# Chapter 1

# Systems of Equations and Matrices

## 1.1 Systems of Linear Equations: Gaussian Elimination

Up until now, when we concerned ourselves with solving different types of equations there was only one equation to solve at a time. Given an equation $f(x) = g(x)$, we could check our solutions geometrically by finding where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The $x$-coordinates of these intersection points correspond to the solutions to the equation $f(x) = g(x)$, and the $y$-coordinates were largely ignored. If we modify the problem and ask for the intersection points of the graphs of $y = f(x)$ and $y = g(x)$, where both the solution to $x$ and $y$ are of interest, we have what is known as a **system of equations**, usually written as

$$\left\{ \begin{array}{rcl} y & = & f(x) \\ y & = & g(x) \end{array} \right.$$

The 'curly bracket' notation means we are to find all **pairs** of points $(x, y)$ which satisfy **both** equations. We begin our study of systems of equations by reviewing some basic notions from Intermediate Algebra.

Definition 1.1. A **linear equation in two variables** is an equation of the form $a_1x + a_2y = c$ where $a_1$, $a_2$ and $c$ are real numbers and at least one of $a_1$ and $a_2$ is nonzero.

For reasons which will become clear later in the section, we are using subscripts in Definition 1.1 to indicate different, but fixed, real numbers and those subscripts have no mathematical meaning beyond that. For example, $3x - \frac{y}{2} = 0.1$ is a linear equation in two variables with $a_1 = 3$, $a_2 = -\frac{1}{2}$ and $c = 0.1$. We can also consider $x = 5$ to be a linear equation in two variables by identifying $a_1 = 1$, $a_2 = 0$, and $c = 5$.[1] If $a_1$ and $a_2$ are both 0, then depending on $c$, we get either an equation which is *always* true, called an **identity**, or an equation which is *never* true, called a **contradiction**. (If $c = 0$, then we get $0 = 0$, which is always true. If $c \neq 0$, then we'd have $0 \neq 0$, which is never true.) Even though identities and contradictions have a large role to play in the upcoming sections, we do not consider them linear equations. The key to identifying linear equations is to note that the variables involved are to the first power and that the coefficients of the variables are numbers. Some examples of equations which are non-linear are $x^2 + y = 1$, $xy = 5$ and $e^{2x} + \ln(y) = 1$. We leave it to the reader to explain why these do not satisfy Definition 1.1. From what we know from Sections **??** and **??**, the graphs of linear equations are lines. If we couple two or more linear equations together, in effect to find the points of intersection of two or more lines, we obtain a **system of linear equations in two variables**. Our first example reviews some of the basic techniques first learned in Intermediate Algebra.

Example 1.1.1. Solve the following systems of equations. Check your answer algebraically and graphically.

[1] Critics may argue that $x = 5$ is clearly an equation in one variable. It can also be considered an equation in 117 variables with the coefficients of 116 variables set to 0. As with many conventions in Mathematics, the context will clarify the situation.

1. $\left\{ \begin{array}{rcr} 2x - y & = & 1 \\ y & = & 3 \end{array} \right.$

2. $\left\{ \begin{array}{rcr} 3x + 4y & = & -2 \\ -3x - y & = & 5 \end{array} \right.$

3. $\left\{ \begin{array}{rcr} \frac{x}{3} - \frac{4y}{5} & = & \frac{7}{5} \\ \frac{2x}{9} + \frac{y}{3} & = & \frac{1}{2} \end{array} \right.$

4. $\left\{ \begin{array}{rcr} 2x - 4y & = & 6 \\ 3x - 6y & = & 9 \end{array} \right.$

5. $\left\{ \begin{array}{rcr} 6x + 3y & = & 9 \\ 4x + 2y & = & 12 \end{array} \right.$

6. $\left\{ \begin{array}{rcr} x - y & = & 0 \\ x + y & = & 2 \\ -2x + y & = & -2 \end{array} \right.$

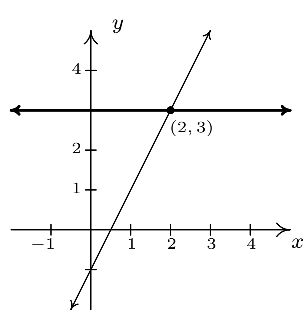
Solution.

1. Our first system is nearly solved for us. The second equation tells us that $y = 3$. To find the corresponding value of $x$, we **substitute** this value for $y$ into the the first equation to obtain $2x - 3 = 1$, so that $x = 2$. Our solution to the system is $(2, 3)$. To check this algebraically, we substitute $x = 2$ and $y = 3$ into each equation and see that they are satisfied. We see $2(2) - 3 = 1$, and $3 = 3$, as required. To check our answer graphically, we graph the lines $2x - y = 1$ and $y = 3$ and verify that they intersect at $(2, 3)$.

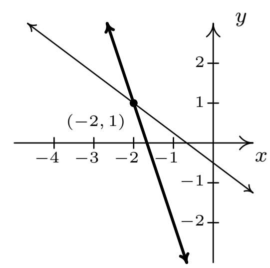
2. To solve the second system, we use the **addition** method to **eliminate** the variable $x$. We take the two equations as given and 'add equals to equals' to obtain

$$\begin{array}{rrcr} & 3x + 4y & = & -2 \\ + & (-3x - y & = & 5) \\ \hline & 3y & = & 3 \end{array}$$

This gives us $y = 1$. We now substitute $y = 1$ into either of the two equations, say $-3x - y = 5$, to get $-3x - 1 = 5$ so that $x = -2$. Our solution is $(-2, 1)$. Substituting $x = -2$ and $y = 1$ into the first equation gives $3(-2) + 4(1) = -2$, which is true, and, likewise, when we check $(-2, 1)$ in the second equation, we get $-3(-2) - 1 = 5$, which is also true. Geometrically, the lines $3x + 4y = -2$ and $-3x - y = 5$ intersect at $(-2, 1)$.




$2x - y = 1$
$\boldsymbol{y = 3}$




$3x + 4y = -2$
$\boldsymbol{-3x - y = 5}$

3. The equations in the third system are more approachable if we clear denominators. We multiply both sides of the first equation by 15 and both sides of the second equation by 18

to obtain the kinder, gentler system

$$\begin{cases} 5x - 12y &= 21 \\ 4x + 6y &= 9 \end{cases}$$

Adding these two equations directly fails to eliminate either of the variables, but we note that if we multiply the first equation by 4 and the second by $-5$, we will be in a position to eliminate the $x$ term

$$\begin{array}{rrcr} & 20x - 48y & = & 84 \\ + & (-20x - 30y & = & -45) \\ \hline & -78y & = & 39 \end{array}$$

From this we get $y = -\frac{1}{2}$. We can temporarily avoid too much unpleasantness by choosing to substitute $y = -\frac{1}{2}$ into one of the equivalent equations we found by clearing denominators, say into $5x - 12y = 21$. We get $5x + 6 = 21$ which gives $x = 3$. Our answer is $\left(3, -\frac{1}{2}\right)$. At this point, we have no choice – in order to check an answer algebraically, we must see if the answer satisfies both of the *original* equations, so we substitute $x = 3$ and $y = -\frac{1}{2}$ into both $\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$ and $\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}$. We leave it to the reader to verify that the solution is correct. Graphing both of the lines involved with considerable care yields an intersection point of $\left(3, -\frac{1}{2}\right)$.

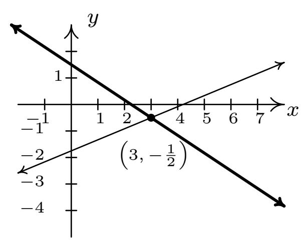
4. An eerie calm settles over us as we cautiously approach our fourth system. Do its friendly integer coefficients belie something more sinister? We note that if we multiply both sides of the first equation by 3 and the both sides of the second equation by $-2$, we are ready to eliminate the $x$

$$\begin{array}{rrcr} & 6x - 12y & = & 18 \\ + & (-6x + 12y & = & -18) \\ \hline & 0 & = & 0 \end{array}$$

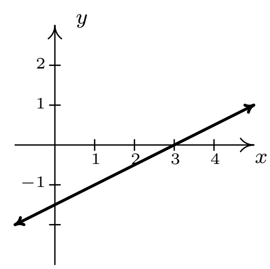
We eliminated not only the $x$, but the $y$ as well and we are left with the identity $0 = 0$. This means that these two different linear equations are, in fact, equivalent. In other words, if an ordered pair $(x, y)$ satisfies the equation $2x - 4y = 6$, it *automatically* satisfies the equation $3x - 6y = 9$. One way to describe the solution set to this system is to use the roster method[2] and write $\{(x, y) : 2x - 4y = 6\}$. While this is correct (and corresponds exactly to what's happening graphically, as we shall see shortly), we take this opportunity to introduce the notion of a **parametric solution**. Our first step is to solve $2x - 4y = 6$ for one of the variables, say $y = \frac{1}{2}x - \frac{3}{2}$. For each value of $x$, the formula $y = \frac{1}{2}x - \frac{3}{2}$ determines the corresponding $y$-value of a solution. Since we have no restriction on $x$, it is called a **free**

[2]See Section **??** for a review of this.

**variable**. We let $x = t$, a so-called 'parameter', and get $y = \frac{1}{2}t - \frac{3}{2}$. Our set of solutions can then be described as $\left\{\left(t, \frac{1}{2}t - \frac{3}{2}\right) : -\infty < t < \infty\right\}$.[3] For specific values of $t$, we can generate solutions. For example, $t = 0$ gives us the solution $\left(0, -\frac{3}{2}\right)$; $t = 117$ gives us $(117, 57)$, and while we can readily check each of these particular solutions satisfy both equations, the question is how do we check our general answer algebraically? Same as always. We claim that for any real number $t$, the pair $\left(t, \frac{1}{2}t - \frac{3}{2}\right)$ satisfies both equations. Substituting $x = t$ and $y = \frac{1}{2}t - \frac{3}{2}$ into $2x - 4y = 6$ gives $2t - 4\left(\frac{1}{2}t - \frac{3}{2}\right) = 6$. Simplifying, we get $2t - 2t + 6 = 6$, which is always true. Similarly, when we make these substitutions in the equation $3x - 6y = 9$, we get $3t - 6\left(\frac{1}{2}t - \frac{3}{2}\right) = 9$ which reduces to $3t - 3t + 9 = 9$, so it checks out, too. Geometrically, $2x - 4y = 6$ and $3x - 6y = 9$ are the same line, which means that they intersect at every point on their graphs. The reader is encouraged to think about how our parametric solution says exactly that.




$\frac{x}{3} - \frac{4y}{5} = \frac{7}{5}$
$\boldsymbol{\frac{2x}{9} + \frac{y}{3} = \frac{1}{2}}$




$2x - 4y = 6$
$\boldsymbol{3x - 6y = 9}$
(Same line.)

5. Multiplying both sides of the first equation by 2 and the both sides of the second equation by $-3$, we set the stage to eliminate $x$

$$\begin{array}{rrcr} & 12x + 6y & = & 18 \\ + & (-12x - 6y & = & -36) \\ \hline & 0 & = & -18 \end{array}$$

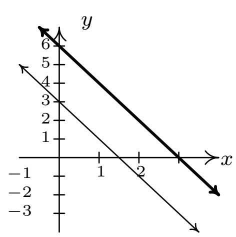
As in the previous example, both $x$ and $y$ dropped out of the equation, but we are left with an irrevocable contradiction, $0 = -18$. This tells us that it is impossible to find a pair $(x, y)$ which satisfies both equations; in other words, the system has no solution. Graphically, we see that the lines $6x + 3y = 9$ and $4x + 2y = 12$ are distinct and parallel, and as such do not intersect.

6. We can begin to solve our last system by adding the first two equations

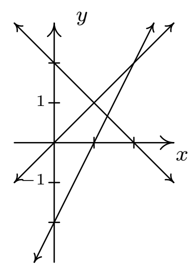
[3] Note that we could have just as easily chosen to solve $2x - 4y = 6$ for $x$ to obtain $x = 2y + 3$. Letting $y$ be the parameter $t$, we have that for any value of $t$, $x = 2t + 3$, which gives $\{(2t + 3, t) : -\infty < t < \infty\}$. There is no one correct way to parameterize the solution set, which is why it is always best to check your answer.

$$\begin{array}{rrcr} & x-y & = & 0 \\ + & (x+y & = & 2) \\ \hline & 2x & = & 2 \end{array}$$

which gives $x = 1$. Substituting this into the first equation gives $1 - y = 0$ so that $y = 1$. We seem to have determined a solution to our system, $(1, 1)$. While this checks in the first two equations, when we substitute $x = 1$ and $y = 1$ into the third equation, we get $-2(1)+(1) = -2$ which simplifies to the contradiction $-1 = -2$. Graphing the lines $x-y = 0$, $x + y = 2$, and $-2x + y = -2$, we see that the first two lines do, in fact, intersect at $(1, 1)$, however, all three lines never intersect at the same point simultaneously, which is what is required if a solution to the system is to be found.




$6x + 3y = 9$
$\mathbf{4x + 2y = 12}$




$y - x = 0$
$y + x = 2$
$-2x + y = -2$

$\square$

A few remarks about Example 1.1.1 are in order. It is clear that some systems of equations have solutions, and some do not. Those which have solutions are called **consistent**, those with no solution are called **inconsistent**. We also distinguish the two different types of behavior among consistent systems. Those which admit free variables are called **dependent**; those with no free variables are called **independent**.[4] Using this new vocabulary, we classify numbers 1, 2 and 3 in Example 1.1.1 as consistent independent systems, number 4 is consistent dependent, and numbers 5 and 6 are inconsistent.[5] The system in 6 above is called **overdetermined**, since we have more equations than variables.[6] Not surprisingly, a system with more variables than equations is called

[4]In the case of systems of linear equations, regardless of the number of equations or variables, consistent independent systems have exactly one solution. The reader is encouraged to think about why this is the case for linear equations in two variables. Hint: think geometrically.

