (c) $\left\{\left(n, 4 - n^2\right) : n = 0, \pm 1, \pm 2\right\}$
   (d) $\left\{\left(\frac{6}{k}, k\right) : k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}$

2. Graph the following relations.

[2] In fact, much of our time in College Algebra will be spent examining the graphs of equations.
[3] The word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern.

(a) $\{(x, -2) : x > -4\}$

(b) $\{(2, y) : y \leq 5\}$

(c) $\{(-2, y) : -3 < y < 4\}$

(d) $\{(x, y) : x \leq 3\}$

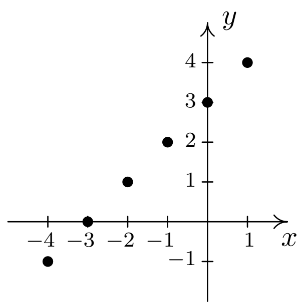
(e) $\{(x, y) : y < 4\}$

(f) $\{(x, y) : x \leq 3,\ y < 2\}$

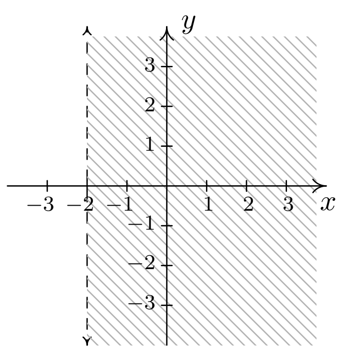
(g) $\{(x, y) : x > 0,\ y < 4\}$

(h) $\{(x, y) : -\sqrt{2} \leq x \leq \frac{2}{3},\ \pi < y \leq \frac{9}{2}\}$

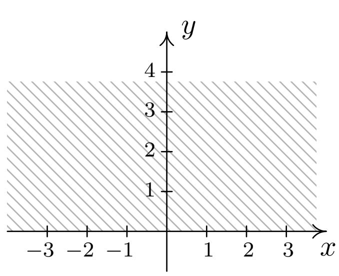
3. Describe the following relations using the roster method.



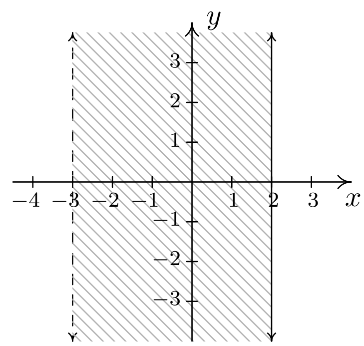
(a) The graph of relation A



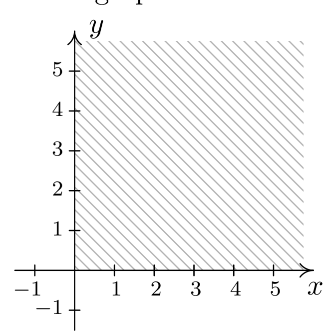
(b) The graph of relation B



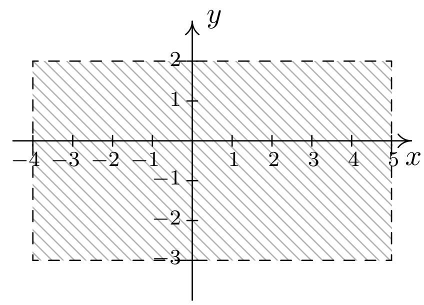
(c) The graph of relation C



(d) The graph of relation D



(e) The graph of relation E



(f) The graph of relation F

4. Graph the following lines.

(a) $x = -2$

(b) $y = 3$

5. What is another name for the line $x = 0$? For $y = 0$?

6. Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the following relations. Please note that in the notation below we are using the **ellipsis**, $\ldots$, to denote that the list does not end, but rather, continues to follow the established pattern indefinitely. For the first two relations, give two examples of points which belong to the relation and two points which do not belong to the relation.

   (a) $\{(x, y) : x$ is an odd integer, and y is an even integer.$\}$

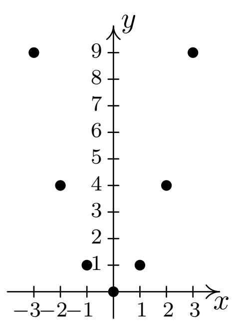
   (b) $\{(x, 1) : x$ is an irrational number $\}$

   (c) $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$

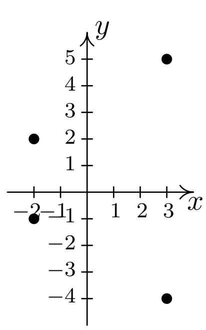
   (d) $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

1.2.2 Answers

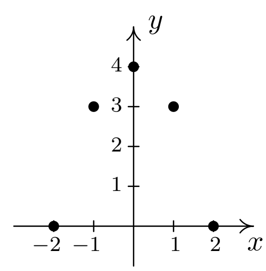
1. (a)



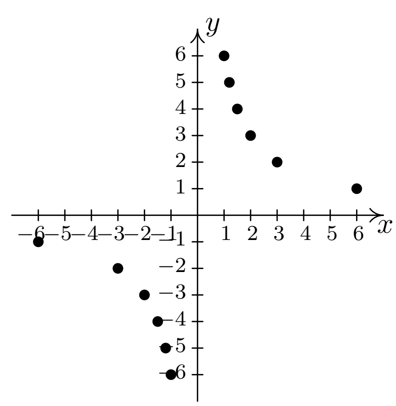
(b)



(c)

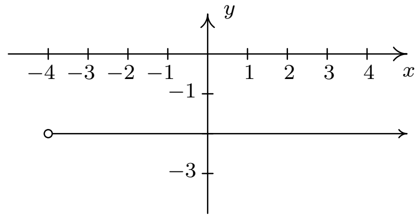


(d)

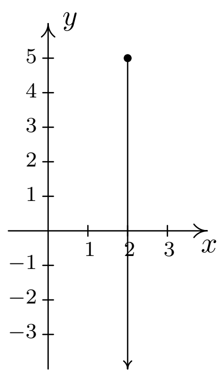


2.

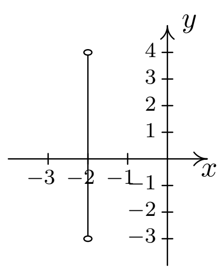
(a)



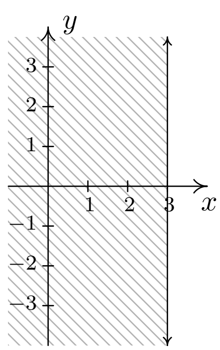
(b)



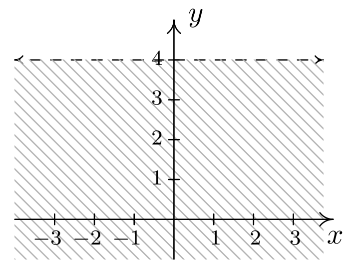
(c)



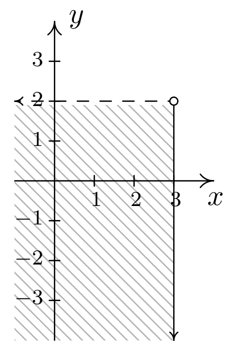
(d)



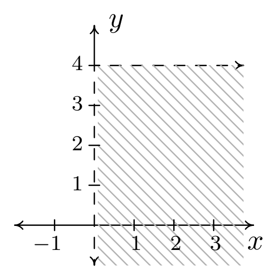
(e)



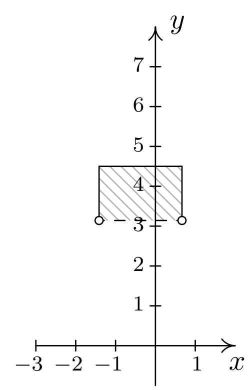
(f)



(g)



(h)



3. (a) $A = \{(-4,-1), (-3,0), (-2,1), (-1,2), (0,3), (1,4)\}$

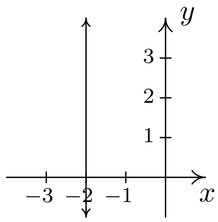
(b) $B = \{(x, y) : x > -2\}$

(c) $C = \{(x, y) : y \geq 0\}$

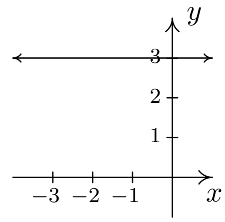
(d) $D = \{(x, y) : -3 < x \leq 2\}$

(e) $E = \{(x, y) : x \geq 0, y \geq 0\}$

(f) $F = \{(x, y) : -4 < x < 5,\ -3 < y < 2\}$



4. (a) The line $x = -2$



(b) The line $y = 3$

5. The line $x = 0$ is the y-axis and the line $y = 0$ is the x-axis.

1.3 Graphs of Equations

In the previous section, we said that an equation in x and y determines a relation.[1] In this section, we begin to explore this topic in greater detail. The main idea of this section is

> **The Fundamental Graphing Principle**
> The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Example 1.3.1. Determine if $(2, -1)$ is on the graph of $x^2 + y^3 = 1$.

Solution. To check, we substitute $x = 2$ and $y = -1$ into the equation and see if the equation is satisfied

$$\begin{aligned} (2)^2 + (-1)^3 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

[1] An inequalities in x and y also determines a relation.

EXAMPLE 1.3.2. Graph $x^2 + y^3 = 1$.

