Chapter 1

Further Topics in Functions

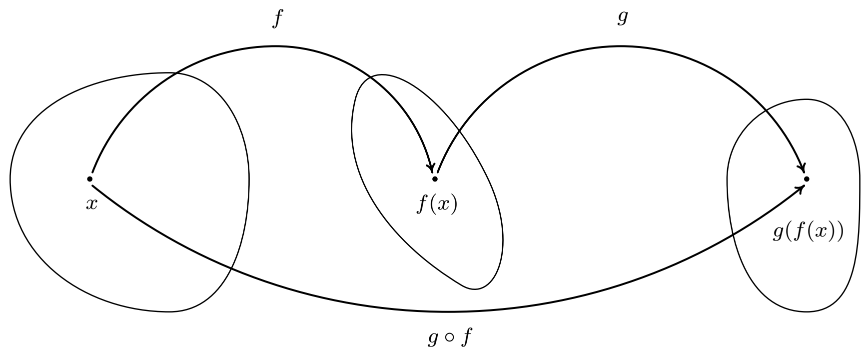
1.1 Function Composition

Before we embark upon any further adventures with functions, we need to take some time to gather our thoughts and gain some perspective. Chapter **??** first introduced us to functions in Section **??**. At that time, functions were specific kinds of relations - sets of points in the plane which passed the Vertical Line Test, Theorem **??**. In Section **??**, we developed the idea that functions are processes - rules which match inputs to outputs - and this gave rise to the concepts of domain and range. We spoke about how functions could be combined in Section **??** using the four basic arithmetic operations, took a more detailed look at their graphs in Section **??** and studied how their graphs behaved under certain classes of transformations in Section **??**. In Chapter **??**, we took a closer look linear functions (Section **??**), and quadratic functions (Section **??**). Linear and quadratic functions were special cases of polynomial functions, which we studied in generality in Chapter **??**. (In Chapter **??** we will learn the Real Factorization Theorem, Theorem **??**, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions.) Our next step was to enlarge our field[1] of study to rational functions in Chapter **??**. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Chapters **??**, **??**, and **??** can be thought of as an exhaustive study of linear and quadratic functions and their arithmetic combinations as described in Section **??**. We now wish to study other algebraic functions, such as $f(x) = \sqrt{x}$ and $g(x) = x^{2/3}$, and the purpose of the first two sections of this chapter is to see how these kinds of functions arise from polynomial and rational functions. To that end, we first study a new way to combine functions as defined below.

DEFINITION 1.1. Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is an element of the domain of f and $f(x)$ is an element of the domain of g.

The quantity $g \circ f$ is also read 'g composed with f' or, more simply 'g of f.' At its most basic level, Definition 1.1 tells us to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$. If we take a step back and look at this from a procedural, 'inputs and outputs' perspective, Defintion 1.1 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have

[1]This is a really bad math pun.



In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. There are two ways to go about evaluating composite functions - 'inside out' and 'outside in' - depending on which function we replace with its formula first. Both ways are demonstrated in the following example.

Example 1.1.1. Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$. Find and simplify the indicated composite functions. State the domain of each.

1. $(g \circ f)(x)$
2. $(f \circ g)(x)$
3. $(g \circ h)(x)$
4. $(h \circ g)(x)$
5. $(h \circ h)(x)$
6. $(h \circ (g \circ f))(x)$
7. $((h \circ g) \circ f)(x)$

Solution.

1. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate the two ways to evaluate this.

   - *inside out*: We insert the expression $f(x)$ into g first to get

     $$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

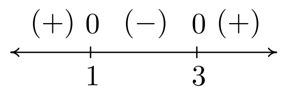
     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

     $$(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} = 2 - \sqrt{(x^2 - 4x) + 3} = 2 - \sqrt{x^2 - 4x + 3}$$

     We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. We accomplish this by following the rule set forth in Section **??**, that is, we find the domain **before** we simplify. To that end, we examine $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$. To keep the square root happy, we solve the inequality $x^2 - 4x + 3 \geq 0$ by creating a sign diagram. If we let $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$. We obtain



Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

2. To find $(f \circ g)(x)$, we find $f(g(x))$.

   - *inside out*: We insert the expression $g(x)$ into f first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(2 - \sqrt{x+3}\right) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= 4 - 4\sqrt{x+3} + \left(\sqrt{x+3}\right)^2 - 8 + 4\sqrt{x+3} \\
&= 4 + x + 3 - 8 \\
&= x - 1
\end{aligned}$$

   - *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= (g(x))^2 - 4\,(g(x)) \\
&= \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right) \\
&= x - 1 && \text{same algebra as before}
\end{aligned}$$

   Thus we get $(f \circ g)(x) = x - 1$. To find the domain of $(f \circ g)$, we look to the step before we did any simplification and find $(f \circ g)(x) = \left(2 - \sqrt{x+3}\right)^2 - 4\left(2 - \sqrt{x+3}\right)$. To keep the square root happy, we set $x + 3 \geq 0$ and find our domain to be $[-3, \infty)$.