[5]The adjectives 'dependent' and 'independent' apply only to *consistent* systems - they describe the type of solutions.

[6]If we think if each variable being an unknown quantity, then ostensibly, to recover two unknown quantities, we need two pieces of information - i.e., two equations. Having more than two equations suggests we have more information than necessary to determine the values of the unknowns. While this is not necessarily the case, it does explain the choice of terminology 'overdetermined'.

**underdetermined**. While the system in number 6 above is overdetermined and inconsistent, there exist overdetermined consistent systems (both dependent and independent) and we leave it to the reader to think about what is happening algebraically and geometrically in these cases. Likewise, there are both consistent and inconsistent underdetermined systems,[7] but a consistent underdetermined system of linear equations is necessarily dependent.[8]

In order to move this section beyond a review of Intermediate Algebra, we now define what is meant by a linear equation in $n$ variables.

DEFINITION 1.2. A **linear equation in $n$ variables**, $x_1$, $x_2$, $\ldots$, $x_n$ is an equation of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = c$ where $a_1$, $a_2$, $\ldots a_n$ and $c$ are real numbers and at least one of $a_1$, $a_2$, $\ldots$, $a_n$ is nonzero.

Instead of using more familiar variables like $x$, $y$, and even $z$ and/or $w$ in Definition 1.2, we use subscripts to distinguish the different variables. We have no idea how many variables may be involved, so we use numbers to distinguish them instead of letters. (There is an endless supply of distinct numbers.) As an example, the linear equation $3x_1 - x_2 = 4$ represents the same relationship between the variables $x_1$ and $x_2$ as the equation $3x - y = 4$ does between the variables $x$ and $y$. In addition, just as we cannot combine the terms in the expression $3x - y$, we cannot combine the terms in the expression $3x_1 - x_2$. Coupling more than one linear equation in $n$ variables results in a **system of linear equations in $n$ variables**. When solving these systems, it becomes increasingly important to keep track of what operations are performed to which equations and to develop a strategy based on the kind of manipulations we've already employed. To this end, we first remind ourselves of the maneuvers which can be applied to a system of linear equations that result in an equivalent system.[9]

THEOREM 1.1. Given a system of equations, the following moves will result in an equivalent system of equations.

- Interchange the position of any two equations.
- Replace an equation with a nonzero multiple of itself.[a]
- Replace an equation with itself plus a nonzero multiple of another equation.

[a] That is, an equation which results from multiplying both sides of the equation by the same nonzero number.

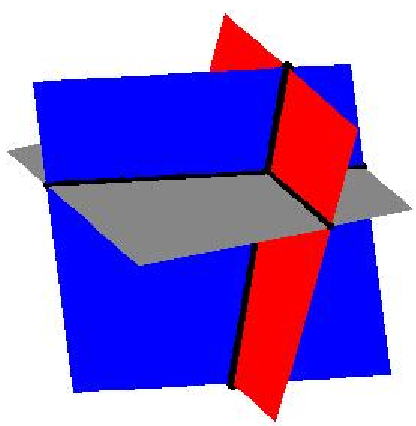
We have seen plenty of instances of the second and third moves in Theorem 1.1 when we solved the systems Example 1.1.1. The first move, while it obviously admits an equivalent system, seems silly. Our perception will change as we consider more equations and more variables in this, and later sections.

Consider the system of equations

[7] We need more than two variables to give an example of the latter.
[8] Again, experience with systems with more variables helps to see this here, as does a solid course in Linear Algebra.
[9] That is, a system with the same solution set.

$$\left\{ \begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array} \right.$$

Clearly $z = -1$, and we substitute this into the second equation $y - \frac{1}{2}(-1) = 4$ to obtain $y = \frac{7}{2}$. Finally, we substitute $y = \frac{7}{2}$ and $z = -1$ into the first equation to get $x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$, so that $x = \frac{8}{3}$. The reader can verify that these values of $x$, $y$ and $z$ satisfy all three original equations. It is tempting for us to write the solution to this system by extending the usual $(x, y)$ notation to $(x, y, z)$ and list our solution as $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$. The question quickly becomes what does an 'ordered triple' like $\left(\frac{8}{3}, \frac{7}{2}, -1\right)$ represent? Just as ordered pairs are used to locate points on the two-dimensional plane, ordered triples can be used to locate points in space.[10] Moreover, just as equations involving the variables $x$ and $y$ describe graphs of one-dimensional lines and curves in the two-dimensional plane, equations involving variables $x$, $y$, and $z$ describe objects called **surfaces** in three-dimensional space. Each of the equations in the above system can be visualized as a plane situated in three-space. Geometrically, the system is trying to find the intersection, or common point, of all three planes. If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line,[11] so our intersection point is where all three of these lines meet.



Since the geometry for equations involving more than two variables is complicated, we will focus our efforts on the algebra. Returning to the system

[10] You were asked to think about this in Exercise **??** in Section **??**.

[11] In fact, these lines are described by the parametric solutions to the systems formed by taking any two of these equations by themselves.

$$\left\{\begin{array}{rcr} x - \frac{1}{3}y + \frac{1}{2}z & = & 1 \\ y - \frac{1}{2}z & = & 4 \\ z & = & -1 \end{array}\right.$$

we note the reason it was so easy to solve is that the third equation is solved for $z$, the second equation involves only $y$ and $z$, and since the coefficient of $y$ is 1, it makes it easy to solve for $y$ using our known value for $z$. Lastly, the coefficient of $x$ in the first equation is 1 making it easy to substitute the known values of $y$ and $z$ and then solve for $x$. We formalize this pattern below for the most general systems of linear equations. Again, we use subscripted variables to describe the general case. The variable with the smallest subscript in a given equation is typically called the **leading variable** of that equation.

DEFINITION 1.3. A system of linear equations with variables $x_1$, $x_2$, $\ldots x_n$ is said to be in **triangular form** provided all of the following conditions hold:

1. The subscripts of the variables in each equation are always increasing from left to right.
2. The leading variable in each equation has coefficient 1.
3. The subscript on the leading variable in a given equation is greater than the subscript on the leading variable in the equation above it.
4. Any equation without variables[a] cannot be placed above an equation with variables.

[a]necessarily an identity or contradiction

In our previous system, if make the obvious choices $x = x_1$, $y = x_2$, and $z = x_3$, we see that the system is in triangular form.[12] An example of a more complicated system in triangular form is

$$\left\{\begin{array}{rcr} x_1 - 4x_3 + x_4 - x_6 & = & 6 \\ x_2 + 2x_3 & = & 1 \\ x_4 + 3x_5 - x_6 & = & 8 \\ x_5 + 9x_6 & = & 10 \end{array}\right.$$

Our goal henceforth will be to transform a given system of linear equations into triangular form using the moves in Theorem 1.1.

EXAMPLE 1.1.2. Use Theorem 1.1 to put the following systems into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

1. $\left\{\begin{array}{rcr} 3x - y + z & = & 3 \\ 2x - 4y + 3z & = & 16 \\ x - y + z & = & 5 \end{array}\right.$

2. $\left\{\begin{array}{rcr} 2x + 3y - z & = & 1 \\ 10x - z & = & 2 \\ 4x - 9y + 2z & = & 5 \end{array}\right.$

3. $\left\{\begin{array}{rcr} 3x_1 + x_2 + x_4 & = & 6 \\ 2x_1 + x_2 - x_3 & = & 4 \\ x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$

[12]If letters are used instead of subscripted variables, Definition 1.3 can be suitably modified using alphabetical order of the variables instead of numerical order on the subscripts of the variables.

Solution.

1. For definitiveness, we label the topmost equation in the system $E1$, the equation beneath that $E2$, and so forth. We now attempt to put the system in triangular form using an algorithm known as **Gaussian Elimination**. What this means is that, starting with $x$, we transform the system so that conditions 2 and 3 in Definition 1.3 are satisfied. Then we move on to the next variable, in this case $y$, and repeat. Since the variables in all of the equations have a consistent ordering from left to right, our first move is to get an $x$ in $E1$'s spot with a coefficient of 1. While there are many ways to do this, the easiest is to apply the first move listed in Theorem 1.1 and interchange $E1$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & 3x-y+z & = & 3 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & x-y+z & = & 5 \end{array} \right. \xrightarrow{\text{Switch } E1 \text{ and } E3} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & 3x-y+z & = & 3 \end{array} \right.$$

To satisfy Definition 1.3, we need to eliminate the $x$'s from $E2$ and $E3$. We accomplish this by replacing each of them with a sum of themselves and a multiple of $E1$. To eliminate the $x$ from $E2$, we need to multiply $E1$ by $-2$ then add; to eliminate the $x$ from $E3$, we need to multiply $E1$ by $-3$ then add. Applying the third move listed in Theorem 1.1 twice, we get

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & 2x-4y+3z & = & 16 \\ (E3) & 3x-y+z & = & 3 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -3E1+E3]{\text{Replace } E2 \text{ with } -2E1+E2} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & -2y+z & = & 6 \\ (E3) & 2y-2z & = & -12 \end{array} \right.$$

Now we enforce the conditions stated in Definition 1.3 for the variable $y$. To that end we need to get the coefficient of $y$ in $E2$ equal to 1. We apply the second move listed in Theorem 1.1 and replace $E2$ with itself times $-\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & -2y+z & = & 6 \\ (E3) & 2y-2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{2}E2} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array} \right.$$

To eliminate the $y$ in $E3$, we add $-2E2$ to it.

$$\left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & 2y-2z & = & -12 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -2E2+E3} \left\{ \begin{array}{lrcr} (E1) & x-y+z & = & 5 \\ (E2) & y-\frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right.$$

Finally, we apply the second move from Theorem 1.1 one last time and multiply $E3$ by $-1$ to satisfy the conditions of Definition 1.3 for the variable $z$.

$$\left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & -z & = & -6 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } -1E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & z & = & 6 \end{array} \right.$$

Now we proceed to substitute. Plugging in $z = 6$ into $E2$ gives $y - 3 = -3$ so that $y = 0$. With $y = 0$ and $z = 6$, $E1$ becomes $x - 0 + 6 = 5$, or $x = -1$. Our solution is $(-1, 0, 6)$. We leave it to the reader to check that substituting the respective values for $x$, $y$, and $z$ into the original system results in three identities. Since we have found a solution, the system is consistent; since there are no free variables, it is independent.

2. Proceeding as we did in 1, our first step is to get an equation with $x$ in the $E1$ position with 1 as its coefficient. Since there is no easy fix, we multiply $E1$ by $\frac{1}{2}$.

$$\left\{ \begin{array}{lrcr} (E1) & 2x + 3y - z & = & 1 \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{2}E1} \left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right.$$

Now it's time to take care of the $x$'s in $E2$ and $E3$.

$$\left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & 10x - z & = & 2 \\ (E3) & 4x - 9y + 2z & = & 5 \end{array} \right. \xrightarrow[\text{Replace } E3 \text{ with } -4E1 + E3]{\text{Replace } E2 \text{ with } -10E1 + E2} \left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

Our next step is to get the coefficient of $y$ in $E2$ equal to 1. To that end, we have

$$\left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y + 4z & = & -3 \\ (E3) & -15y + 4z & = & 3 \end{array} \right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{1}{15}E2} \left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array} \right.$$

Finally, we rid $E3$ of $y$.

$$\left\{ \begin{array}{lrcr} (E1) & x + \frac{3}{2}y - \frac{1}{2}z & = & \frac{1}{2} \\ (E2) & y - \frac{4}{15}z & = & \frac{1}{5} \\ (E3) & -15y + 4z & = & 3 \end{array} \right. \xrightarrow{\text{Replace } E3 \text{ with } 15E2 + E3} \left\{ \begin{array}{lrcr} (E1) & x - y + z & = & 5 \\ (E2) & y - \frac{1}{2}z & = & -3 \\ (E3) & 0 & = & 6 \end{array} \right.$$

The last equation, $0 = 6$, is a contradiction so the system has no solution. According to Theorem 1.1, since this system has no solutions, neither does the original, thus we have an inconsistent system.