SOLUTION. To efficiently generate points on the graph of this equation, we first solve for y

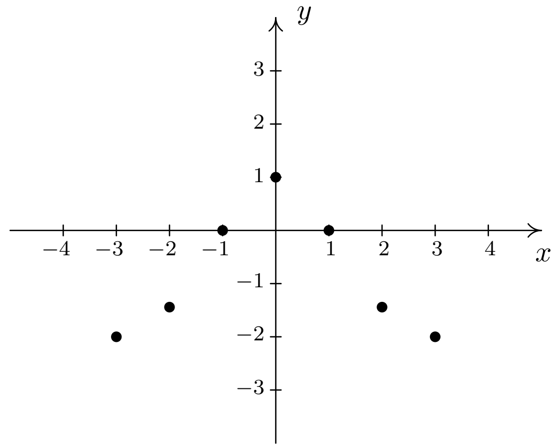
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for x, determine the corresponding value y, and plot the resulting point, (x, y). For example, for $x = -3$, we substitute

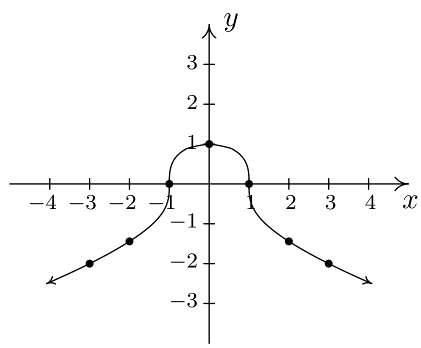
$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown below.

x	y	(x, y)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$



Remember, these points constitute only a small **sampling** of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots.' Doing so would result in a curve similar to the one pictured below.



Don't worry if you don't get all of the little bends and curves just right − Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can reach for a graphing calculator, input the formula as shown above, and graph.[2] □

Of all of the points on the graph of an equation, the places where the graph crosses the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

DEFINITION 1.3. Suppose the graph of an equation is given.

- A point at which a graph meets the y-axis is called an ***y*-intercept** of the graph.

In our previous example the graph had two x-intercepts, $(-1, 0)$ and $(1, 0)$, and one y-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all! Since x-intercepts lie on the x-axis, we can find them by setting $y = 0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

Steps for finding the intercepts of the graph of an equation

Given an equation involving x and y:

- the x-intercepts always have the form $(x, 0)$; to find the x-intercepts of the graph, set $y = 0$ and solve for x.
- y-intercepts always have the form $(0, y)$; to find the y-intercepts of the graph, set $x = 0$ and solve for y.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the y-axis. To actually prove this analytically, we assume (x, y) is a generic point on the graph of the equation. That is, we assume $x^2 + y^3 = 1$. As we learned in Section 1.1, the point symmetric to (x, y) about the y-axis is $(-x, y)$. To show the graph is symmetric about the y-axis, we need to show that $(-x, y)$ is on the graph whenever (x, y) is. In

[2]Remember: At UW we don't allow calculators on exams. But using them intelligently outside of class can be a great benefit.

other words, we need to show $(-x, y)$ satisfies the equation $x^2 + y^3 = 1$ whenever (x, y) does. Substituting gives

$$\begin{aligned} (-x)^2 + (y)^3 &\stackrel{?}{=} 1 \\ x^2 + y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

When we substituted $(-x, y)$ into the equation $x^2 + y^3 = 1$, we obtained the original equation back when we simplified. This means $(-x, y)$ satisfies the equation and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.1. The results are summarized below.

Steps for testing if the graph of an equation possesses symmetry

To test the graph of an equation for symmetry

- About the y-axis: Substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- About the x-axis: Substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- About the origin: Substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as evidenced in the next example.

EXAMPLE 1.3.3. Find the x- and y-intercepts (if any) of the graph of $(x-2)^2 + y^2 = 1$. Test for symmetry. Plot additional points as needed to complete the graph.

SOLUTION. To look for x-intercepts, we set $y = 0$ and solve:

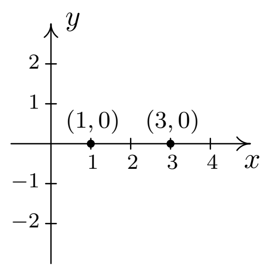
$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (x-2)^2 + 0^2 &= 1 \\ (x-2)^2 &= 1 \\ \sqrt{(x-2)^2} &= \sqrt{1} \qquad \text{extract square roots} \\ x - 2 &= \pm 1 \\ x &= 2 \pm 1 \\ x &= 3, 1 \end{aligned}$$

We get **two** answers for x which correspond to **two** x-intercepts: $(1, 0)$ and $(3, 0)$. Turning our attention to y-intercepts, we set $x = 0$ and solve:

$$\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(0-2)^2 + y^2 &= 1 \\
4 + y^2 &= 1 \\
y^2 &= -3
\end{aligned}$$

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has **no** y-intercepts.

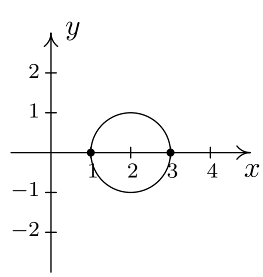
Plotting the data we have so far, we get



Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the y-axis or the origin. If the graph possessed either of these symmetries, then the fact that $(1, 0)$ is on the graph would mean $(-1, 0)$ would have to be on the graph. (Why?) Since $(-1, 0)$ would be another x-intercept (and we've found all of these), the graph can't have y-axis or origin symmetry. The only symmetry left to test is symmetry about the x-axis. To that end, we substitute $(x, -y)$ into the equation and simplify

$$\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(x-2)^2 + (-y)^2 &\stackrel{?}{=} 1 \\
(x-2)^2 + y^2 &\stackrel{\checkmark}{=} 1
\end{aligned}$$

Since we have obtained our original equation, we know the graph is symmetric about the x-axis. This means we can cut our 'plug and plot' time in half: whatever happens below the x-axis is reflected above the x-axis, and vice-versa. Proceeding as we did in the previous example, we obtain



$\square$

A couple of remarks are in order. First, it is entirely possible to choose a value for x which does not correspond to a point on the graph. For example, in the previous example, if we solve for y as is our custom, we get:

$$y = \pm\sqrt{1-(x-2)^2}.$$

Upon substituting $x = 0$ into the equation, we would obtain

$$y = \pm\sqrt{1-(0-2)^2} = \pm\sqrt{1-4} = \pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an x-coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book developing techniques which allow us to graph entire families of equations quickly.[3] Second, it is instructive to show what would have happened had we tested the equation in the last example for symmetry about the y-axis. Substituting $(-x, y)$ into the equation yields

$$\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(-x-2)^2 + y^2 &\stackrel{?}{=} 1 \\
((-1)(x+2))^2 + y^2 &\stackrel{?}{=} 1 \\
(x+2)^2 + y^2 &\stackrel{?}{=} 1.
\end{aligned}$$

This last equation does not **appear** to be equivalent to our original equation. However, to **prove** it is not symmetric about the y-axis, we need to find a point (x, y) on the graph whose reflection $(-x, y)$ is not. Our x-intercept $(1, 0)$ fits this bill nicely, since if we substitute $(-1, 0)$ into the equation we get

$$\begin{aligned}
(x-2)^2 + y^2 &\stackrel{?}{=} 1 \\
(-1-2)^2 + 0^2 &\neq 1 \\
9 &\neq 1.
\end{aligned}$$

This proves that $(-1, 0)$ is not on the graph.

1.3.1 Exercises

1. For each equation given below

   - Find the x- and y-intercept(s) of the graph, if any exist.
   - Following the procedure in Example 1.3.2, create a table of sample points on the graph of the equation.
   - Plot the sample points and create a rough sketch of the graph of the equation.
   - Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 1.3.3

[3] Without the use of a calculator, if you can believe it!

(a) $y = x^2 + 1$
(b) $y = x^2 - 2x - 8$
(c) $y = x^3 - x$
(d) $y = \frac{x^3}{4} - 3x$
(e) $y = \sqrt{x-2}$
(f) $y = 2\sqrt{x+4} - 2$
(g) $3x - y = 7$
(h) $3x - 2y = 10$
(i) $(x+2)^2 + y^2 = 16$
(j) $x^2 - y^2 = 1$
(k) $4y^2 - 9x^2 = 36$
(l) $x^3y = -4$

2. The procedures which we have outlined in the Examples of this section and used in the exercises given above all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing these equations. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here.

(a) $x^3 + y^3 - 3xy = 0$ Folium of Descartes
(b) $x^4 = x^2 + y^2$ Kampyle of Eudoxus
(c) $y^2 = x^3 + 3x^2$ Tschirnhausen cubic
(d) $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg

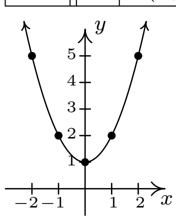
1.3.2 Answers

1. (a) $y = x^2 + 1$

The graph has no x-intercepts

y-intercept: $(0, 1)$

x	y	(x, y)
-2	5	$(-2, 5)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	5	$(2, 5)$



The graph is not symmetric about the x-axis (e.g. $(2, 5)$ is on the graph but $(2, -5)$ is not)

The graph is symmetric about the y-axis

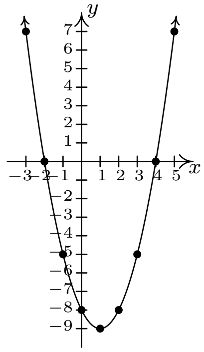
The graph is not symmetric about the origin (e.g. $(2, 5)$ is on the graph but $(-2, -5)$ is not)

(b) $y = x^2 - 2x - 8$

x-intercepts: $(4, 0)$, $(-2, 0)$

y-intercept: $(0, -8)$

x	y	(x, y)
-3	7	$(-3, 7)$
-2	0	$(-2, 0)$
-1	-5	$(-1, -5)$
0	-8	$(0, -8)$
1	-9	$(1, -9)$
2	-8	$(2, -8)$
3	-5	$(3, -5)$
4	0	$(4, 0)$
5	7	$(5, 7)$



The graph is not symmetric about the x-axis (e.g. $(-3, 7)$ is on the graph but $(-3, -7)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-3, 7)$ is on the graph but $(3, 7)$ is not)

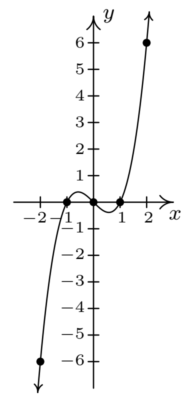
The graph is not symmetric about the origin (e.g. $(-3, 7)$ is on the graph but $(3, -7)$ is not)

(c) $y = x^3 - x$

x-intercepts: $(-1, 0), (0, 0), (1, 0)$

y-intercept: $(0, 0)$

x	y	(x, y)
-2	-6	$(-2, -6)$
-1	0	$(-1, 0)$
0	0	$(0, 0)$
1	0	$(1, 0)$
2	6	$(2, 6)$



The graph is not symmetric about the x-axis. (e.g. $(2, 6)$ is on the graph but $(2, -6)$ is not)

The graph is not symmetric about the y-axis. (e.g. $(2, 6)$ is on the graph but $(-2, 6)$ is not)

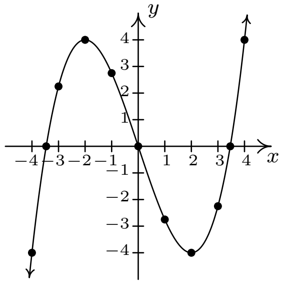
The graph is symmetric about the origin.