3. For our last system, we begin by multiplying $E1$ by $\frac{1}{3}$ to get a coefficient of 1 on $x_1$.

$$\left\{\begin{array}{lrcl} (E1) & 3x_1 + x_2 + x_4 & = & 6 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Replace } E1 \text{ with } \frac{1}{3}E1} \left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

Next we eliminate $x_1$ from $E2$

$$\left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & 2x_1 + x_2 - x_3 & = & 4 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -2E1+E2]{\text{Replace } E2} \left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right.$$

We switch $E2$ and $E3$ to get a coefficient of 1 for $x_2$.

$$\left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \\ (E3) & x_2 - 3x_3 - 2x_4 & = & 0 \end{array}\right. \xrightarrow{\text{Switch } E2 \text{ and } E3} \left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right.$$

Finally, we eliminate $x_2$ in $E3$.

$$\left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & \frac{1}{3}x_2 - x_3 - \frac{2}{3}x_4 & = & 0 \end{array}\right. \xrightarrow[\text{with } -\frac{1}{3}E2+E3]{\text{Replace } E3} \left\{\begin{array}{lrcl} (E1) & x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_4 & = & 2 \\ (E2) & x_2 - 3x_3 - 2x_4 & = & 0 \\ (E3) & 0 & = & 0 \end{array}\right.$$

Equation $E3$ reduces to $0 = 0$,which is always true. Since we have no equations with $x_3$ or $x_4$ as leading variables, they are both free, which means we have a consistent dependent system. We parametrize the solution set by letting $x_3 = s$ and $x_4 = t$ and obtain from $E2$ that $x_2 = 3s + 2t$. Substituting this and $x_4 = t$ into $E1$, we have $x_1 + \frac{1}{3}(3s + 2t) + \frac{1}{3}t = 2$ which gives $x_1 = 2 - s - t$. Our solution is the set $\{(2-s-t, 2s+3t, s, t) : -\infty < s, t < \infty\}$.[13] We leave it to the reader to verify that the substitutions $x_1 = 2 - s - t$, $x_2 = 3s + 2t$, $x_3 = s$ and $x_4 = t$ satisfy the equations in the original system. □

Like all algorithms, Gaussian Elimination has the advantage of always producing what we need, but it can also be inefficient at times. For example, when solving 2 above, it is clear after we eliminated the $x$'s in the second step to get the system

[13] Here, any choice of $s$ and $t$ will determine a solution which is a point in 4-dimensional space. Yeah, we have trouble visualizing that, too.

$$\left\{\begin{array}{lrcr} (E1) & x+\frac{3}{2}y-\frac{1}{2}z & = & \frac{1}{2} \\ (E2) & -15y+4z & = & -3 \\ (E3) & -15y+4z & = & 3 \end{array}\right.$$

that equations $E2$ and $E3$ when taken together form a contradiction since we have identical left hand sides and different right hand sides. The algorithm takes two more steps to reach this contradiction. We also note that substitution in Gaussian Elimination is delayed until all the elimination is done, thus it gets called **back-substitution**. This may also be inefficient in many cases. Rest assured, the technique of substitution as you may have learned it in Intermediate Algebra will once again take center stage in Section 1.7. Lastly, we note that the system in 3 above is underdetermined, and as it is consistent, we have free variables in our answer. We close this section with a standard 'mixture' type application of systems of linear equations.

Example 1.1.3. Lucas needs to create a 500 milliliters (mL) of a 40% acid solution. He has stock solutions of 30% and 90% acid as well as all of the distilled water he wants. Set-up and solve a system of linear equations which determines all of the possible combinations of the stock solutions and water which would produce the required solution.

Solution. We are after three unknowns, the amount (in mL) of the 30% stock solution (which we'll call $x$), the amount (in mL) of the 90% stock solution (which we'll call $y$) and the amount (in mL) of water (which we'll call $w$). We now need to determine some relationships between these variables. Our goal is to produce 500 milliliters of a 40% acid solution. This product has two defining characteristics. First, it must be 500 mL; second, it must be 40% acid. We take each of these qualities in turn. First, the total volume of 500 mL must be the sum of the contributed volumes of the two stock solutions and the water. That is

$$\text{amount of 30\% stock solution} + \text{amount of 90\% stock solution} + \text{amount of water} = 500\,\text{mL}$$

Using our defined variables, this reduces to $x+y+w=500$. Next, we need to make sure the final solution is 40% acid. Since water contains no acid, the acid will come from the stock solutions only. We find 40% of 500 mL to be 200 mL which means the final solution must contain 200 mL of acid. We have

$$\text{amount of acid in 30\% stock solution} + \text{amount of acid 90\% stock solution} = 200\,\text{mL}$$

The amount of acid in $x$ mL of 30% stock is $0.30x$ and the amount of acid in $y$ mL of 90% solution is $0.90y$. We have $0.30x+0.90y=200$. Converting to fractions,[14] our system of equations becomes

$$\left\{\begin{array}{rcl} x+y+w & = & 500 \\ \frac{3}{10}x+\frac{9}{10}y & = & 200 \end{array}\right.$$

We first eliminate the $x$ from the second equation

[14] We do this only because we believe students can use all of the practice with fractions they can get!

$$\left\{\begin{array}{lrcl} (E1) & x+y+w & = & 500 \\ (E2) & \frac{3}{10}x+\frac{9}{10}y & = & 200 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } -\frac{3}{10}E1+E2} \left\{\begin{array}{lrcl} (E1) & x+y+w & = & 500 \\ (E2) & \frac{3}{5}y-\frac{3}{10}w & = & 50 \end{array}\right.$$

Next, we get a coefficient of 1 on the leading variable in $E2$

$$\left\{\begin{array}{lrcl} (E1) & x+y+w & = & 500 \\ (E2) & \frac{3}{5}y-\frac{3}{10}w & = & 50 \end{array}\right. \xrightarrow{\text{Replace } E2 \text{ with } \frac{5}{3}E2} \left\{\begin{array}{lrcl} (E1) & x+y+w & = & 500 \\ (E2) & y-\frac{1}{2}w & = & \frac{250}{3} \end{array}\right.$$

Notice that we have no equation to determine $w$, and as such, $w$ is free. We set $w=t$ and from $E2$ get $y=\frac{1}{2}t+\frac{250}{3}$. Substituting into $E1$ gives $x+\left(\frac{1}{2}t+\frac{250}{3}\right)+t=500$ so that $x=-\frac{3}{2}t+\frac{1250}{3}$. This system is consistent, dependent and its solution set is $\left\{\left(-\frac{3}{2}t+\frac{1250}{3},\frac{1}{2}t+\frac{250}{3},t\right) : -\infty < t < \infty\right\}$. While this answer checks algebraically, we have neglected to take into account that $x$, $y$ and $w$, being amounts of acid and water, need to be nonnegative. That is, $x \geq 0$, $y \geq 0$ and $w \geq 0$. The constraint $x \geq 0$ gives us $-\frac{3}{2}t+\frac{1250}{3} \geq 0$, or $t \leq \frac{2500}{9}$. From $y \geq 0$, we get $\frac{1}{2}t+\frac{250}{3} \geq 0$ or $t \geq -\frac{500}{3}$. The condition $z \geq 0$ yields $t \geq 0$, and we see that when we take the set theoretic intersection of these intervals, we get $0 \leq t \leq \frac{2500}{9}$. Our final answer is $\left\{\left(-\frac{3}{2}t+\frac{1250}{3},\frac{1}{2}t+\frac{250}{3},t\right) : 0 \leq t \leq \frac{2500}{9}\right\}$. Of what practical use is our answer? Suppose there is only 100 mL of the 90% solution remaining and it is due to expire. Can we use all of it to make our required solution? We would have $y=100$ so that $\frac{1}{2}t+\frac{250}{3}=100$, and we get $t=\frac{100}{3}$. This means the amount of 30% solution required is $x=-\frac{3}{2}t+\frac{1250}{3}=-\frac{3}{2}\left(\frac{100}{3}\right)+\frac{1250}{3}=\frac{1100}{3}$ mL, and for the water, $w=t=\frac{100}{3}$ mL. The reader is invited to check that mixing these three amounts of our constituent solutions produces the required 40% acid mix. $\square$

### 1.1.1 Exercises

1. Put the following systems of linear equations into triangular form and then solve the system if possible. Classify each system as consistent independent, consistent dependent, or inconsistent.

(a) $\left\{ \begin{array}{rcr} -5x + y & = & 17 \\ x + y & = & 5 \end{array} \right.$

(b) $\left\{ \begin{array}{rcr} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

(c) $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 5 \end{array} \right.$

(d) $\left\{ \begin{array}{rcr} 4x - y + z & = & 5 \\ 2y + 6z & = & 30 \\ x + z & = & 6 \end{array} \right.$

(e) $\left\{ \begin{array}{rcr} x + y + z & = & -17 \\ y - 3z & = & 0 \end{array} \right.$

(f) $\left\{ \begin{array}{rcr} x - 2y + 3z & = & 7 \\ -3x + y + 2z & = & -5 \\ 2x + 2y + z & = & 3 \end{array} \right.$

(g) $\left\{ \begin{array}{rcr} 3x - 2y + z & = & -5 \\ x + 3y - z & = & 12 \\ x + y + 2z & = & 0 \end{array} \right.$

(h) $\left\{ \begin{array}{rcr} 2x - y + z & = & -1 \\ 4x + 3y + 5z & = & 1 \\ 5y + 3z & = & 4 \end{array} \right.$

(i) $\left\{ \begin{array}{rcr} x - y + z & = & -4 \\ -3x + 2y + 4z & = & -5 \\ x - 5y + 2z & = & -18 \end{array} \right.$

(j) $\left\{ \begin{array}{rcr} 2x - 4y + z & = & -7 \\ x - 2y + 2z & = & -2 \\ -x + 4y - 2z & = & 3 \end{array} \right.$

(k) $\left\{ \begin{array}{rcr} 2x - y + z & = & 1 \\ 2x + 2y - z & = & 1 \\ 3x + 6y + 4z & = & 9 \end{array} \right.$

(l) $\left\{ \begin{array}{rcr} x - 3y - 4z & = & 3 \\ 3x + 4y - z & = & 13 \\ 2x - 19y - 19z & = & 2 \end{array} \right.$

(m) $\left\{ \begin{array}{rcr} x + y + z & = & 4 \\ 2x - 4y - z & = & -1 \\ x - y & = & 2 \end{array} \right.$

(n) $\left\{ \begin{array}{rcr} x - y + z & = & 8 \\ 3x + 3y - 9z & = & -6 \\ 7x - 2y + 5z & = & 39 \end{array} \right.$

(o) $\left\{ \begin{array}{rcr} 2x - 3y + z & = & -1 \\ 4x - 4y + 4z & = & -13 \\ 6x - 5y + 7z & = & -25 \end{array} \right.$

(p) $\left\{ \begin{array}{rcr} 2x_1 + x_2 - 12x_3 - x_4 & = & 16 \\ -x_1 + x_2 + 12x_3 - 4x_4 & = & -5 \\ 3x_1 + 2x_2 - 16x_3 - 3x_4 & = & 25 \\ x_1 + 2x_2 - 5x_4 & = & 11 \end{array} \right.$

(q) $\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$

(r) $\left\{ \begin{array}{rcr} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_1 + x_2 + 5x_3 - 3x_4 & = & 0 \\ x_2 + 5x_3 - 3x_4 & = & 1 \\ x_1 - 2x_2 - 10x_3 + 6x_4 & = & -1 \end{array} \right.$

2. Find two other forms of the parametric solution to Exercise 1c above by reorganizing the equations so that $x$ or $y$ can be the free variable.

3. At The Old Home Fill'er Up and Keep on a-Truckin' Cafe, Mavis mixes two different types of coffee beans to produce a house blend. The first type costs \$3 per pound and the second costs \$8 per pound. How much of each type does Mavis use to make 50 pounds of a blend which costs \$6 per pound?

4. At The Crispy Critter's Head Shop and Patchouli Emporium along with their dried up weeds, sunflower seeds and astrological postcards they sell an herbal tea blend. By weight, Type I herbal tea is 30% peppermint, 40% rose hips and 30% chamomile, Type II has percents 40%, 20% and 40%, respectively, and Type III has percents 35%, 30% and 35%, respectively. How much of each Type of tea is needed to make 2 pounds of a new blend of tea that is equal parts peppermint, rose hips and chamomile?

5. Discuss with your classmates how you would approach Exercise 4 above if they needed to use up a pound of Type I tea to make room on the shelf for a new canister.

6. Discuss with your classmates why it is impossible to mix a 20% acid solution with a 40% acid solution to produce a 60% acid solution. If you were to try to make 100 mL of a 60% acid solution using stock solutions at 20% and 40%, respectively, what would the triangular form of the resulting system look like?

### 1.1.2 Answers

1. Because triangular form is not unique, we give only one possible answer to that part of the question. Yours may be different and still be correct.

   (a) $\left\{\begin{array}{rcr} x+y & = & 5 \\ y & = & 7 \end{array}\right.$ Consistent independent Solution $(-2,7)$

   (b) $\left\{\begin{array}{rcr} x-\frac{5}{3}y-\frac{7}{3}z & = & -\frac{7}{3} \\ y+\frac{5}{4}z & = & 2 \\ z & = & 0 \end{array}\right.$ Consistent independent Solution $(1,2,0)$

   (c) $\left\{\begin{array}{rcr} x-\frac{1}{4}y+\frac{1}{4}z & = & \frac{5}{4} \\ y+3z & = & 15 \\ 0 & = & 0 \end{array}\right.$ Consistent dependent Solution $(-t+5,-3t+15,t)$ for all real numbers $t$

   (d) $\left\{\begin{array}{rcr} x-\frac{1}{4}y+\frac{1}{4}z & = & \frac{5}{4} \\ y+3z & = & 15 \\ 0 & = & 1 \end{array}\right.$ Inconsistent No solution

   (e) $\left\{\begin{array}{rcr} x+y+z & = & -17 \\ y-3z & = & 0 \end{array}\right.$ Consistent dependent Solution $(-4t-17,3t,t)$ for all real numbers $t$

   (f) $\left\{\begin{array}{rcr} x-2y+3z & = & 7 \\ y-\frac{11}{5}z & = & -\frac{16}{5} \\ z & = & 1 \end{array}\right.$ Consistent independent Solution $(2,-1,1)$

   (g) $\left\{\begin{array}{rcr} x+y+2z & = & 0 \\ y-\frac{3}{2}z & = & 6 \\ z & = & -2 \end{array}\right.$ Consistent independent Solution $(1,3,-2)$

   (h) $\left\{\begin{array}{rcr} x-\frac{1}{2}y+\frac{1}{2}z & = & -\frac{1}{2} \\ y+\frac{3}{5}z & = & \frac{3}{5} \\ 0 & = & 1 \end{array}\right.$ Inconsistent no solution