(d) $y = \frac{x^3}{4} - 3x$

x-intercepts: $\left(\pm 2\sqrt{3}, 0\right)$

y-intercept: $(0, 0)$

x	y	(x,y)
-4	-4	$(-4,-4)$
-3	$\frac{9}{4}$	$\left(-3,\frac{9}{4}\right)$
-2	4	$(-2,4)$
-1	$\frac{11}{4}$	$\left(-1,\frac{11}{4}\right)$
0	0	$(0,0)$
1	$-\frac{11}{4}$	$\left(1,-\frac{11}{4}\right)$
2	-4	$(2,-4)$
3	$-\frac{9}{4}$	$\left(3,-\frac{9}{4}\right)$
4	4	$(4,4)$



The graph is not symmetric about the x-axis (e.g. $(-4,-4)$ is on the graph but $(-4,4)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-4,-4)$ is on the graph but $(4,-4)$ is not)

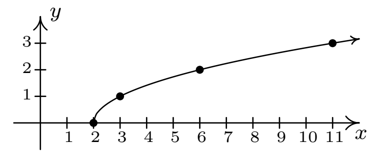
The graph is symmetric about the origin

(e) $y=\sqrt{x-2}$

x-intercept: $(2,0)$

The graph has no y-intercepts

x	y	(x,y)
2	0	$(2,0)$
3	1	$(3,1)$
6	2	$(6,2)$
11	3	$(11,3)$



The graph is not symmetric about the x-axis (e.g. $(3,1)$ is on the graph but $(3,-1)$ is not)

The graph is not symmetric about the y-axis (e.g. $(3,1)$ is on the graph but $(-3,1)$ is not)

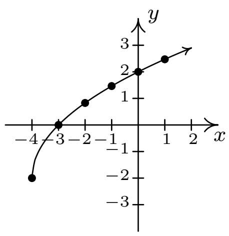
The graph is not symmetric about the origin (e.g. $(3,1)$ is on the graph but $(-3,-1)$ is not)

(f) $y=2\sqrt{x+4}-2$

x-intercept: $(-3,0)$

y-intercept: $(0,2)$

x	y	(x, y)
-4	-2	$(-4, -2)$
-3	0	$(-3, 0)$
-2	$2\sqrt{2}-2$	$\left(-2, \sqrt{2}-2\right)$
-1	$2\sqrt{3}-2$	$\left(-2, \sqrt{3}-2\right)$
0	2	$(0, 2)$
1	$2\sqrt{5}-2$	$\left(-2, \sqrt{5}-2\right)$



The graph is not symmetric about the x-axis (e.g. $(-4, -2)$ is on the graph but $(-4, 2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(-4, -2)$ is on the graph but $(4, -2)$ is not)

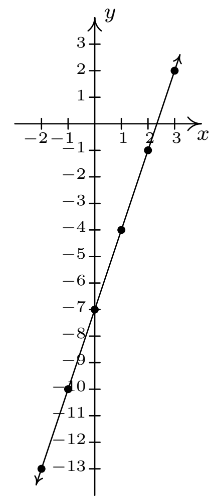
The graph is not symmetric about the origin (e.g. $(-4, -2)$ is on the graph but $(4, 2)$ is not)

(g) $3x - y = 7$
Re-write as: $y = 3x - 7$.

x-intercept: $\left(\frac{7}{3}, 0\right)$

y-intercept: $(0, -7)$

x	y	(x, y)
-2	-13	$(-2, -13)$
-1	-10	$(-1, -10)$
0	-7	$(0, -7)$
1	-4	$(1, -4)$
2	-1	$(2, -1)$
3	2	$(3, 2)$



The graph is not symmetric about the x-axis (e.g. $(3, 2)$ is on the graph but $(3, -2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(3, 2)$ is on the graph but $(-3, 2)$ is not)

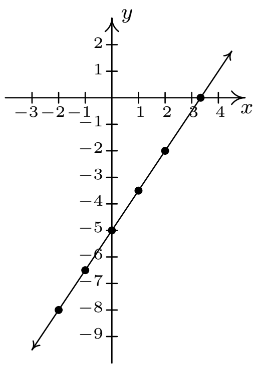
The graph is not symmetric about the origin (e.g. $(3, 2)$ is on the graph but $(-3, -2)$ is not)

(h) $3x - 2y = 10$
Re-write as: $y = \frac{3x-10}{2}$.

x-intercepts: $\left(\frac{10}{3}, 0\right)$

y-intercept: $(0, -5)$

x	y	(x,y)
-2	-8	$(-2,-8)$
-1	$-\frac{13}{2}$	$\left(-1,-\frac{13}{2}\right)$
0	-5	$(0,-5)$
1	$-\frac{7}{2}$	$\left(1,-\frac{7}{2}\right)$
2	-2	$(2,-2)$



The graph is not symmetric about the x-axis (e.g. $(2,-2)$ is on the graph but $(2,2)$ is not)

The graph is not symmetric about the y-axis (e.g. $(2,-2)$ is on the graph but $(-2,-2)$ is not)

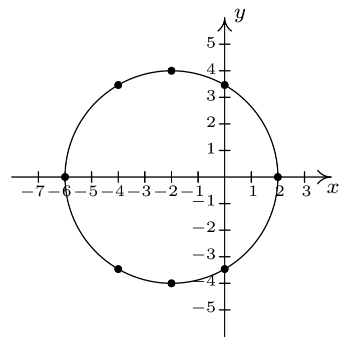
The graph is not symmetric about the origin (e.g. $(2,-2)$ is on the graph but $(-2,2)$ is not)

(i) $(x+2)^2+y^2=16$
Re-write as $y=\pm\sqrt{16-(x+2)^2}$.

x-intercepts: $(-6,0)$, $(2,0)$

y-intercepts: $\left(0,\pm 2\sqrt{3}\right)$

x	y	(x,y)
-6	0	$(-6,0)$
-4	$\pm 2\sqrt{3}$	$\left(-4,\pm 2\sqrt{3}\right)$
-2	± 4	$(-2,\pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0,\pm 2\sqrt{3}\right)$
2	0	$(2,0)$



The graph is symmetric about the x-axis

The graph is not symmetric about the y-axis (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

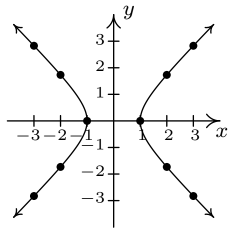
The graph is not symmetric about the origin (e.g. $(-6,0)$ is on the graph but $(6,0)$ is not)

(j) $x^2-y^2=1$
Re-write as: $y=\pm\sqrt{x^2-1}$.

x-intercepts: $(-1,0),(1,0)$

The graph has no y-intercepts

x	y	(x,y)
-3	$\pm\sqrt{8}$	$(-3,\pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2,\pm\sqrt{3})$
-1	0	$(-1,0)$
1	0	$(1,0)$
2	$\pm\sqrt{3}$	$(2,\pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3,\pm\sqrt{8})$



The graph is symmetric about the x-axis

The graph is symmetric about the y-axis

The graph is symmetric about the origin

The graph is symmetric about the y-axis

The graph is symmetric about the origin

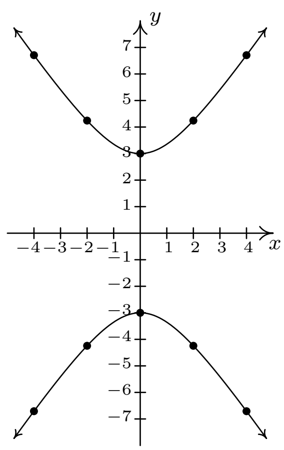
(k) $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\frac{\sqrt{9x^2+36}}{2}$.

The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$



The graph is symmetric about the x-axis

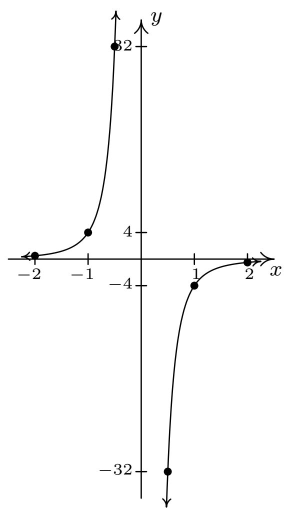
(l) $x^3 y = -4$

Re-write as: $y = -\dfrac{4}{x^3}$.

The graph has no x-intercepts

The graph has no y-intercepts

x	y	(x, y)
-2	$\frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$\left(-\frac{1}{2}, 32\right)$
$\frac{1}{2}$	-32	$\left(\frac{1}{2}, -32\right)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$\left(2, -\frac{1}{2}\right)$



The graph is not symmetric about the x-axis (e.g. $(1,-4)$ is on the graph but $(1,4)$ is not)

The graph is not symmetric about the y-axis (e.g. $(1,-4)$ is on the graph but $(-1,-4)$ is not)

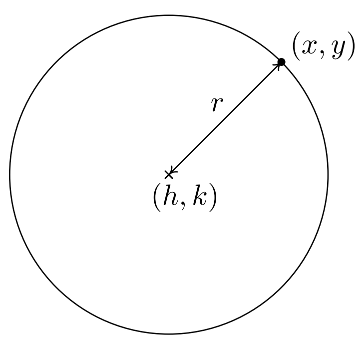
The graph is symmetric about the origin

1.4 Three Interesting Curves

1.4.1 Circles

Recall from geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 1.4. A **circle** with center (h,k) and radius $r > 0$ is the set of all points (x,y) in the plane whose distance to (h,k) is r.



From the picture, we see that a point (x,y) is on the circle if and only if its distance to (h,k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

EQUATION 1.3. **The Standard Equation of a Circle:** The equation of a circle with center (h, k) and radius $r > 0$ is $(x - h)^2 + (y - k)^2 = r^2$.

EXAMPLE 1.4.1. Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

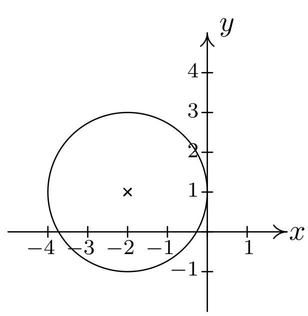
SOLUTION. Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

$$\begin{aligned} (x - (-2))^2 + (y - 3)^2 &= (5)^2 \\ (x + 2)^2 + (y - 3)^2 &= 25 \end{aligned}$$

□

EXAMPLE 1.4.2. Graph $(x + 2)^2 + (y - 1)^2 = 4$. Find the center and radius.

SOLUTION. From the standard form of a circle, Equation 1.3, we have that $x+2$ is $x-h$, so $h = -2$ and $y - 1$ is $y - k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



□

If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x + 2)^2 + (y - 1)^2 = 4$, we'd be contending with $x^2 + 4x + y^2 - 2y + 1 = 0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 1.3 by following the steps given below.

To Put a Circle into Standard Form

1. Group the same variables together on one side of the equation and put the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

EXAMPLE 1.4.3. Complete the square to find the center and radius of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$.

SOLUTION.

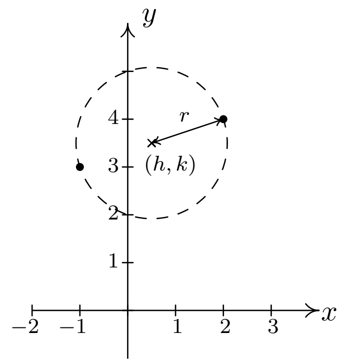
$$\begin{array}{rcll}
3x^2 - 6x + 3y^2 + 4y - 4 & = & 0 & \\
3x^2 - 6x + 3y^2 + 4y & = & 4 & \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \dfrac{4}{3}y\right) & = & 4 & \text{factor out leading coefficients} \\
3\left(x^2 - 2x + 1\right) + 3\left(y^2 + \dfrac{4}{3}y + \dfrac{4}{9}\right) & = & 4 + 3(1) + 3\left(\dfrac{4}{9}\right) & \text{complete the square in } x,\ y \\
3(x-1)^2 + 3\left(y + \dfrac{2}{3}\right)^2 & = & \dfrac{25}{3} & \text{factor} \\
(x-1)^2 + \left(y + \dfrac{2}{3}\right)^2 & = & \dfrac{25}{9} & \text{divide both sides by 3}
\end{array}$$

From Equation 1.3, we identify $x - 1$ as $x - h$, so $h = 1$, and $y + \frac{2}{3}$ as $y - k$, so $k = -\frac{2}{3}$. Hence, the center is $(h, k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$. □

It is possible to obtain equations like $(x-3)^2 + (y+1)^2 = 0$ or $(x-3)^2 + (y+1)^2 = -1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

EXAMPLE 1.4.4. Write the standard equation of the circle which has $(-1, 3)$ and $(2, 4)$ as the endpoints of a diameter.

SOLUTION. We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right)\\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{10}{4}$ □

We close this section with the most important[4] circle in all of mathematics: the **Unit Circle**.

DEFINITION 1.5. The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

[4]While this may seem like an opinion, it is indeed a fact.

Example 1.4.5. Find the points on the unit circle with y-coordinate $\dfrac{\sqrt{3}}{2}$.

Solution. We replace y with $\dfrac{\sqrt{3}}{2}$ in the equation $x^2 + y^2 = 1$ to get

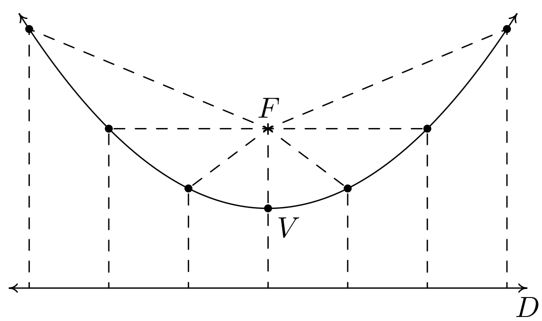
$$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1.$$

From this we get $x^2 + \dfrac{3}{4} = 1$ so $x^2 = \dfrac{1}{4}$ so $x = \pm\dfrac{1}{2}$. The points are $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. These two points are the intersection of the horizontal line $y = \dfrac{\sqrt{3}}{2}$ and the unit circle $x^2 + y^2 = 1$, $\square$

1.4.2 Parabolas

Definition 1.6. Let F be a point in the plane and D be a line not containing F. The set of all points equidistant from F and D is called the **parabola** with **focus** F and **directrix** D.

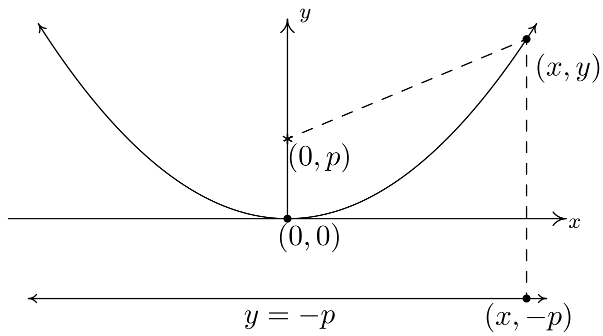
Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section **??**. Let p denote the directed[5] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes

[5] We'll talk more about what 'directed' means later.



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation 1.1, we get

$$
\begin{array}{rcll}
\sqrt{(x-0)^2+(y-p)^2} & = & \sqrt{(x-x)^2+(y-(-p))^2} & \\
\sqrt{x^2+(y-p)^2} & = & \sqrt{(y+p)^2} & \\
x^2+(y-p)^2 & = & (y+p)^2 & \text{square both sides} \\
x^2+y^2-2py+p^2 & = & y^2+2py+p^2 & \text{expand quantities} \\
x^2 & = & 4py & \text{gather like terms}
\end{array}
$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation **??** with $a = \frac{1}{4p}$ and vertex $(0, 0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

What if we choose to place the vertex at an arbitrary point (h, k)? We can either use transformations (vertical and horizontal shifts from Section **??**) or re-derive the equation from Definition 1.6 to arrive at the following.

EQUATION 1.4. **The Standard Equation of a Vertical[a] Parabola:** The equation of a (vertical) parabola with vertex (h, k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

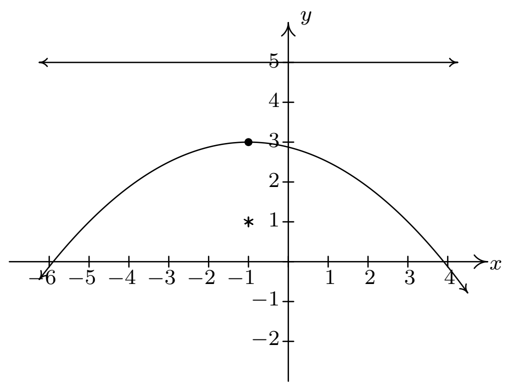
If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a]That is, a parabola which opens either upwards or downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

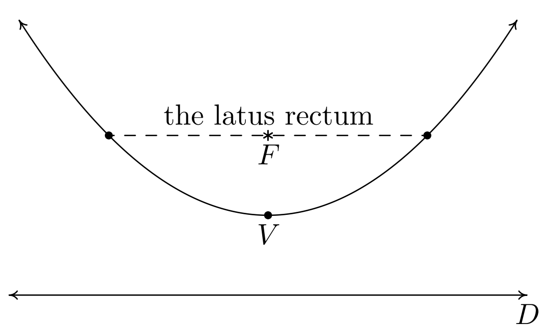
Example 1.4.6. Graph $(x + 1)^2 = -8(y - 3)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 1.4. Here, $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 3$ so $k = 3$. Hence, the vertex is $(-1, 3)$. We also see that $4p = -8$ so $p = -2$. Since $p < 0$, the focus will be below the vertex and the parabola will open downwards. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. If we start at $(-1, 3)$ and move down 2 units, we arrive at the focus $(-1, 1)$. The directrix, then, is 2 units above the vertex and if we move 2 units up from $(-1, 3)$, we'd be on the horizontal line $y = 5$.