   (i) $\left\{\begin{array}{rcr} x-y+z & = & -4 \\ y-7z & = & 17 \\ z & = & -2 \end{array}\right.$ Consistent independent Solution $(1,3,-2)$

   (j) $\left\{\begin{array}{rcr} x-2y+2z & = & -2 \\ y & = & \frac{1}{2} \\ z & = & 1 \end{array}\right.$ Consistent independent Solution $\left(-3,\frac{1}{2},1\right)$

(k) $\left\{\begin{array}{rcr} x - \frac{1}{2}y + \frac{1}{2}z & = & \frac{1}{2} \\ y - \frac{2}{3}z & = & 0 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $\left(\frac{1}{3}, \frac{2}{3}, 1\right)$

(l) $\left\{\begin{array}{rcr} x - 3y - 4z & = & 3 \\ y + \frac{11}{13}z & = & \frac{4}{13} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(\frac{19}{13}t + \frac{51}{13}, -\frac{11}{13}t + \frac{4}{13}, t\right)$
for all real numbers $t$

(m) $\left\{\begin{array}{rcr} x + y + z & = & 4 \\ y + \frac{1}{2}z & = & \frac{3}{2} \\ 0 & = & 1 \end{array}\right.$

Inconsistent
no solution

(n) $\left\{\begin{array}{rcr} x - y + z & = & 8 \\ y - 2z & = & -5 \\ z & = & 1 \end{array}\right.$

Consistent independent
Solution $(4, -3, 1)$

(o) $\left\{\begin{array}{rcr} x - \frac{3}{2}y + \frac{1}{2}z & = & -\frac{1}{2} \\ y + z & = & -\frac{11}{2} \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $\left(-2t - \frac{35}{4}, -t - \frac{11}{2}, t\right)$
for all real numbers $t$

(p) $\left\{\begin{array}{rcr} x_1 + \frac{2}{3}x_2 - \frac{16}{3}x_3 - x_4 & = & \frac{25}{3} \\ x_2 + 4x_3 - 3x_4 & = & 2 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}\right.$

Consistent dependent
Solution $(8s - t + 7, -4s + 3t + 2, s, t)$
for all real numbers $s$ and $t$

(q) $\left\{\begin{array}{rcr} x_1 - x_3 & = & -2 \\ x_2 - \frac{1}{2}x_4 & = & 0 \\ x_3 - \frac{1}{2}x_4 & = & 1 \\ x_4 & = & 4 \end{array}\right.$

Consistent independent
Solution $(1, 2, 3, 4)$

(r) $\left\{\begin{array}{rcr} x_1 - x_2 - 5x_3 + 3x_4 & = & -1 \\ x_2 + 5x_3 - 3x_4 & = & \frac{1}{2} \\ 0 & = & 1 \\ 0 & = & 0 \end{array}\right.$

Inconsistent
No solution

2. If $x$ is the free variable then the solution is $(t, 3t, -t+5)$ and if $y$ is the free variable then the solution is $\left(\frac{1}{3}t, t, -\frac{1}{3}t + 5\right)$.

3. Mavis needs 20 pounds of \$3 per pound coffee and 30 pounds of \$8 per pound coffee.

4. $\frac{4}{3} - \frac{1}{2}t$ pounds of Type I, $\frac{2}{3} - \frac{1}{2}t$ pounds of Type II and $t$ pounds of Type III where $0 \leq t \leq \frac{4}{3}$.

## 1.2 Determinants and Cramer's Rule

### 1.2.1 Definition and Properties of the Determinant

In this section we assign to each square matrix $A$ a real number, called the **determinant** of $A$, which will eventually lead us to yet another technique for solving consistent independent systems of linear equations. The determinant is defined recursively, that is, we define it for $1 \times 1$ matrices and give a rule by which we can reduce determinants of $n \times n$ matrices to a sum of determinants of $(n-1) \times (n-1)$ matrices.[1] This means we will be able to evaluate the determinant of a $2 \times 2$ matrix as a sum of the determinants of $1 \times 1$ matrices; the determinant of a $3 \times 3$ matrix as a sum of the determinants of $2 \times 2$ matrices, and so forth. To explain how we will take an $n \times n$ matrix and distill from it an $(n-1) \times (n-1)$, we use the following notation.

Definition 1.4. Given an $n \times n$ matrix $A$ where $n > 1$, the matrix $A_{ij}$ is the $(n-1) \times (n-1)$ matrix formed by deleting the $i$th row of $A$ and the $j$th column of $A$.

For example, using the matrix $A$ below, we find the matrix $A_{23}$ by deleting the second row and third column of $A$.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Delete } R2 \text{ and } C3} A_{23} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

We are now in the position to define the determinant of a matrix.

Definition 1.5. Given an $n \times n$ matrix $A$ the **determinant of $A$**, denoted $\det(A)$, is defined as follows

- If $n = 1$, then $A = [a_{11}]$ and $\det(A) = \det([a_{11}]) = a_{11}$.
- If $n > 1$, then $A = [a_{ij}]_{n \times n}$ and

$$\det(A) = \det\left([a_{ij}]_{n \times n}\right) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

There are two commonly used notations for the determinant of a matrix $A$: '$\det(A)$' and '$|A|$' We have chosen to use the notation $\det(A)$ as opposed to $|A|$ because we find that the latter is often confused with absolute value, especially in the context of a $1 \times 1$ matrix. In the expansion $a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n}\det(A_{1n})$, the notation '$+ - \ldots + (-1)^{1+n} a_{1n}$' means that the signs alternate and the final sign is dictated by the sign of the quantity $(-1)^{1+n}$. Since the entries $a_{11}$, $a_{12}$ and so forth up through $a_{1n}$ comprise the first row of $A$, we say we are finding the determinant of $A$ by 'expanding along the first row'. Later in the section, we will develop a formula for $\det(A)$ which allows us to find it by expanding along any row.

Applying Definition 1.13 to the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$ we get

[1] We will talk more about the term 'recursively' in Section ??.

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}\right) \\
&= 4\det(A_{11}) - (-3)\det(A_{12}) \\
&= 4\det([1]) + 3\det([2]) \\
&= 4(1) + 3(2) \\
&= 10
\end{aligned}$$

For a generic $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we get

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= a\det(A_{11}) - b\det(A_{12}) \\
&= a\det([d]) - b\det([c]) \\
&= ad - bc
\end{aligned}$$

This formula is worth remembering

Equation 1.1. For a $2 \times 2$ matrix,

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Applying Definition 1.13 to the $3 \times 3$ matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$ we obtain

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}\right) \\
&= 3\det(A_{11}) - 1\det(A_{12}) + 2\det(A_{13}) \\
&= 3\det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 2 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\right) \\
&= 3((-1)(4) - (5)(1)) - ((0)(4) - (5)(2)) + 2((0)(1) - (-1)(2)) \\
&= 3(-9) - (-10) + 2(2) \\
&= -13
\end{aligned}$$

To evaluate the determinant of a $4 \times 4$ matrix, we would have to evaluate the determinants of *four* $3 \times 3$ matrices, each of which involves the finding the determinants of *three* $2 \times 2$ matrices. As you can see, our method of evaluating determinants quickly gets out of hand and many of you may be reaching for the calculator. There is some mathematical machinery which can assist us in calculating determinants and we present that here. Before we state the theorem, we need some more terminology.

DEFINITION 1.6. Let $A$ be an $n \times n$ matrix and $A_{ij}$ be defined as in Definition 1.12. The ***ij* minor** of $A$, denoted $M_{ij}$ is defined by $M_{ij} = \det(A_{ij})$. The ***ij* cofactor** of $A$, denoted $C_{ij}$ is defined by $C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij})$.

We note that in Definition 1.13, the sum

$$a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \ldots + (-1)^{1+n} a_{1n} \det(A_{1n})$$

can be rewritten as

$$a_{11}(-1)^{1+1} \det(A_{11}) + a_{12}(-1)^{1+2} \det(A_{12}) + \ldots + a_{1n}(-1)^{1+n} \det(A_{1n})$$

which, in the language of cofactors is

$$a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$$

We are now ready to state our main theorem concerning determinants.

THEOREM 1.2. **Properties of the Determinant:** Let $A = [a_{ij}]_{n \times n}$.

- We may find the determinant by expanding along any row. That is, for any $1 \leq k \leq n$,

$$\det(A) = a_{k1}C_{k1} + a_{k2}C_{k2} + \ldots + a_{kn}C_{kn}$$

- If $A'$ is the matrix obtained from $A$ by:
  - interchanging any two rows, then $\det(A') = -\det(A)$.
  - replacing a row with a nonzero multiple (say $c$) of itself, then $\det(A') = c\det(A)$
  - replacing a row with itself plus a multiple of another row, then $\det(A') = \det(A)$
- If $A$ has two identical rows, or a row consisting of all 0's, then $\det(A) = 0$.
- If $A$ is upper or lower triangular,[a] then $\det(A)$ is the product of the entries on the main diagonal.[b]
- If $B$ is an $n \times n$ matrix, then $\det(AB) = \det(A)\det(B)$.
- $\det(A^n) = \det(A)^n$ for all natural numbers $n$.
- $A$ is invertible if and only if $\det(A) \neq 0$. In this case, $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$.

[a] See Exercise 5 in 1.3.

[b] See page 36 in Section 1.3.

Unfortunately, while we can easily *demonstrate* the results in Theorem 1.7, the proofs of most of these properties are beyond the scope of this text. We could prove these properties for generic $2 \times 2$ or even $3 \times 3$ matrices by brute force computation, but this manner of proof belies the elegance and symmetry of the determinant. We will prove what few properties we can after we have developed some more tools such as the Principle of Mathematical Induction in Section ??.[2] For the moment, let us demonstrate some of the properties listed in Theorem 1.7 on the matrix $A$ below. (Others will be discussed in the Exercises.)

$$A = \left[ \begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array} \right]$$

We found $\det(A) = -13$ by expanding along the first row. Theorem 1.7 guarantees that we will get the same result if we expand along the second row. (Doing so would take advantage of the 0 there.)

$$\begin{array}{rcl} \det \left( \left[ \begin{array}{rrr} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{array} \right] \right) & = & 0C_{21} + (-1)C_{22} + 5C_{23} \\ & = & (-1)(-1)^{2+2} \det\left(A_{22}\right) + 5(-1)^{2+3} \det\left(A_{23}\right) \\ & = & -\det \left( \left[ \begin{array}{rr} 3 & 2 \\ 2 & 4 \end{array} \right] \right) - 5 \det \left( \left[ \begin{array}{rr} 3 & 1 \\ 2 & 1 \end{array} \right] \right) \\ & = & -((3)(4) - (2)(2)) - 5((3)(1) - (2)(1)) \\ & = & -8 - 5 \\ & = & -13 \checkmark \end{array}$$

In general, the sign of $(-1)^{i+j}$ in front of the minor in the expansion of the determinant follows an alternating pattern. Below is the pattern for $2 \times 2$, $3 \times 3$ and $4 \times 4$ matrices, and it extends naturally to higher dimensions.

$$\left[ \begin{array}{cc} + & - \\ - & + \end{array} \right] \qquad \left[ \begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \right] \qquad \left[ \begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array} \right]$$

The reader is cautioned, however, against reading too much into these sign patterns. In the example above, we expanded the $3 \times 3$ matrix $A$ by its second row and the term which corresponds to the second entry ended up being negative even though the sign attached to the minor is $(+)$. These signs represent only the signs of the $(-1)^{i+j}$ in the formula; the sign of the corresponding entry as well as the minor itself determine the ultimate sign of the term in the expansion of the determinant.

To illustrate some of the other properties in Theorem 1.7, we use row operations to transform our $3 \times 3$ matrix $A$ into an upper triangular matrix, keeping track of the row operations, and labeling

[2] For a very elegant treatment, take a course in Linear Algebra. There, you will most likely see the treatment of determinants logically reversed than what is presented here. Specifically, the determinant is defined as a function which takes a square matrix to a real number and satisfies some of the properties in Theorem 1.7. From that function, a formula for the determinant is developed.

each successive matrix.[3]

$$\underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}}_{A} \xrightarrow[\text{with } -\frac{2}{3}R1+R3]{\text{Replace } R3} \underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & \frac{1}{3} & \frac{8}{3} \end{bmatrix}}_{B} \xrightarrow[\frac{1}{3}R2+R3]{\text{Replace } R3 \text{ with}} \underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & \frac{13}{3} \end{bmatrix}}_{C}$$

Theorem 1.7 guarantees us that $\det(A) = \det(B) = \det(C)$ since we are replacing a row with itself plus a multiple of another row moving from one matrix to the next. Furthermore, since $C$ is upper triangular, $\det(C)$ is the product of the entries on the main diagonal, in this case $\det(C) = (3)(-1)\left(\frac{13}{3}\right) = -13$. This demonstrates the utility of using row operations to assist in calculating determinants. This also sheds some light on the connection between a determinant and invertibility. Recall from Section 1.4 that in order to find $A^{-1}$, we attempt to transform $A$ to $I_n$ using row operations

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \xrightarrow{\text{Gauss Jordan Elimination}} \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

As we apply our allowable row operations on $A$ to put it into reduced row echelon form, the determinant of the intermediate matrices can vary from the determinant of $A$ by at most a *nonzero* multiple. This means that if $\det(A) \neq 0$, then the determinant of $A$'s reduced row echelon form must also be nonzero, which, according to Definition 1.4 means that all the main diagonal entries on $A$'s reduced row echelon form must be 1. That is, $A$'s reduced row echelon form is $I_n$, and $A$ is invertible. Conversely, if $A$ is invertible, then $A$ can be transformed into $I_n$ using row operations. Since $\det(I_n) = 1 \neq 0$, our same logic implies $\det(A) \neq 0$. Basically, we have established that the determinant *determines* whether or not the matrix $A$ is invertible.[4]

It is worth noting that when we first introduced the notion of a matrix inverse, it was in the context of solving a linear matrix equation. In effect, we were trying to 'divide' both sides of the matrix equation $AX = B$ by the matrix $A$. Just like we cannot divide a real number by 0, Theorem 1.7 tells us we cannot 'divide' by a matrix whose *determinant* is 0. We also know that if the coefficient matrix of a system of linear equations is invertible, then system is consistent and independent. It follows, then, that if the determinant of said coefficient is not zero, the system is consistent and independent.