□

Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[6] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[7] that the length of the latus rectum is $|4p|$, which, in light of Equation 1.4, is easy to find. In our last example, for instance, when graphing $(x + 1)^2 = -8(y - 3)$, we can use the fact that the length of the latus rectum is $|-8| = 8$, which means the parabola is 8 units wide at the

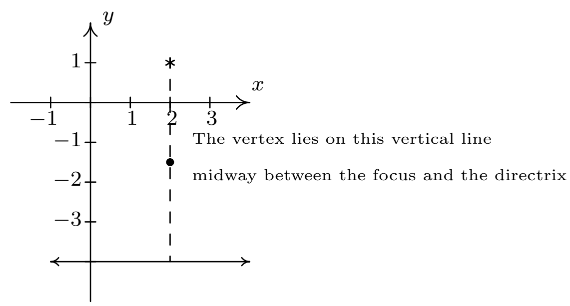
[6] No, I'm not making this up.

[7] Consider this an exercise to show what follows.

focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

EXAMPLE 1.4.7. Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

SOLUTION. Sketching the data yields,



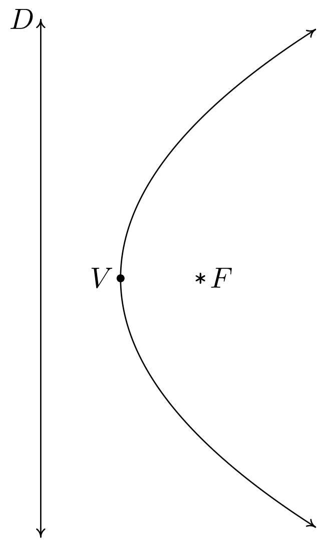
From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1 - (-4) = 5$, which means the vertex lies $5/2$ units (halfway) below the focus. Starting at $(2, 1)$ and moving down $5/2$ units leaves us at $(2, -3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p = 5/2$. Plugging all of this data into Equation 1.4 give us

$$\begin{aligned}(x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right)\\(x-2)^2 &= 10\left(y + \frac{3}{2}\right)\end{aligned}$$

□

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[8] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex. A typical 'horizontal' parabola is sketched below.

[8] plural of 'directrix'



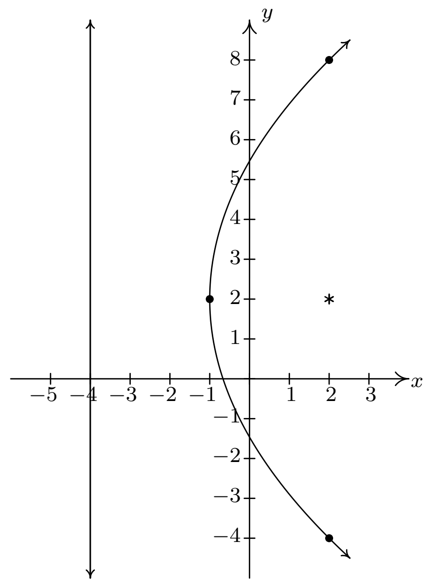
Equation 1.5. **The Standard Equation of a Horizontal Parabola:** The equation of a (horizontal) parabola with vertex (h, k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Example 1.4.8. Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution. We recognize this as the form given in Equation 1.5. Here, $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 2$ so $k = 2$. Hence, the vertex is $(-1, 2)$. We also see that $4p = 12$ so $p = 3$. Since $p > 0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p| = 3$, which means the focus is 3 units to the right. If we start at $(-1, 2)$ and move right 3 units, we arrive at the focus $(2, 2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1, 2)$, we'd be on the vertical line $x = -4$. Since the length of the latus rectum is $|4p| = 12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



As with circles, not all parabolas will come to us in the forms in Equations 1.4 or 1.5. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Put a Parabola into Standard Form

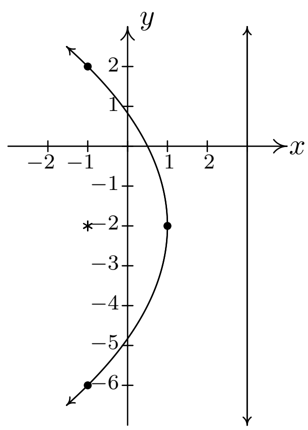
1. Group the variable which is squared on one side of the equation and put the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 1.4.9. Consider the equation $y^2 + 4y + 8x = 4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

Solution. We need to get a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

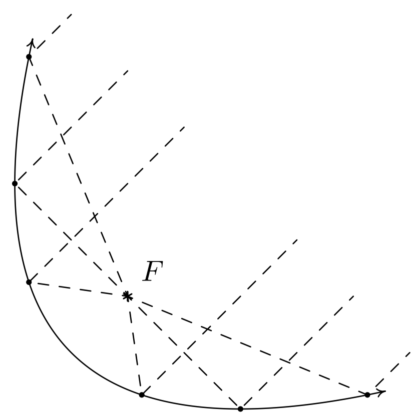
$$\begin{aligned} y^2 + 4y + 8x &= 4 \\ y^2 + 4y &= -8x + 4 \\ y^2 + 4y + 4 &= -8x + 4 + 4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x + 8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

Now that the equation is in the form given in Equation 1.5, we see that $x - h$ is $x - 1$ so $h = 1$, and $y - k$ is $y + 2$ so $k = -2$. Hence, the vertex is $(1, -2)$. We also see that $4p = -8$ so that $p = -2$. Since $p < 0$, the focus will be the left of the vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units to the left of 1, so if we start at $(1, -2)$ and move left 2 units, we arrive at the focus $(-1, -2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1, -2)$, we'd be on the vertical line $x = 3$. Since the length of the latus rectum is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.



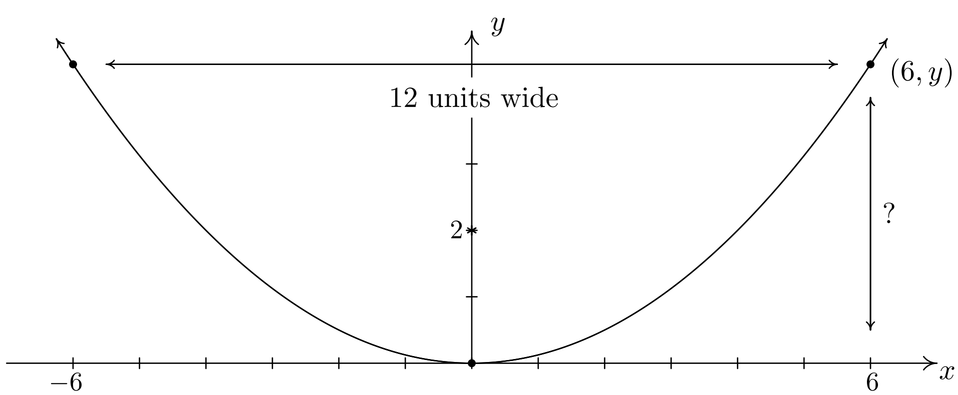
In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.

Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.



EXAMPLE 1.4.10. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

SOLUTION. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



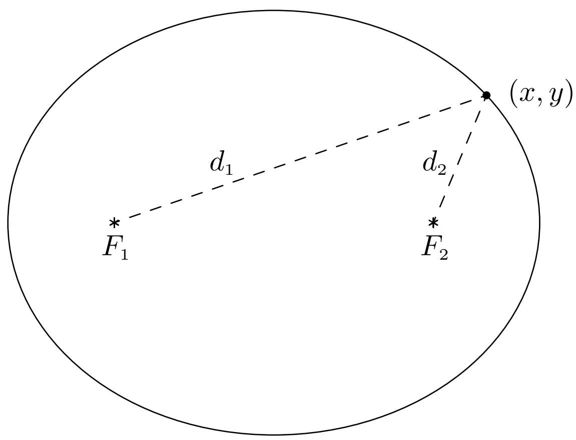
Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = 36/8 = 9/2 = 4.5$. Hence, the dish will be $9/2$ or 4.5 feet deep. $\square$

1.4.3 Ellipses

In the definition of a circle, Definition 1.4, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 1.7. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of the distance from F_1 to (x, y) and the distance from F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

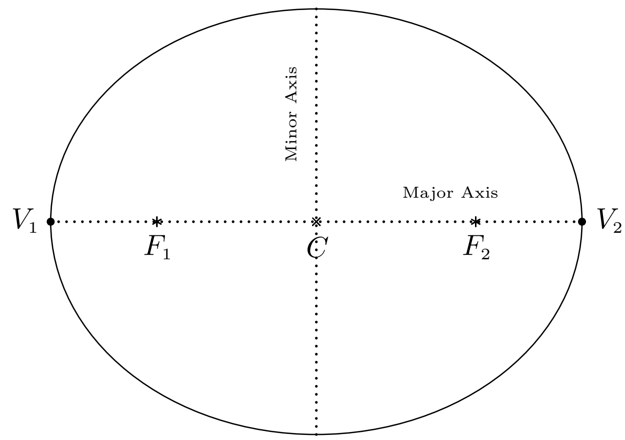
[a]the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