### 1.2.2 Cramer's Rule and Matrix Adjoints

In this section, we introduce a theorem which enables us to solve a system of linear equations by means of determinants only. As usual, the theorem is stated in full generality, using numbered unknowns $x_1$, $x_2$, etc., instead of the more familiar letters $x$, $y$, $z$, etc. The proof of the general case is best left to a course in Linear Algebra.

[3]Essentially, we follow the Gauss Jordan algorithm but we don't care about getting leading 1's.

[4]As we will see in Section 1.5.2, determinants (specifically cofactors) are deeply connected with the inverse of a matrix.

THEOREM 1.3. **Cramer's Rule:** Suppose $AX = B$ is the matrix form of a system of $n$ linear equations in $n$ unknowns where $A$ is the coefficient matrix, $X$ is the unknowns matrix, and $B$ is the constant matrix. If $\det(A) \neq 0$, then the corresponding system is consistent and independent and the solution for unknowns $x_1, x_2, \ldots x_n$ is given by:

$$x_j = \frac{\det(A_j)}{\det(A)},$$

where $A_j$ is the matrix $A$ whose $j$th column has been replaced by the constants in $B$.

In words, Cramer's Rule tells us we can solve for each unknown, one at a time, by finding the ratio of the determinant of $A_j$ to that of the determinant of the coefficient matrix. The matrix $A_j$ is found by replacing the column in the coefficient matrix which holds the coefficients of $x_j$ with the constants of the system. The following example fleshes out this method.

EXAMPLE 1.2.1. Use Cramer's Rule to solve for the indicated unknowns.

1. Solve $\left\{\begin{array}{rcr} 2x_1 - 3x_2 & = & 4 \\ 5x_1 + x_2 & = & -2 \end{array}\right.$ for $x_1$ and $x_2$

2. Solve $\left\{\begin{array}{rcr} 2x - 3y + z & = & -1 \\ x - y + z & = & 1 \\ 3x - 4z & = & 0 \end{array}\right.$ for $z$.

SOLUTION.

1. Writing this system in matrix form, we find

$$A = \left[\begin{array}{rr} 2 & -3 \\ 5 & 1 \end{array}\right] \qquad X = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \qquad B = \left[\begin{array}{r} 4 \\ -2 \end{array}\right]$$

   To find the matrix $A_1$, we remove the column of the coefficient matrix $A$ which holds the coefficients of $x_1$ and replace it with the corresponding entries in $B$. Likewise, we replace the column of $A$ which corresponds to the coefficients of $x_2$ with the constants to form the matrix $A_2$. This yields

$$A_1 = \left[\begin{array}{rr} 4 & -3 \\ -2 & 1 \end{array}\right] \qquad A_2 = \left[\begin{array}{rr} 2 & 4 \\ 5 & -2 \end{array}\right]$$

   Computing determinants, we get $\det(A) = 17$, $\det(A_1) = -2$ and $\det(A_2) = -24$, so that

$$x_1 = \frac{\det(A_1)}{\det(A)} = -\frac{2}{17} \qquad x_2 = \frac{\det(A_2)}{\det(A)} = -\frac{24}{17}$$

   The reader can check that the solution to the system is $\left(-\frac{2}{17}, -\frac{24}{17}\right)$.

2. To use Cramer's Rule to find $z$, we identify $x_3$ as $z$. We have

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad A_3 = A_z = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

Expanding both $\det(A)$ and $\det(A_z)$ along the third rows (to take advantage of the 0's) gives

$$z = \frac{\det(A_z)}{\det(A)} = \frac{-12}{-10} = \frac{6}{5}$$

The reader is encouraged to solve this system for $x$ and $y$ similarly and check the answer. $\square$

Our last application of determinants is to develop an alternative method for finding the inverse of a matrix.[5] Let us consider the $3 \times 3$ matrix $A$ which we so extensively studied in Section 1.5.1

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

We found through a variety of methods that $\det(A) = -13$. To our surprise and delight, its inverse below has a remarkable number of 13's in the denominators of its entries. This is no coincidence.

$$A^{-1} = \begin{bmatrix} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{bmatrix}$$

Recall that to find $A^{-1}$, we are essentially solving the matrix equation $AX = I_3$, where $X = [x_{ij}]_{3 \times 3}$ is a $3 \times 3$ matrix. Because of how matrix multiplication is defined, the first column of $I_3$ is the product of $A$ with the first column of $X$, the second column of $I_3$ is the product of $A$ with the second column of $X$ and the third column of $I_3$ is the product of $A$ with the third column of $X$.[6] In other words, we are solving three equations

$$A \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can solve each of these systems using Cramer's Rule. Focusing on the first system, we have

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 1 & 4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 5 \\ 2 & 0 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

[5] We are developing a *method* in the forthcoming discussion. As with the discussion in Section 1.4 when we developed the first algorithm to find matrix inverses, we ask that you indulge us.

[6] The reader is encouraged to stop and think this through.

If we expand $\det(A_1)$ along the first row, we get

$$\begin{aligned}\det(A_1) &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 0 & 5 \\ 0 & 4 \end{bmatrix}\right) + 2\det\left(\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1 & 5 \\ 1 & 4 \end{bmatrix}\right)\end{aligned}$$

Amazingly, this is none other than the $C_{11}$ cofactor of $A$. The reader is invited to check this, as well as the claims that $\det(A_2) = C_{12}$ and $\det(A_3) = C_{13}$.[7] (To see this, though it seems unnatural to do so, expand along the first row.) Cramer's Rule tells us

$$x_{11} = \frac{\det(A_1)}{\det(A)} = \frac{C_{11}}{\det(A)}, \quad x_{21} = \frac{\det(A_2)}{\det(A)} = \frac{C_{12}}{\det(A)}, \quad x_{31} = \frac{\det(A_3)}{\det(A)} = \frac{C_{13}}{\det(A)}$$

So the first column of the inverse matrix $X$ is:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{\det(A)} \\ \dfrac{C_{12}}{\det(A)} \\ \dfrac{C_{13}}{\det(A)} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

Notice the reversal of the subscripts going from the unknown to the corresponding cofactor of $A$. This trend continues and we get

$$\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \end{bmatrix} \qquad \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} C_{31} \\ C_{32} \\ C_{33} \end{bmatrix}$$

Putting all of these together, we have obtained a new and surprising formula for $A^{-1}$, namely

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

To see that this does indeed yield $A^{-1}$, we find all of the cofactors of $A$

$$\begin{array}{rclrclrcl} C_{11} &=& -9, & C_{21} &=& -2, & C_{31} &=& 7 \\ C_{12} &=& 10, & C_{22} &=& 8, & C_{32} &=& -15 \\ C_{13} &=& 2, & C_{23} &=& -1, & C_{33} &=& -3 \end{array}$$

And, as promised,

[7] In a solid Linear Algebra course you will learn that the properties in Theorem 1.7 hold equally well if the word 'row' is replaced by the word 'column'. We're not going to get into column operations in this text, but they do make some of what we're trying to say easier to follow.

$$A^{-1} = \frac{1}{\det(A)} \left[ \begin{array}{ccc} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{array} \right] = -\frac{1}{13} \left[ \begin{array}{rrr} -9 & -2 & 7 \\ 10 & 8 & -15 \\ 2 & -1 & -3 \end{array} \right] = \left[ \begin{array}{rrr} \frac{9}{13} & \frac{2}{13} & -\frac{7}{13} \\ -\frac{10}{13} & -\frac{8}{13} & \frac{15}{13} \\ -\frac{2}{13} & \frac{1}{13} & \frac{3}{13} \end{array} \right]$$

To generalize this to invertible $n \times n$ matrices, we need another definition and a theorem. Our definition gives a special name to the cofactor matrix, and the theorem tells us how to use it along with $\det(A)$ to find the inverse of a matrix.

DEFINITION 1.7. Let $A$ be an $n \times n$ matrix, and $C_{ij}$ denote the $ij$ cofactor of $A$. The **adjoint** of $A$, denoted $\text{adj}(A)$ is the matrix whose $ij$-entry is the $ji$ cofactor of $A$, $C_{ji}$. That is

$$\text{adj}(A) = \left[ \begin{array}{cccc} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{array} \right]$$

This new notation greatly shortens the statement of the formula for the inverse of a matrix.

THEOREM 1.4. Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

For $2 \times 2$ matrices, Theorem 1.9 reduces to a fairly simple formula.

EQUATION 1.2. For an invertible $2 \times 2$ matrix,

$$\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]^{-1} = \frac{1}{ad - bc} \left[ \begin{array}{rr} d & -b \\ -c & a \end{array} \right]$$

The proof of Theorem 1.9 is, like so many of the results in this section, best left to a course in Linear Algebra. In such a course, not only do you gain some more sophisticated proof techniques, you also gain a larger perspective. The authors assure you that persistence pays off. If you stick around a few semesters and take a course in Linear Algebra, you'll see just how pretty all things matrix really are - in spite of the tedious notation and sea of subscripts. Within the scope of this text, we will prove a few results involving determinants in Section **??** once we have the Principle of Mathematical Induction well in hand. Until then, make sure you have a handle on the *mechanics* of matrices and the theory will come eventually.

### 1.2.3 Exercises

1. Compute the determinant of the following matrices. (Some of these matrices appeared in Exercise 1 in Section 1.4.)

   (a) $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

   (b) $C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$

   (c) $Q = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}$

   (d) $L = \begin{bmatrix} \frac{1}{x^3} & \frac{\ln(x)}{x^3} \\ -\frac{3}{x^4} & \frac{1-3\ln(x)}{x^4} \end{bmatrix}$

   (e) $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

   (f) $G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$

   (g) $V = \begin{bmatrix} i & j & k \\ -1 & 0 & 5 \\ 9 & -4 & -2 \end{bmatrix}$

   (h) $H = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 2 & -2 & 8 & 7 \\ -5 & 0 & 16 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$

2. Use Cramer's Rule to solve the system of linear equations.

   (a) $\left\{ \begin{array}{rcl} 3x + 7y & = & 26 \\ 5x + 12y & = & 39 \end{array} \right.$

   (b) $\left\{ \begin{array}{rcl} x + y + z & = & 3 \\ 2x - y + z & = & 0 \\ -3x + 5y + 7z & = & 7 \end{array} \right.$

3. Use Cramer's Rule to solve for $x_4$ in the following system of linear equations.

$$\left\{ \begin{array}{rcr} x_1 - x_3 & = & -2 \\ 2x_2 - x_4 & = & 0 \\ x_1 - 2x_2 + x_3 & = & 0 \\ -x_3 + x_4 & = & 1 \end{array} \right.$$

4. Find the inverse of the following matrices using their determinants and adjoints.

   (a) $B = \begin{bmatrix} 12 & -7 \\ -5 & 3 \end{bmatrix}$

   (b) $F = \begin{bmatrix} 4 & 6 & -3 \\ 3 & 4 & -3 \\ 1 & 2 & 6 \end{bmatrix}$

5. Let $R = \begin{bmatrix} -7 & 3 \\ 11 & 2 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -5 \\ 6 & 9 \end{bmatrix}$ $T = \begin{bmatrix} 11 & 2 \\ -7 & 3 \end{bmatrix}$, and $U = \begin{bmatrix} -3 & 15 \\ 6 & 9 \end{bmatrix}$

   (a) Show that $\det(RS) = \det(R)\det(S)$

   (b) Show that $\det(T) = -\det(R)$

   (c) Show that $\det(U) = -3\det(S)$

6. For $M$ and $N$ below, show that $\det(M) = 0$ and $\det(N) = 0$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

7. Let $A$ be an arbitrary invertible $3 \times 3$ matrix.

   (a) Show that $\det(I_3) = 1$.[8]

   (b) Using the facts that $AA^{-1} = I_3$ and $\det(AA^{-1}) = \det(A)\det(A^{-1})$, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

8. The purpose of this exercise is to introduce you to the eigenvalues and eigenvectors of a matrix.[9] We begin with an example using a $2 \times 2$ matrix and then guide you through some exercises using a $3 \times 3$ matrix.

   Consider the matrix

$$C = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix}$$

   from Exercise 1 above. We know that $\det(C) = 0$ which means that $CX = 0_{2\times 2}$ does not have a unique solution. So there is a nonzero matrix $Y$ such that $CY = 0_{2\times 2}$. In fact, every matrix of the form

$$Y = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix}$$

   is a solution to $CX = 0_{2\times 2}$, so there are infinitely many matrices such that $CX = 0_{2\times 2}$. But consider the matrix

$$X_{41} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

   It is NOT a solution to $CX = 0_{2\times 2}$, but rather,

$$CX_{41} = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 123 \\ 287 \end{bmatrix} = 41 \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

   In fact, if $Z$ is of the form

$$Z = \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix}$$

[8] If you think about it for just a moment, you'll see that $\det(I_n) = 1$ for any natural number $n$. The formal proof of this fact requires the Principle of Mathematical Induction (Section **??**) so we'll stick with $n = 3$ for the time being.