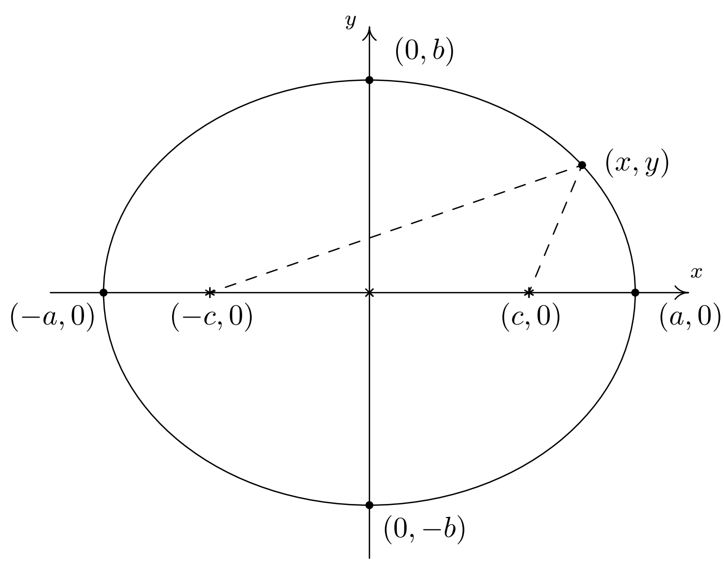
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0,0)$, its major axis along the x-axis, and has foci $(c,0)$ and $(-c,0)$ and vertices $(-a,0)$ and $(a,0)$. We will label the y-intercepts of the ellipse as $(0,b)$ and $(0,-b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 1.7. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\
(a+c)+(a-c) &= d \\
2a &= d
\end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0,b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\
\sqrt{(0-(-c))^2+(b-0)^2} + \sqrt{(0-c)^2+(b-0)^2} &= 2a \\
\sqrt{b^2+c^2} + \sqrt{b^2+c^2} &= 2a \\
2\sqrt{b^2+c^2} &= 2a \\
\sqrt{b^2+c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x,y) on the ellipse. Applying Definition 1.7, we get

$$\begin{aligned}
\text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} &= 2a \\
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to do some rearranging, squaring, and more rearranging.[9]

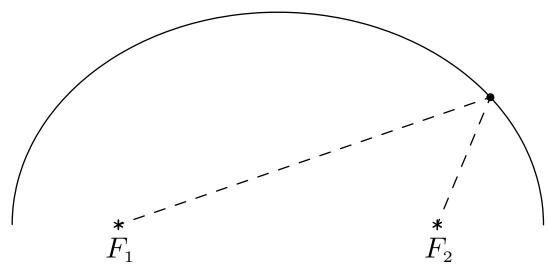
$$\begin{aligned}
\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a - \sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a - \sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 + (x-c)^2 - (x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2 - 4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2 - cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2-cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 &= a^4 - 2a^2cx + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned}
\left(a^2-c^2\right)x^2 + a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2 + a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1
\end{aligned}$$

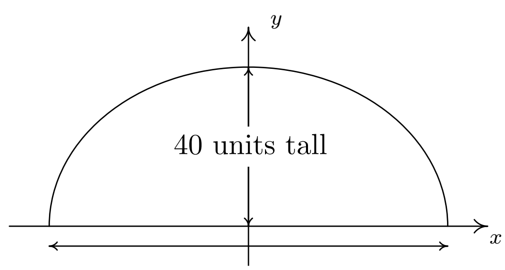
[9] In other words, tons and tons of Intermediate Algebra. Stay sharp, this is not for the faint of heart.

As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below as sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus. Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them.



EXAMPLE 1.4.11. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

SOLUTION. Graphing the data yields



It's most convenient to imagine this ellipse centered at $(0, 0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$. Hence, our ellipse has the equation

$$\frac{x^2}{50^2} + \frac{y^2}{40^2} = 1.$$

We're looking for the foci, and we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, so that the foci are 30 units from the center. That means they are $50 - 30 = 20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery. $\square$

1.4.4 Exercises

1. Find the standard equation of the circle given the center and radius and sketch its graph.

   (a) Center $(-1, -5)$, radius 10
   (b) Center $(4, -2)$, radius 3
   (c) Center $\left(-3, \frac{7}{13}\right)$, radius $\frac{1}{2}$
   (d) Center (π, e^2), radius $\sqrt[3]{91}$
   (e) Center $\left(-e, \sqrt{2}\right)$, radius π

2. Complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

   (a) $x^2 - 4x + y^2 + 10y = -25$
   (b) $-2x^2 - 36x - 2y^2 - 112 = 0$
   (c) $x^2 + y^2 + 8x - 10y - 1 = 0$
   (d) $x^2 + y^2 + 5x - y - 1 = 0$
   (e) $4x^2 + 4y^2 - 24y + 36 = 0$
   (f) $x^2 + x + y^2 - \frac{6}{5}y = 1$

3. Find the standard equation of the circle which satisfies the following criteria:

   (a) center $(3, 5)$, passes through $(-1, -2)$
   (b) center $(3, 6)$, passes through $(-1, 4)$
   (c) endpoints of a diameter: $(3, 6)$ and $(-1, 4)$
   (d) endpoints of a diameter: $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$

4. Verify the following points lie on the Unit Circle: $(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$

5. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[10] Find an equation for the wheel assuming that its center lies on the y-axis.

6. Sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

   (a) $(y - 2)^2 = -12(x + 3)$
   (b) $(y + 4)^2 = 4x$
   (c) $(x - 3)^2 = -16y$
   (d) $\left(x + \frac{7}{3}\right)^2 = 2\left(y + \frac{5}{2}\right)$
   (e) $(x - 1)^2 = 4(y + 3)$
   (f) $(x + 2)^2 = -20(y - 5)$
   (g) $(y - 4)^2 - 18(x - 2)$
   (h) $\left(y + \frac{3}{2}\right)^2 = -7\left(x + \frac{9}{2}\right)$

7. Put the equation into standard form and identify the vertex, focus and directrix.

[10] Source: Cedar Point's webpage.

(a) $y^2 - 10y - 27x + 133 = 0$
(b) $25x^2 + 20x + 5y - 1 = 0$
(c) $x^2 + 2x - 8y + 49 = 0$
(d) $2y^2 + 4y + x - 8 = 0$
(e) $x^2 - 10x + 12y + 1 = 0$
(f) $3y^2 - 27y + 4x + \frac{211}{4} = 0$

8. Find an equation for the parabola which fits the given criteria.

(a) Vertex $(7, 0)$, focus $(0, 0)$
(b) Vertex $(-8, -9)$, Both $(0, 0)$, $(-16, 0)$ are points on the curve
(c) Focus $(10, 1)$, directrix $x = 5$
(d) The endpoints of latus rectum are $(-2, -7)$ and $(-4, -7)$

9. Graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the foci and the eccentricity.

(a) $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
(b) $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
(c) $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
(d) $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
(e) $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
(f) $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
(g) $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$

10. Put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the foci and the eccentricity.

(a) $12x^2 + 3y^2 - 30y + 39 = 0$
(b) $5x^2 + 18y^2 - 30x + 72y + 27 = 0$
(c) $x^2 - 2x + 2y^2 - 12y + 3 = 0$
(d) $9x^2 + 4y^2 - 4y - 8 = 0$
(e) $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
(f) $6x^2 + 5y^2 - 24x + 20y + 14 = 0$

11. Find the standard form of the equation of the ellipse which has the given properties.

(a) Center $(3, 7)$, Vertex $(3, 2)$, Focus $(3, 3)$
(b) All points on the ellipse are in Quadrant IV except $(0, -9)$ and $(8, 0)$[11]
(c) Foci $(0, \pm 4)$, Point on curve $\left(2, \frac{5\sqrt{5}}{3}\right)$
(d) Vertex $(-10, 5)$, Focus $(-2, 5)$, Eccentricity $\frac{1}{2}$

[11] One might also say that the ellipse is "tangent to the axes" at those two points.

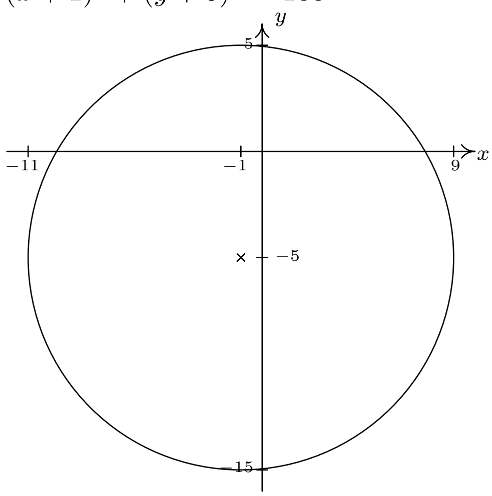
12. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farther from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.

13. With the help of your classmates, research whispering galleries and other ways ellipses have been used in architecture and design.

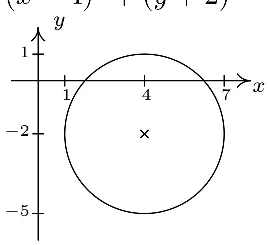
14. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

1.4.5 Answers

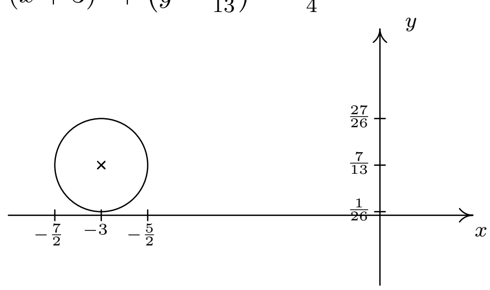
1. (a) $(x+1)^2+(y+5)^2=100$



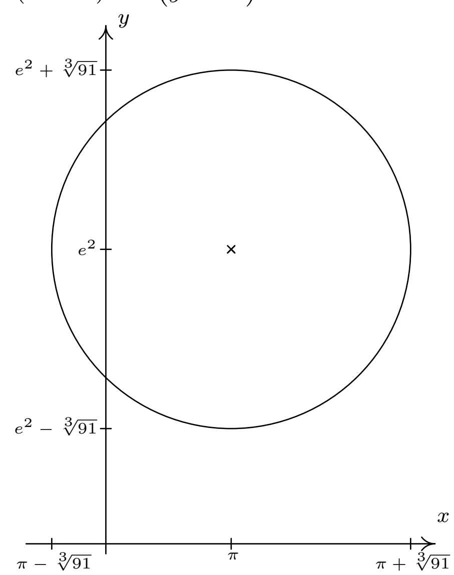
(b) $(x-4)^2+(y+2)^2=9$



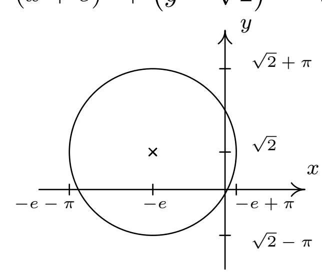
(c) $(x+3)^2+\left(y-\frac{7}{13}\right)^2=\frac{1}{4}$



(d) $(x-\pi)^2+\left(y-e^2\right)^2=91^{\frac{2}{3}}$



(e) $(x+e)^2+\left(y-\sqrt{2}\right)^2=\pi^2$



2. (a) $(x-2)^2+(y+5)^2=4$
Center $(2,-5)$, radius $r=2$

(b) $(x+9)^2+y^2=25$
Center $(-9,0)$, radius $r=5$

(c) $(x+4)^2+(y-5)^2=42$
Center $(-4,5)$, radius $r=\sqrt{42}$

(d) $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$

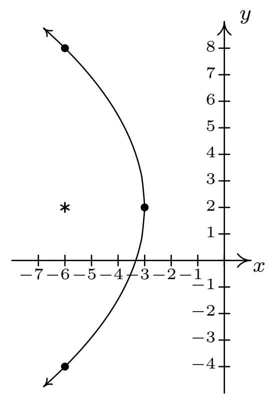
(e) $x^2+(y-3)^2=0$
This is not a circle.

(f) $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$

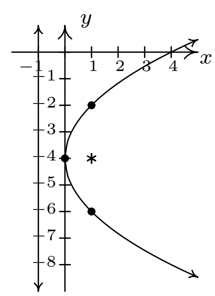
3. (a) $(x-3)^2 + (y-5)^2 = 65$
   (b) $(x-3)^2 + (y-6)^2 = 20$
   (c) $(x-1)^2 + (y-5)^2 = 5$
   (d) $(x-1)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{13}{2}$

5. $x^2 + (y-72)^2 = 4096$

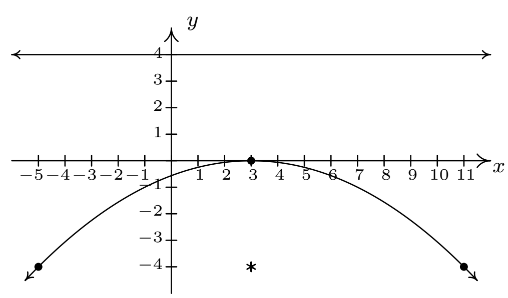
6. (a) $(y-2)^2 = -12(x+3)$
   Vertex $(-3, 2)$
   Focus $(-6, 2)$
   Directrix $x = 0$
   Endpoints of latus rectum $(-6, 8)$, $(-6, -4)$



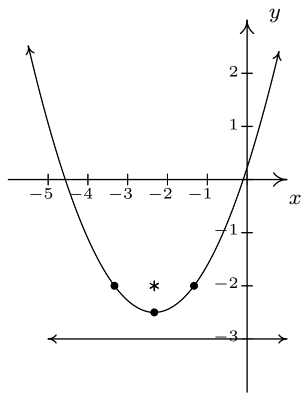
   (b) $(y+4)^2 = 4x$
   Vertex $(0, -4)$
   Focus $(1, -4)$
   Directrix $x = -1$
   Endpoints of latus rectum $(1, -2)$, $(1, -6)$



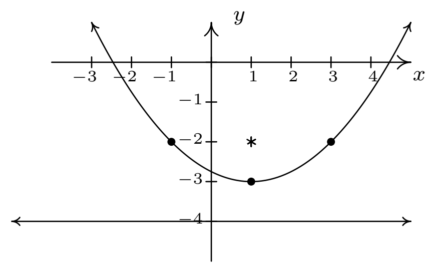
   (c) $(x-3)^2 = -16y$
   Vertex $(3, 0)$
   Focus $(3, -4)$
   Directrix $y = 4$
   Endpoints of latus rectum $(-5, -4)$, $(11, -4)$



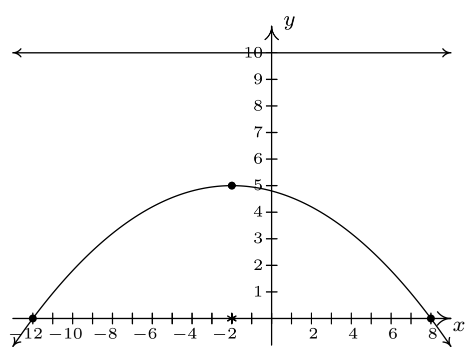
   (d) $\left(x + \frac{7}{3}\right)^2 = 2\left(y + \frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3}, -\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3}, -2\right)$
   Directrix $y = -3$
   Endpoints of latus rectum $\left(-\frac{10}{3}, -2\right)$, $\left(-\frac{4}{3}, -2\right)$



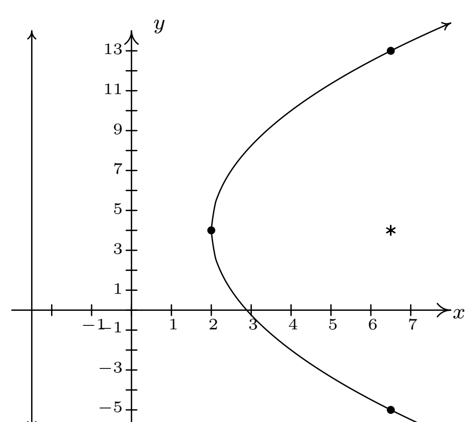
(e) $(x-1)^2 = 4(y+3)$
Vertex $(1, -3)$
Focus $(1, -2)$
Directrix $y = -4$
Endpoints of latus rectum $(3, -2)$, $(-1, -2)$



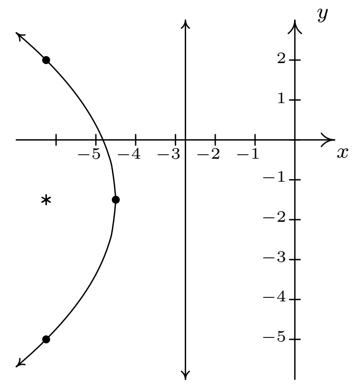
(f) $(x+2)^2 = -20(y-5)$
Vertex $(-2, 5)$
Focus $(-2, 0)$
Directrix $y = 10$
Endpoints of latus rectum $(-12, 0)$, $(8, 0)$



(g) $(y-4)^2 - 18(x-2)$
Vertex $(2, 4)$
Focus $\left(\frac{13}{2}, 4\right)$
Directrix $x = -\frac{5}{2}$
Endpoints of latus rectum $\left(-\frac{13}{2}, -5\right)$, $\left(\frac{13}{2}, 13\right)$



(h) $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$
Vertex $\left(-\frac{9}{2}, -\frac{3}{2}\right)$
Focus $\left(-\frac{25}{4}, -\frac{3}{2}\right)$
Directrix $x = -\frac{11}{4}$
Endpoints of latus rectum $\left(-\frac{25}{4}, 2\right)$, $\left(-\frac{25}{4}, -5\right)$



7. (a) $(y-5)^2 = 27(x-4)$
Vertex $(4,5)$
Focus $\left(\frac{43}{4}, 5\right)$
Directrix $x = -\frac{11}{4}$

(b) $\left(x+\frac{2}{5}\right)^2 = -\frac{1}{5}(y-1)$
Vertex $\left(-\frac{2}{5}, 1\right)$
Focus $\left(-\frac{2}{5}, \frac{19}{20}\right)$
Directrix $y = \frac{21}{20}$

(c) $(x+1)^2 = 8(y-6)$
Vertex $(-1,6)$
Focus $(-1,8)$
Directrix $y = 4$

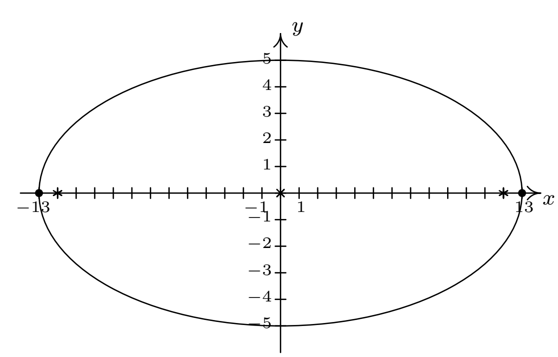
(d) $(y+1)^2 = -\frac{1}{2}(x-10)$
Vertex $(10,-1)$
Focus $\left(\frac{79}{8}, -1\right)$
Directrix $x = \frac{81}{8}$

(e) $(x-5)^2 = -12(y-2)$
Vertex $(5,2)$
Focus $(5,-1)$
Directrix $y = 5$

(f) $\left(y-\frac{9}{2}\right)^2 = -\frac{4}{3}(x-2)$
Vertex $\left(2, \frac{9}{2}\right)$
Focus $\left(\frac{5}{3}, \frac{9}{2}\right)$
Directrix $x = \frac{7}{3}$

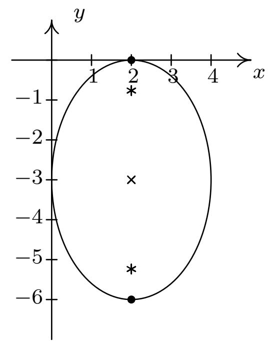
8. (a) $y^2 = -28(x-7)$
(b) $(x+8)^2 = \frac{64}{9}(y+9)$
(c) $(y-1)^2 = 10\left(x-\frac{15}{2}\right)$
(d) $(x-1)^2 = 6\left(y+\frac{17}{2}\right)$ or
$(x-1)^2 = -6\left(y+\frac{11}{2}\right)$