[9] This material is usually given its own chapter in a Linear Algebra book so clearly we're not able to tell you everything you need to know about eigenvalues and eigenvectors. They are a nice application of determinants, though, so we're going to give you enough background so that you can start playing around with them.

then

$$CZ = \begin{bmatrix} 6 & 15 \\ 14 & 35 \end{bmatrix} \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{123}{7}t \\ 41t \end{bmatrix} = 41 \begin{bmatrix} \frac{3}{7}t \\ t \end{bmatrix} = 41Z$$

for all $t$. The big question is "How did we know to use 41?"

We need a number $\lambda$ such that $CX = \lambda X$ has nonzero solutions. We have demonstrated that $\lambda = 0$ and $\lambda = 41$ both worked. Are there others? If we look at the matrix equation more closely, what we *really* wanted was a nonzero solution to $(C - \lambda I_2)X = 0_{2\times 2}$ which we know exists if and only if the determinant of $C - \lambda I_2$ is zero.[10] So we computed

$$\det(C - \lambda I_2) = \det\left(\begin{bmatrix} 6-\lambda & 15 \\ 14 & 35-\lambda \end{bmatrix}\right) = (6-\lambda)(35-\lambda) - 14\cdot 15 = \lambda^2 - 41\lambda$$

This is called the **characteristic polynomial** of the matrix $C$ and it has two zeros: $\lambda = 0$ and $\lambda = 41$. That's how we knew to use 41 in our work above. The fact that $\lambda = 0$ showed up as one of the zeros of the characteristic polynomial just means that $C$ itself had determinant zero which we already knew. Those two numbers are called the **eigenvalues** of $C$. The corresponding matrix solutions to $CX = \lambda X$ are called the **eigenvectors** of $C$ and the 'vector' portion of the name will make more sense after you've studied vectors.

Okay, you should be mostly ready to start on your own. In the following exercises, you'll be using the matrix

$$G = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 11 \\ 3 & 4 & 19 \end{bmatrix}$$

from Exercise 1 above.

(a) Show that the characteristic polynomial of $G$ is $p(\lambda) = -\lambda(\lambda-1)(\lambda-22)$. That is, compute the determinant of $G - \lambda I_3$.

(b) Let $G_0 = G$. Find the parametric description of the solution to the system of linear equations given by $GX = 0_{3\times 3}$.

(c) Let $G_1 = G - I_3$. Find the parametric description of the solution to the system of linear equations given by $G_1 X = 0_{3\times 3}$. Show that any solution to $G_1 X = 0_{3\times 3}$ also has the property that $GX = 1X$.

(d) Let $G_{22} = G - 22I_3$. Find the parametric description of the solution to the system of linear equations given by $G_{22}X = 0_{3\times 3}$. Show that any solution to $G_{22}X = 0_{3\times 3}$ also has the property that $GX = 22X$.

[10]Think about this.

### 1.2.4 Answers

1. (a) $\det(B) = 1$
   (b) $\det(C) = 0$
   (c) $\det(Q) = x^2$
   (d) $\det(L) = \frac{1}{x^7}$
   (e) $\det(F) = -12$
   (f) $\det(G) = 0$
   (g) $\det(V) = 20i + 43j + 4k$
   (h) $\det(H) = -2$

2. (a) $x = 39,\ y = -13$
   (b) $x = 1,\ y = 2,\ z = 0$

3. $x_4 = 4$

4. (a) $B^{-1} = \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix}$

   (b) $F^{-1} = \begin{bmatrix} -\frac{5}{2} & \frac{7}{2} & \frac{1}{2} \\ \frac{7}{4} & -\frac{9}{4} & -\frac{1}{4} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

## 1.3 Systems of Non-Linear Equations and Inequalities

In this section, we study systems of non-linear equations and inequalities. Unlike the systems of linear equations for which we have developed several algorithmic solution techniques, there is no general algorithm to solve systems of non-linear equations. Moreover, all of the usual hazards of non-linear equations like extraneous solutions and unusual function domains are once again present. Along with the tried and true techniques of substitution and elimination, we shall often need equal parts tenacity and ingenuity to see a problem through to the end. You may find it necessary to review topics throughout the text which pertain to solving equations involving the various functions we have studied thus far. To get the section rolling we begin with a fairly routine example.

Example 1.3.1. Solve the following systems of equations. Verify your answers algebraically and graphically.

1. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2+9y^2 & = & 36 \end{array}\right.$

2. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ 4x^2-9y^2 & = & 36 \end{array}\right.$

3. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ y-2x & = & 0 \end{array}\right.$

4. $\left\{\begin{array}{rcr} x^2+y^2 & = & 4 \\ y-x^2 & = & 0 \end{array}\right.$

Solution:

1. Since both equations contain $x^2$ and $y^2$ only, we can eliminate one of the variables as we did in Section 1.1.

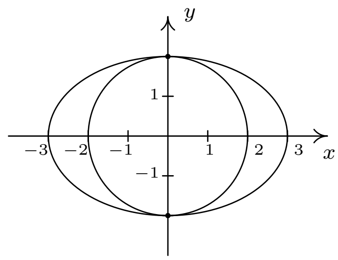
$$\left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2+9y^2 & = & 36 \end{array}\right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 5y^2 & = & 20 \end{array}\right.$$

From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated $x$ values, we substitute each value of $y$ into one of the equations to find the resulting value of $x$. Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x = 0$. Our solution is thus $\{(0, 2), (0, -2)\}$. To check this algebraically, we need to show that both points satisfy both of the original equations. We leave it to the reader to verify this. To check our answer graphically, we sketch both equations and look for their points of intersection. The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2, whereas the graph of $4x^2+9y^2 = 36$, when written in the standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is easily recognized as an ellipse centered at $(0, 0)$ with a major axis along the $x$-axis of length 6 and a minor axis along the $y$-axis of length 4. We see from the graph that the two curves intersect at their $y$-intercepts only, $(0, \pm 2)$.

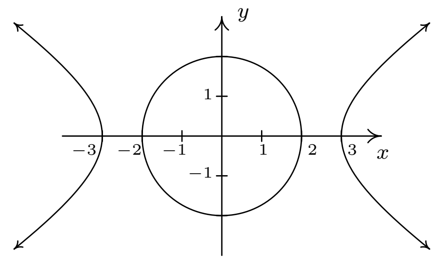
2. We proceed as before to eliminate one of the variables

$$\left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & 4x^2-9y^2 & = & 36 \end{array}\right. \xrightarrow[-4E1+E2]{\text{Replace } E2 \text{ with}} \left\{\begin{array}{lrcr} (E1) & x^2+y^2 & = & 4 \\ (E2) & -13y^2 & = & 20 \end{array}\right.$$

Since the equation $-13y^2 = 20$ admits no real solution, the system is inconsistent. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0,0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4. We see that the circle and the hyperbola have no points in common.




Graphs for $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 + 9y^2 & = & 36 \end{array} \right.$




Graphs for $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 4 \\ 4x^2 - 9y^2 & = & 36 \end{array} \right.$

3. Since there are no like terms among the two equations, elimination won't do us any good. We turn to substitution and from the equation $y - 2x = 0$, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Solving, we find $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$. Returning to the equation we used for the substitution, $y = 2x$, we find $y = \frac{4\sqrt{5}}{5}$ when $x = \frac{2\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$. Similarly, we find the other solution to be $\left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)$. We leave it to the reader that both points satisfy both equations, so that our final answer is $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)\right\}$. The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$ is a line through the origin with slope 2. Though we cannot verify the numerical values of the points of intersection from our sketch, we do see that we have two solutions: one in Quadrant I and one in Quadrant III as required.

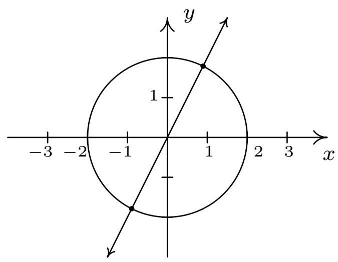
4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination.[1]

$$\left\{ \begin{array}{rrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & y - x^2 & = & 0 \end{array} \right. \xrightarrow[E1 + E2]{\text{Replace } E2 \text{ with}} \left\{ \begin{array}{rrcr} (E1) & x^2 + y^2 & = & 4 \\ (E2) & y^2 + y & = & 4 \end{array} \right.$$

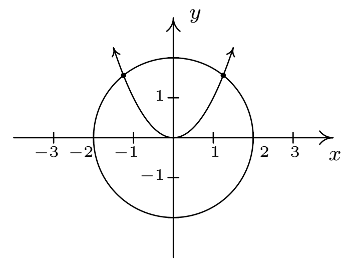
From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$ which gives $y = \frac{-1\pm\sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations to head off any extraneous solutions we may encounter. We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly twice, and both of these points have a positive $y$ value. Of the two solutions for $y$, only $y = \frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this

[1]We encourage the reader to solve the system using substitution to see that you get the same solution.

into $y - x^2 = 0$ and solve for $x$. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$. Our solution is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$, which we leave to the reader to verify.




Graphs for $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ y - 2x & = & 0 \end{array}\right.$




Graphs for $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ y - x^2 & = & 36 \end{array}\right.$

□

A couple of remarks about Example 1.7.1 are in order. First note that, unlike systems of linear equations, it is possible for a system of non-linear equations to have more than one solution without having infinitely many solutions. In fact, while we characterize systems of nonlinear equations as being 'consistent' or 'inconsistent,' we generally don't use the labels 'dependent' or 'independent'. Secondly, as we saw with number 4, sometimes making a quick sketch of the problem situation can save a lot of time and effort. While in general the curves in a system of non-linear equations may not be easily visualized, it sometimes pays to take advantage when they are. Our next example provides some considerable review of many of the topics introduced in this text.

Example 1.3.2. Solve the following systems of equations. Verify your answers algebraically and graphically, as appropriate.

1. $\left\{\begin{array}{rcl} x^2 + 2xy - 16 & = & 0 \\ y^2 + 2xy - 16 & = & 0 \end{array}\right.$

2. $\left\{\begin{array}{rcl} y + 4e^{2x} & = & 1 \\ y^2 + 2e^{x} & = & 1 \end{array}\right.$

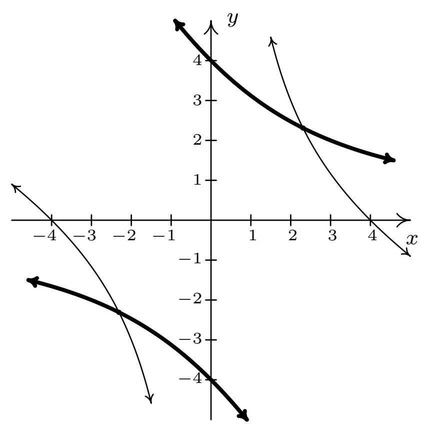
3. $\left\{\begin{array}{rcl} z(x-2) & = & x \\ yz & = & y \\ (x-2)^2 + y^2 & = & 1 \end{array}\right.$

Solution.

1. At first glance, it doesn't appear as though elimination will do us any good since it's clear that we cannot completely eliminate one of the variables. The alternative, solving one of the equations for one variable and substituting it into the other, is full of unpleasantness. Returning to elimination, we note that it is possible to eliminate the troublesome $xy$ term, and the constant term as well, by elimination and doing so we get a more tractable relationship between $x$ and $y$

$$\left\{\begin{array}{lrcl} (E1) & x^2 + 2xy - 16 & = & 0 \\ (E2) & y^2 + 2xy - 16 & = & 0 \end{array}\right. \xrightarrow[-E1+E2]{\text{Replace } E2 \text{ with}} \left\{\begin{array}{lrcl} (E1) & x^2 + 2xy - 16 & = & 0 \\ (E2) & y^2 - x^2 & = & 0 \end{array}\right.$$

We get $y^2 - x^2 = 0$ or $y = \pm x$. Substituting $y = x$ into $E1$ we get $x^2 + 2x^2 - 16 = 0$ so that $x^2 = \frac{16}{3}$ or $x = \pm\frac{4\sqrt{3}}{3}$. On the other hand, when we substitute $y = -x$ into $E1$, we get $x^2 - 2x^2 - 16 = 0$ or $x^2 = -16$ which gives no real solutions. Substituting each of $x = \pm\frac{4\sqrt{3}}{3}$ into the substitution equation $y = x$ yields the solution $\left\{\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{4\sqrt{3}}{3}\right)\right\}$. We leave it to the reader to show that both points satisfy both equations and now turn to verifying our solution graphically. We begin by solving $x^2+2xy-16 = 0$ for $y$ to obtain $y = \frac{16-x^2}{2x}$. This function is easily graphed using the techniques of Section **??**. Solving the second equation, $y^2 + 2xy - 16 = 0$, for $y$, however, is more complicated. We use the quadratic formula to obtain $y = -x \pm \sqrt{x^2 + 16}$ which would require the use of Calculus or a calculator to graph. Believe it or not, we don't need either because the equation $y^2+2xy-16 = 0$ can be obtained from the equation $x^2 + 2xy - 16 = 0$ by interchanging $y$ and $x$. Thinking back to Section **??**, this means we can obtain the graph of $y^2 + 2xy - 16 = 0$ by reflecting the graph of $x^2 + 2xy - 16 = 0$ across the line $y = x$. Doing so confirms that the two graphs intersect twice: once in Quadrant I, and once in Quadrant III as required.