9. (a) $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$
Center $(0,0)$
Major axis along $y = 0$
Minor axis along $x = 0$
Vertices $(13,0)$, $(-13,0)$
Foci $(12,0)$, $(-12,0)$
$e = \dfrac{12}{13}$

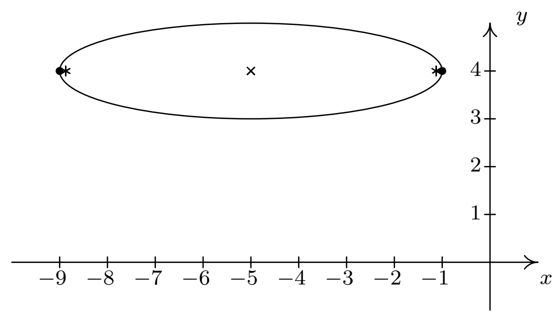


(b) $\dfrac{(x-2)^2}{4} + \dfrac{(y+3)^2}{9} = 1$
Center $(2,-3)$
Major axis along $x = 2$
Minor axis along $y = -3$
Vertices $(2,0)$, $(2,-6)$

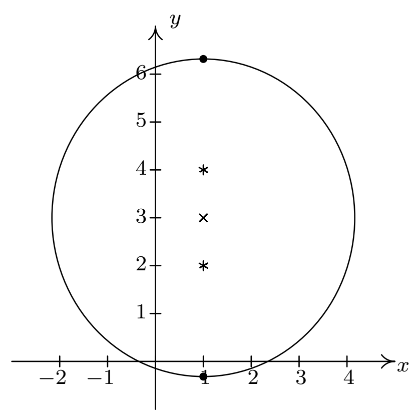
Foci $(2, -3+\sqrt{5})$, $(2, -3-\sqrt{5})$
$e = \dfrac{\sqrt{5}}{3}$



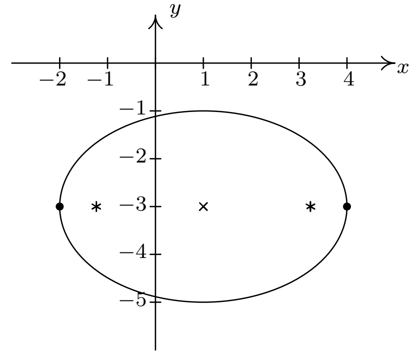
(c) $\dfrac{(x+5)^2}{16} + \dfrac{(y-4)^2}{1} = 1$
Center $(-5, 4)$
Major axis along $y = 4$
Minor axis along $x = -5$
Vertices $(-9, 4)$, $(-1, 4)$
Foci $(-5+\sqrt{15}, 4)$, $(-5-\sqrt{15}, 4)$
$e = \dfrac{\sqrt{15}}{4}$



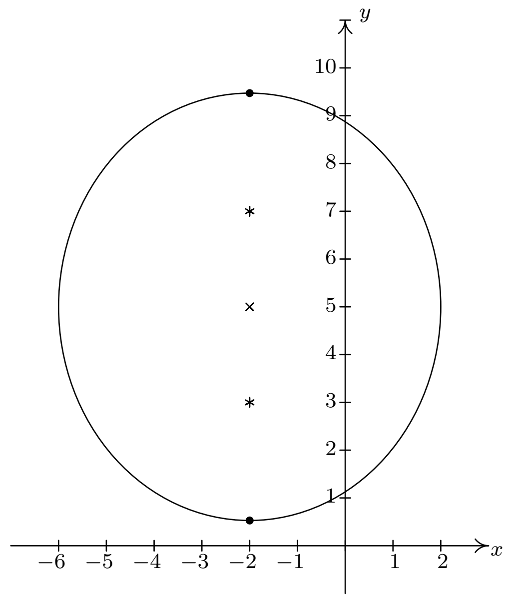
(d) $\dfrac{(x-1)^2}{10} + \dfrac{(y-3)^2}{11} = 1$
Center $(1, 3)$
Major axis along $x = 1$
Minor axis along $y = 3$
Vertices $(1, 3+\sqrt{11})$, $(1, 3-\sqrt{11})$
Foci $(1, 2)$, $(1, 4)$
$e = \dfrac{\sqrt{11}}{11}$



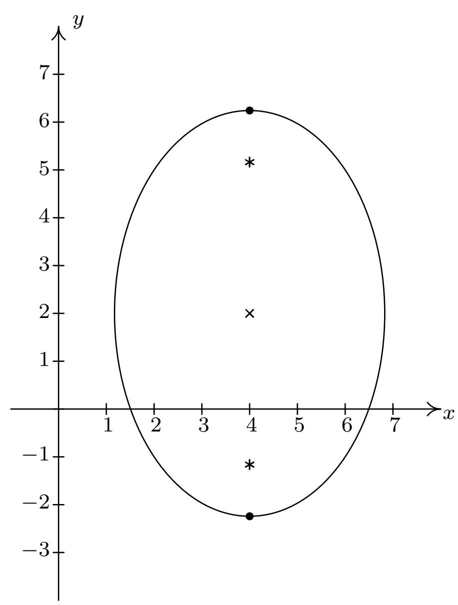
(e) $\dfrac{(x-1)^2}{9} + \dfrac{(y+3)^2}{4} = 1$
Center $(1, -3)$
Major axis along $y = -3$
Minor axis along $x = 1$
Vertices $(4, -3)$, $(-2, -3)$
Foci $(1+\sqrt{5}, -3)$, $(1-\sqrt{5}, -3)$
$e = \dfrac{\sqrt{5}}{3}$



(f) $\dfrac{(x+2)^2}{16} + \dfrac{(y-5)^2}{20} = 1$
Center $(-2, 5)$
Major axis along $x = -2$
Minor axis along $y = 5$
Vertices $(-2, 5 + 2\sqrt{5})$, $(-2, 5 - 2\sqrt{5})$
Foci $(-2, 7)$, $(-2, 3)$
$e = \dfrac{\sqrt{5}}{5}$



(g) $\dfrac{(x-4)^2}{8} + \dfrac{(y-2)^2}{18} = 1$
Center $(4, 2)$
Major axis along $x = 4$
Minor axis along $y = 2$
Vertices $(4, 2 + 3\sqrt{2})$, $(4, 2 - 3\sqrt{2})$
Foci $(4, 2 + \sqrt{10})$, $(4, 2 - \sqrt{10})$
$e = \dfrac{\sqrt{5}}{3}$



10. (a) $\dfrac{x^2}{3} + \dfrac{(y-5)^2}{12} = 1$
Center $(0, 5)$
Major axis along $x = 0$
Minor axis along $y = 5$
Vertices $(0, 5 - 2\sqrt{3}), (0, 5 + 2\sqrt{3})$
Foci $(0, 2), (0, 8)$
$e = \frac{\sqrt{3}}{2}$

(b) $\dfrac{(x-3)^2}{18} + \dfrac{(y+2)^2}{5} = 1$
Center $(3, -2)$
Major axis along $y = -2$
Minor axis along $x = 3$
Vertices $(3 - 3\sqrt{2}, -2), (3 + 3\sqrt{2}, -2)$
Foci $(3 - \sqrt{13}, -2), (3 + \sqrt{13}, -2)$
$e = \frac{\sqrt{26}}{6}$

(c) $\dfrac{(x-1)^2}{16}+\dfrac{(y-3)^2}{8}=1$
Center $(1,3)$
Major Axis along $y=3$
Minor Axis along $x=1$
Vertices $(5,3)$, $(-3,3)$
Foci $(1+2\sqrt{2},3)$, $(1-2\sqrt{2},3)$
$e=\frac{\sqrt{2}}{2}$

(d) $\dfrac{x^2}{1}+\dfrac{4\left(y-\frac{1}{2}\right)^2}{9}=1$
Center $\left(0,\frac{1}{2}\right)$
Major Axis along $x=0$ (the y-axis)
Minor Axis along $y=\frac{1}{2}$
Vertices $(0,2)$, $(0,-1)$
Foci $\left(0,\frac{1+\sqrt{5}}{2}\right)$, $\left(0,\frac{1-\sqrt{5}}{2}\right)$
$e=\frac{\sqrt{5}}{3}$

(e) $\dfrac{(x-3)^2}{25}+\dfrac{(y-1)^2}{9}=1$
Center $(3,1)$
Major Axis along $y=1$
Minor Axis along $x=3$
Vertices $(8,1)$, $(-2,1)$
Foci $(7,1)$, $(-1,1)$
$e=\frac{4}{5}$

(f) $\dfrac{(x-2)^2}{5}+\dfrac{(y+2)^2}{6}=1$
Center $(2,-2)$
Major Axis along $x=2$
Minor Axis along $y=-2$
Vertices $\left(2,-2+\sqrt{6}\right)$, $(2,-2-\sqrt{6})$
Foci $(2,-1)$, $(2,-3)$
$e=\frac{\sqrt{6}}{6}$

11. (a) $\dfrac{(x-3)^2}{9}+\dfrac{(y-7)^2}{25}=1$
(b) $\dfrac{(x-8)^2}{64}+\dfrac{(y+9)^2}{81}=1$
(c) $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1$
(d) $\dfrac{(x-6)^2}{256}+\dfrac{(y-5)^2}{192}=1$

12. Distance from the sun to aphelion ≈ 1.0167 AU.
Distance from the sun to perihelion ≈ 0.9833 AU.