The graphs of $x^2 + 2xy - 16 = 0$ and $\boldsymbol{y^2 + 2xy - 16 = 0}$

2. Unlike the previous problem, there seems to be no avoiding substitution and a bit of algebraic unpleasantness. Solving $y + 4e^{2x} = 1$ for $y$, we get $y = 1 - 4e^{2x}$ which, when substituted into the second equation, yields $\left(1 - 4e^{2x}\right)^2 + 2e^x = 1$. After expanding and gathering like terms, we get $16e^{4x} - 8e^{2x} + 2e^x = 0$. Factoring gives us $2e^x\left(8e^{3x} - 4e^x + 1\right) = 0$, and since $2e^x \neq 0$ for any real $x$, we are left with solving $8e^{3x} - 4e^x + 1 = 0$. We have three terms, and even though this is not a 'quadratic in disguise', we can benefit from the substitution $u = e^x$. The equation becomes $8u^3-4u+1 = 0$. Using the techniques set forth in Section **??**, we find $u = \frac{1}{2}$ is a zero and use synthetic division to factor the left hand side as $\left(u - \frac{1}{2}\right)\left(8u^2 + 4u - 2\right)$. We use the quadratic formula to solve $8u^2 + 4u - 2 = 0$ and find $u = \frac{-1\pm\sqrt{5}}{4}$. Since $u = e^x$, we now must solve $e^x = \frac{1}{2}$ and $e^x = \frac{-1\pm\sqrt{5}}{4}$. From $e^x = \frac{1}{2}$, we get $x = \ln\left(\frac{1}{2}\right) = -\ln(2)$. As for $e^x = \frac{-1\pm\sqrt{5}}{4}$, we first note that $\frac{-1-\sqrt{5}}{4} < 0$, so $e^x = \frac{-1-\sqrt{5}}{4}$ has no real solutions. We are

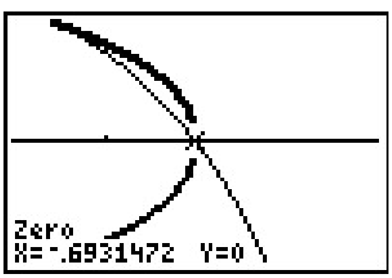
left with $e^x = \frac{-1+\sqrt{5}}{4}$, so that $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$. We now return to $y = 1 - 4e^{2x}$ to find the accompanying $y$ values for each of our solutions for $x$. For $x = -\ln(2)$, we get

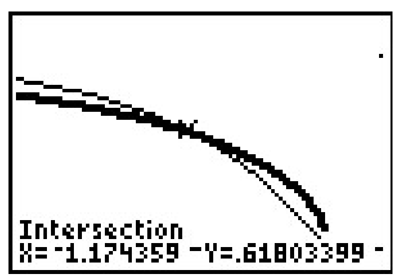
$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{-2\ln(2)} \\ &= 1 - 4e^{\ln(1/4)} \\ &= 1 - 4\left(\tfrac{1}{4}\right) \\ &= 0 \end{aligned}$$

For $x = \ln\left(\frac{-1+\sqrt{5}}{4}\right)$, we have

$$\begin{aligned} y &= 1 - 4e^{2x} \\ &= 1 - 4e^{2\ln\left(\frac{-1+\sqrt{5}}{4}\right)} \\ &= 1 - 4e^{\ln\left(\frac{-1+\sqrt{5}}{4}\right)^2} \\ &= 1 - 4\left(\frac{-1+\sqrt{5}}{4}\right)^2 \\ &= 1 - 4\left(\frac{3-\sqrt{5}}{8}\right) \\ &= \frac{-1+\sqrt{5}}{2} \end{aligned}$$

We get two solutions, $\left\{(0, -\ln(2)), \left(\ln\left(\frac{-1+\sqrt{5}}{4}\right), \frac{-1+\sqrt{5}}{2}\right)\right\}$. It is a good review of the properties of logarithms to verify both solutions, so we leave that to the reader. We are able to sketch $y = 1 - 4e^{2x}$ using transformations, but the second equation is more difficult and we resort to the calculator. We note that to graph $y^2 + 2e^x = 1$, we need to graph both the positive and negative roots, $y = \pm\sqrt{1 - 2e^x}$. After some careful zooming,[2] we confirm our solutions.




The graphs of $y = 1 - 4e^{2x}$ and $\boldsymbol{y = \pm\sqrt{1 - 2e^x}}$.

3. Our last system involves three variables and gives some insight on how to keep such systems organized. Labeling the equations as before, we have

[2] The calculator has trouble confirming the solution $(-\ln(2), 0)$ due to its issues in graphing square root functions. If we mentally connect the two branches of the thicker curve, we see the intersection.

$$\begin{cases} E1 & z(x-2) &=& x \\ E2 & yz &=& y \\ E3 & (x-2)^2+y^2 &=& 1 \end{cases}$$

The easiest equation to start with appears to be $E2$. While it may be tempting to divide both sides of $E2$ by $y$, we caution against this practice because it presupposes $y \neq 0$. Instead, we take $E2$ and rewrite it as $yz - y = 0$ so $y(z-1) = 0$. From this, we get two cases: $y = 0$ or $z = 1$. We take each case in turn.

Case 1: $y = 0$. Substituting $y = 0$ into $E1$ and $E3$, we get

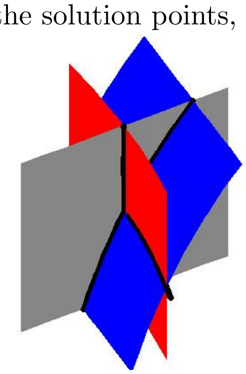
$$\begin{cases} E1 & z(x-2) &=& x \\ E3 & (x-2)^2 &=& 1 \end{cases}$$

Solving $E3$ for $x$ gives $x = 1$ or $x = 3$. Substituting these values into $E1$ gives $z = -1$ when $x = 1$ and $z = 3$ when $x = 3$. We obtain two solutions, $(1, 0, -1)$ and $(3, 0, 3)$.

Case 2: $z = 1$. Substituting $z = 1$ into $E1$ and $E3$ gives us

$$\begin{cases} E1 & (1)(x-2) &=& x \\ E3 & (1-2)^2+y^2 &=& 1 \end{cases}$$

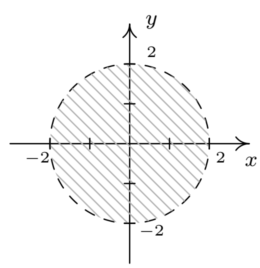
Equation $E1$ gives us $x - 2 = x$ or $-2 = 0$, which is a contradiction. This means we have no solution to the system in this case, even though $E3$ is solvable and gives $y = 0$. Hence, our final answer is $\{(1, 0, -1), (3, 0, 3)\}$. These points are easy enough to check algebraically in our three original equations, so that is left to the reader. As for verifying these solutions graphically, they require plotting surfaces in three dimensions and looking for intersection points. While this is beyond the scope of this book, we provide a snapshot of the graphs of our three equations near one of the solution points, $(1, 0, -1)$.



$\square$

Example 1.7.2 showcases some of the ingenuity and tenacity mentioned at the beginning of the section. Sometimes you just have to look at a system the right way to find the most efficient method to solve it. Sometimes you just have to try something.

We close this section discussing how non-linear inequalities can be used to describe regions in the plane which we first introduced in Section **??**. Before we embark on some examples, a little motivation is in order. Suppose we wish to solve $x^2 < 4 - y^2$. If we mimic the algorithms for solving nonlinear inequalities in one variable, we would gather all of the terms on one side and leave a 0 on the other to obtain $x^2 + y^2 - 4 < 0$. Then we would find the zeros of the left hand side, that is, where is $x^2 + y^2 - 4 = 0$, or $x^2 + y^2 = 4$. Instead of obtaining a few *numbers* which divide the real number *line* into *intervals*, we get an equation of a *curve*, in this case, a circle, which divides the *plane* into two *regions* - the 'inside' and 'outside' of the circle - with the circle itself as the boundary between the two. Just like we used test *values* to determine whether or not an interval belongs to the solution of the inequality, we use test *points* in the each of the regions to see which of these belong to our solution set.[3] We choose $(0, 0)$ to represent the region inside the circle and $(0, 3)$ to represent the points outside of the circle. When we substitute $(0, 0)$ into $x^2 + y^2 - 4 < 0$, we get $-4 < 4$ which is true. This means $(0, 0)$ and all the other points inside the circle are part of the solution. On the other hand, when we substitute $(0, 3)$ into the same inequality, we get $5 < 0$ which is false. This means $(0, 3)$ along with all other points outside the circle are not part of the solution. What about points on the circle itself? Choosing a point on the circle, say $(0, 2)$, we get $0 < 0$, which means the circle itself does not satisfy the inequality.[4] As a result, we leave the circle dashed in the final diagram.




The solution to $x^2 < 4 - y^2$

We put this technique to good use in the following example.

Example 1.3.3. Sketch the solution to the following nonlinear inequalities in the plane.

1. $y^2 - 4 \leq x < y + 2$

2. $\left\{ \begin{array}{rcl} x^2 + y^2 & \geq & 4 \\ x^2 - 2x + y^2 - 2y & \leq & 0 \end{array} \right.$

Solution.

1. The inequality $y^2 - 4 \leq x < y + 2$ is a compound inequality. It translates as $y^2 - 4 \leq x$ and $x < y + 2$. As usual, we solve each inequality and take the set theoretic intersection to determine the region which satisfies both inequalities. To solve $y^2 - 4 \leq x$, we write

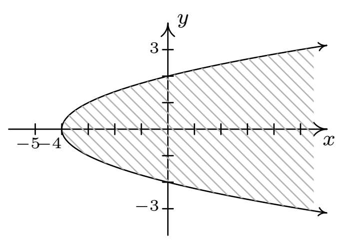
[3] The theory behind why all this works is, surprisingly, the same theory which guarantees that sign diagrams work the way they do - continuity and the Intermediate Value Theorem - but in this case, applied to functions of more than one variable.

[4] Another way to see this is that points on the circle satisfy $x^2 + y^2 - 4 = 0$, so they do not satisfy $x^2 + y^2 - 4 < 0$.

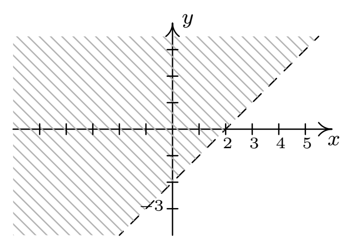
$y^2 - x - 4 \leq 0$. The curve $y^2 - x - 4 = 0$ describes a parabola since exactly one of the variables is squared. Rewriting this in standard form, we get $y^2 = x + 4$ and we see that the vertex is $(-4, 0)$ and the parabola opens to the right. Using the test points $(-5, 0)$ and $(0, 0)$, we find that the solution to the inequality includes the region to the right of, or 'inside', the parabola. The points on the parabola itself are also part of the solution, since the vertex $(-4, 0)$ satisfies the inequality. We now turn our attention to $x < y + 2$. Proceeding as before, we write $x - y - 2 < 0$ and focus our attention on $x - y - 2 = 0$, which is the line $y = x - 2$. Using the test points $(0, 0)$ and $(0, -4)$, we find points in the region above the line $y = x - 2$ satisfy the inequality. The points on the line $y = x - 2$ do not satisfy the inequality, since the $y$-intercept $(0, -2)$ does not. We see that these two regions do overlap, and to make the graph more precise, we seek the intersection of these two curves. That is, we need to solve the system of nonlinear equations

$$\left\{ \begin{array}{lrcl} (E1) & y^2 & = & x + 4 \\ (E2) & y & = & x - 2 \end{array} \right.$$

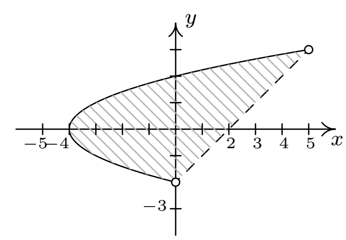
Solving $E1$ for $x$, we get $x = y^2 - 4$. Substituting this into $E2$ gives $y = y^2 - 4 - 2$, or $y^2 - y - 6 = 0$. We find $y = -2$ and $y = 3$ and since $x = y^2 - 4$, we get that the graphs intersect at $(0, -2)$ and $(5, 3)$. Putting all of this together, we get our final answer below.




$y^2 - 4 \leq x$




$x < y + 2$



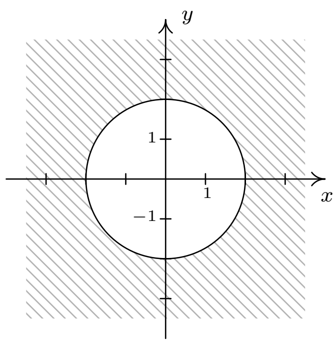

$y^2 - 4 \leq x < y + 2$

2. To solve this system of inequalities, we need to find all of the points $(x, y)$ which satisfy both inequalities. To do this, we solve each inequality separately and take the set theoretic intersection of the solution sets. We begin with the inequality $x^2 + y^2 \geq 4$ which we rewrite as $x^2 + y^2 - 4 \geq 0$. The points which satisfy $x^2 + y^2 - 4 = 0$ form our friendly circle $x^2 + y^2 = 4$. Using test points $(0, 0)$ and $(0, 3)$ we find that our solution comprises the region outside the circle. As far as the circle itself, the point $(0, 2)$ satisfies the inequality, so the circle itself is part of the solution set. Moving to the inequality $x^2 - 2x + y^2 - 2y \leq 0$, we start with $x^2 - 2x + y^2 - 2y = 0$. Completing the squares, we obtain $(x - 1)^2 + (y - 1)^2 = 2$, which is a circle centered at $(1, 1)$ with a radius of $\sqrt{2}$. Choosing $(1, 1)$ to represent the inside of the circle, $(1, 3)$ as a point outside of the circle and $(0, 0)$ as a point on the circle, we find that the solution to the inequality is the inside of the circle, including the circle itself. Our final answer, then, consists of the points on or outside of the circle $x^2 + y^2 = 4$ which lie on or

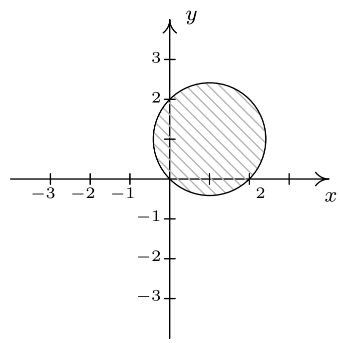
inside the circle $(x-1)^2+(y-1)^2=2$. To produce the most accurate graph, we need to find where these circles intersect. To that end, we solve the system

$$\left\{\begin{array}{lrcl} (E1) & x^2+y^2 & = & 4 \\ (E2) & x^2-2x+y^2-2y & = & 0 \end{array}\right.$$

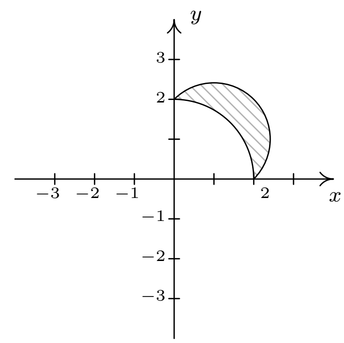
We can eliminate both the $x^2$ and $y^2$ by replacing $E2$ with $-E1+E2$. Doing so produces $-2x-2y=-4$. Solving this for $y$, we get $y=2-x$. Substituting this into $E1$ gives $x^2+(2-x)^2=4$ which simplifies to $x^2+4-4x+x^2=4$ or $2x^2-4x=0$. Factoring yields $2x(x-2)$ which gives $x=0$ or $x=2$. Substituting these values into $y=2-x$ gives the points $(0,2)$ and $(2,0)$. The intermediate graphs and final solution are below.



$x^2+y^2 \geq 4$



$x^2-2x+y^2-2y \leq 0$



Solution to the system.

□

### 1.3.1 Exercises

1. Solve the following systems of nonlinear equations. Sketch the graph of both equations on the same set of axes to verify the solution set.

(a) $\left\{\begin{array}{rcl} x^2 - y & = & 4 \\ x^2 + y^2 & = & 4 \end{array}\right.$

(b) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 4 \\ x^2 - y & = & 5 \end{array}\right.$

(c) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 16 \\ 16x^2 + 4y^2 & = & 64 \end{array}\right.$

(d) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 16 \\ 9x^2 - 16y^2 & = & 144 \end{array}\right.$

(e) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 16 \\ \frac{1}{9}y^2 - \frac{1}{16}x^2 & = & 1 \end{array}\right.$

(f) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 16 \\ x - y & = & 2 \end{array}\right.$

2. Solve the following systems of nonlinear equations. Use a graph to help you avoid any potential extraneous solutions.

(a) $\left\{\begin{array}{rcl} x^2 - y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

(b) $\left\{\begin{array}{rcl} \sqrt{x+1} - y & = & 0 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

(c) $\left\{\begin{array}{rcl} x + 2y^2 & = & 2 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

(d) $\left\{\begin{array}{rcl} (x-2)^2 + y^2 & = & 1 \\ x^2 + 4y^2 & = & 4 \end{array}\right.$

(e) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 25 \\ y - x & = & 1 \end{array}\right.$

(f) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 25 \\ x^2 + (y-3)^2 & = & 10 \end{array}\right.$

(g) $\left\{\begin{array}{rcl} y & = & x^3 + 8 \\ y & = & 10x - x^2 \end{array}\right.$

(h) $\left\{\begin{array}{rcl} x^2 + y^2 & = & 25 \\ 4x^2 - 9y & = & 0 \\ 3y^2 - 16x & = & 0 \end{array}\right.$

3. Consider the system of nonlinear equations below

$$\left\{\begin{array}{rcl} \dfrac{4}{x} + \dfrac{3}{y} & = & 1 \\ \dfrac{3}{x} + \dfrac{2}{y} & = & -1 \end{array}\right.$$

If we let $u = \frac{1}{x}$ and $v = \frac{1}{y}$ then the system becomes

$$\left\{\begin{array}{rcl} 4u + 3v & = & 1 \\ 3u + 2v & = & -1 \end{array}\right.$$

This associated system of linear equations can then be solved using any of the techniques presented earlier in the chapter to find that $u = -5$ and $v = 7$. Thus $x = \frac{1}{u} = -\frac{1}{5}$ and $y = \frac{1}{v} = \frac{1}{7}$.

We say that the original system is **linear in form** because its equations are not linear but a few basic substitutions reveal a structure that we can treat like a system of linear equations. Each system given below is linear in form. Make the appropriate substitutions to help you solve for $x$ and $y$.

(a) $\left\{\begin{array}{rcl} 4x^3 + 3\sqrt{y} & = & 1 \\ 3x^3 + 2\sqrt{y} & = & -1 \end{array}\right.$

(b) $\left\{\begin{array}{rcl} 4e^x + 3e^{-y} & = & 1 \\ 3e^x + 2e^{-y} & = & -1 \end{array}\right.$

(c) $\left\{\begin{array}{rcl} 4\ln(x) + 3y^2 & = & 1 \\ 3\ln(x) + 2y^2 & = & -1 \end{array}\right.$

4. Solve the following system

$$\left\{ \begin{array}{rcr} x^2 + \sqrt{y} + \log_2(z) & = & 6 \\ 3x^2 - 2\sqrt{y} + 2\log_2(z) & = & 5 \\ -4x^2 + \sqrt{y} - 3\log_2(z) & = & 11 \end{array} \right.$$

5. Sketch the solution to each system of nonlinear inequalities in the plane.

   (a) $\left\{ \begin{array}{rcl} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{array} \right.$

   (b) $\left\{ \begin{array}{rcl} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{array} \right.$

   (c) $\left\{ \begin{array}{rcl} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{array} \right.$

   (d) $\left\{ \begin{array}{rcr} y & > & 10x - x^2 \\ y & < & x^3 + 8 \end{array} \right.$

   (e) $\left\{ \begin{array}{rcl} x + 2y^2 & > & 2 \\ x^2 + 4y^2 & \leq & 4 \end{array} \right.$

   (f) $\left\{ \begin{array}{rcr} x^2 + y^2 & \geq & 25 \\ y - x & \leq & 1 \end{array} \right.$

6. Systems of nonlinear equations show up in third semester Calculus in the midst of some really cool problems. The system below came from a problem in which we were asked to find the dimensions of a rectangular box with a volume of 1000 cubic inches that has minimal surface area. The variables $x$, $y$ and $z$ are the dimensions of the box and $\lambda$ is called a Lagrange multiplier. With the help of your classmates, solve the system.[5]

$$\left\{ \begin{array}{rcr} 2y + 2z & = & \lambda yz \\ 2x + 2z & = & \lambda xz \\ 2y + 2x & = & \lambda xy \\ xyz & = & 1000 \end{array} \right.$$

7. According to Theorem **??** in Section **??**, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In this exercise, we present a method for obtaining that factorization.

   (a) Show that $p$ has no real zeros.

   (b) Because $p$ has no real zeros, its factorization must be of the form $(x^2+ax+b)(x^2+cx+d)$ where each factor is an irreducible quadratic. Expand this quantity and gather like terms together.

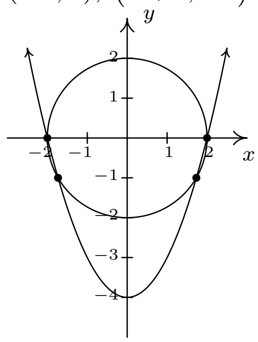
   (c) Create and solve the system of nonlinear equations which results from equating the coefficients of the expansion found above with those of $x^4 + 4$. You should get four equations in the four unknowns $a$, $b$, $c$ and $d$. Write $p(x)$ in factored form.

8. Factor $q(x) = x^4 + 6x^2 - 5x + 6$.

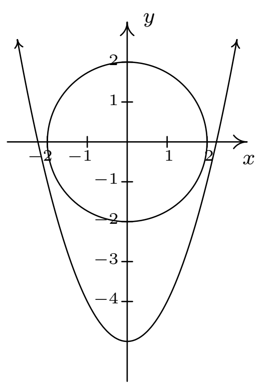
[5]If using $\lambda$ bothers you, change it to $w$ when you solve the system.

### 1.3.2 Answers

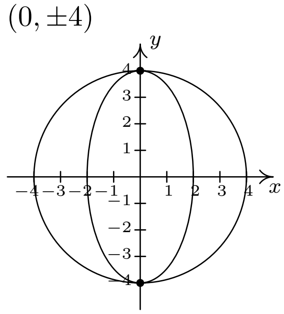
1. (a) $(\pm 2, 0)$, $\left(\pm\sqrt{3}, -1\right)$



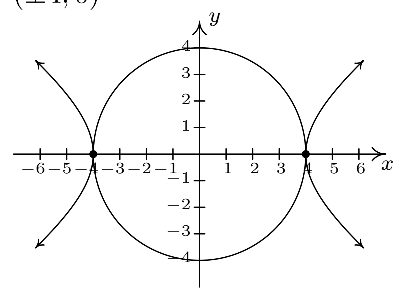
(b) No solution



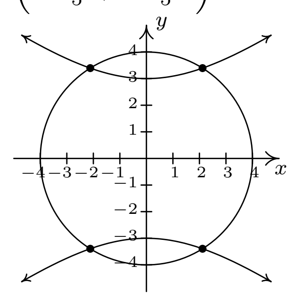
(c) $(0, \pm 4)$



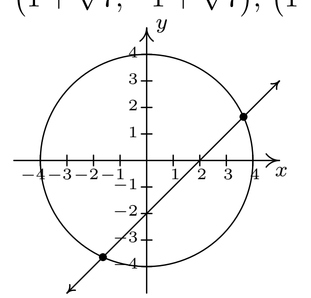
(d) $(\pm 4, 0)$



(e) $\left(\pm\frac{4\sqrt{7}}{5}, \pm\frac{12\sqrt{2}}{5}\right)$



(f) $\left(1+\sqrt{7}, -1+\sqrt{7}\right)$, $\left(1-\sqrt{7}, -1-\sqrt{7}\right)$

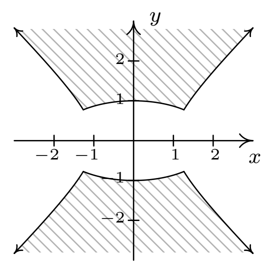


2. (a) $\left(\pm\frac{2\sqrt{10}}{5}, \pm\frac{\sqrt{15}}{5}\right)$
   (b) $(0, 1)$
   (c) $(0, \pm 1)$, $(2, 0)$
   (d) $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{9}\right)$
   (e) $(3, 4)$, $(-4, -3)$
   (f) $(\pm 3, 4)$
   (g) $(-4, -56)$, $(1, 9)$, $(2, 16)$
   (h) $(3, 4)$

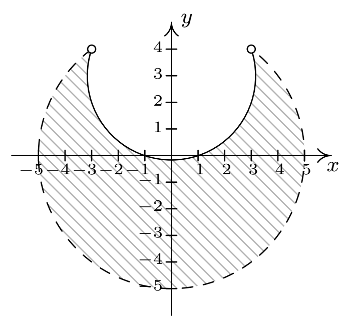
3. (a) $\left(-\sqrt[3]{5}, 49\right)$ (b) No solution (c) $\left(e^{-5}, \sqrt{7}\right)$

4. $(1,4,8),\ (-1,4,8)$

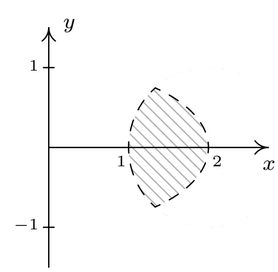
5. (a) $\begin{cases} x^2 - y^2 & \leq & 1 \\ x^2 + 4y^2 & \geq & 4 \end{cases}$



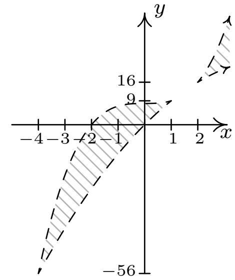

(b) $\begin{cases} x^2 + y^2 & < & 25 \\ x^2 + (y-3)^2 & \geq & 10 \end{cases}$



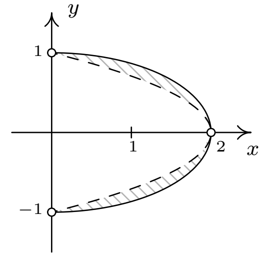

(c) $\begin{cases} (x-2)^2 + y^2 & < & 1 \\ x^2 + 4y^2 & < & 4 \end{cases}$



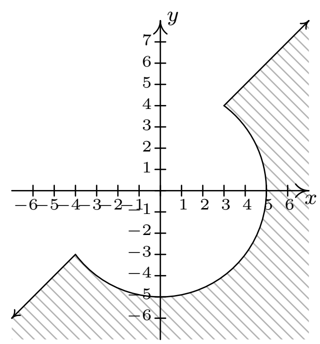
(d) $\begin{cases} y > 10x - x^2 \\ y < x^3 + 8 \end{cases}$




(e) $\begin{cases} x + 2y^2 > 2 \\ x^2 + 4y^2 \leq 4 \end{cases}$




(f) $\begin{cases} x^2 + y^2 \geq 25 \\ y - x \leq 1 \end{cases}$




6. $x = 10,\ y = 10,\ z = 10, \lambda = \frac{2}{5}$

7. (c) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

8. $x^4 + 6x^2 - 5x + 6 = (x^2 - x + 1)(x^2 + x + 6)$