3. To find $(g \circ h)(x)$, we compute $g(h(x))$.

   - *inside out*: We insert the expression $h(x)$ into g first to get

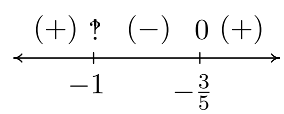
$$\begin{aligned}
(g \circ h)(x) &= g(h(x)) \\
&= g\left(\frac{2x}{x+1}\right) \\
&= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\
&= 2 - \sqrt{\frac{2x}{x+1} + \frac{3(x+1)}{x+1}} && \text{get common denominators}
\end{aligned}$$

$$= \quad 2 - \sqrt{\frac{5x+3}{x+1}}$$

- *outside in*: We use the formula for $g(x)$ first to get

$$\begin{aligned} (g \circ h)(x) &= g(h(x)) \\ &= 2 - \sqrt{h(x)+3} \\ &= 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3} \\ &= 2 - \sqrt{\frac{5x+3}{x+1}} && \text{get common denominators as before} \end{aligned}$$

Hence, $(g \circ h)(x) = 2 - \sqrt{\frac{5x+3}{x+1}}$. To find the domain, we look to the step before we began to simplify: $(g \circ h)(x) = 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3}$. To avoid division by zero, we need $x \neq -1$. To keep the radical happy, we need to solve $\frac{2x}{x+1} + 3 \geq 0$. Getting common denominators as before, this reduces to $\frac{5x+3}{x+1} \geq 0$. Defining $r(x) = \frac{5x+3}{x+1}$, we have that r is undefined at $x = -1$ and $r(x) = 0$ at $x = -\frac{3}{5}$. We get



Our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

4. We find $(h \circ g)(x)$ by finding $h(g(x))$.

- *inside out*: We insert the expression $g(x)$ into h first to get

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h\left(2 - \sqrt{x+3}\right) \\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}} \end{aligned}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$(h \circ g)(x) \quad = \quad h(g(x))$$

$$
\begin{aligned}
&= \frac{2\,(g(x))}{(g(x))+1} \\
&= \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1} \\
&= \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}
\end{aligned}
$$

Hence, $(h \circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. To find the domain of $h \circ g$, we look to the step before any simplification: $(h \circ g)(x) = \frac{2\left(2-\sqrt{x+3}\right)}{\left(2-\sqrt{x+3}\right)+1}$. To keep the square root happy, we require $x+3 \geq 0$ or $x \geq -3$. Setting the denominator equal to zero gives $\left(2-\sqrt{x+3}\right)+1 = 0$ or $\sqrt{x+3} = 3$. Squaring both sides gives us $x+3 = 9$, or $x = 6$. Since $x = 6$ checks in the original equation, $\left(2-\sqrt{x+3}\right)+1 = 0$, we know $x = 6$ is the only zero of the denominator. Hence, the domain of $h \circ g$ is $[-3, 6) \cup (6, \infty)$.

5. To find $(h \circ h)(x)$, we substitute the function h into itself, $h(h(x))$.

   - *inside out*: We insert the expression $h(x)$ into h to get

$$
\begin{aligned}
(h \circ h)(x) &= h(h(x)) \\
&= h\left(\frac{2x}{x+1}\right) \\
&= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1} \\
&= \frac{\frac{4x}{x+1}}{\frac{2x}{x+1}+1} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{\frac{4x}{x+1} \cdot (x+1)}{\left(\frac{2x}{x+1}\right) \cdot (x+1) + 1 \cdot (x+1)} \\
&= \frac{\frac{4x}{\cancelto{1}{(x+1)}} \cdot \cancel{(x+1)}}{\frac{2x}{\cancelto{1}{(x+1)}} \cdot \cancel{(x+1)} + x + 1} \\
&= \frac{4x}{3x+1}
\end{aligned}
$$

- *outside in*: This approach yields

$$
\begin{aligned}
(h \circ h)(x) &= h(h(x)) \\
&= \frac{2(h(x))}{h(x)+1} \\
&= \frac{2\left(\dfrac{2x}{x+1}\right)}{\left(\dfrac{2x}{x+1}\right)+1} \\
&= \frac{4x}{3x+1} \qquad \text{same algebra as before}
\end{aligned}
$$

To find the domain of $h \circ h$, we analyze $(h \circ h)(x) = \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1}$. To keep the denominator $x+1$ happy, we need $x \neq -1$. Setting the denominator $\frac{2x}{x+1}+1=0$ gives $x=-\frac{1}{3}$. Our domain is $(-\infty,-1) \cup \left(-1,-\frac{1}{3}\right) \cup \left(-\frac{1}{3},\infty\right)$.

6. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 1 that $(g \circ f)(x) = 2-\sqrt{x^2-4x+3}$. We now proceed as usual.

   - *inside out*: We insert the expression $(g \circ f)(x)$ into h first to get

$$
\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\
&= h\left(2-\sqrt{x^2-4x+3}\right) \\
&= \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1} \\
&= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}
\end{aligned}
$$

   - *outside in*: We use the formula for $h(x)$ first to get

$$
\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) \\
&= \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x))+1} \\
&= \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1}
\end{aligned}
$$

$$= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}$$

So we get $(h \circ (g \circ f))(x) = \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}}$. To find the domain, we look at the step before we began to simplify, $(h \circ (g \circ f))(x) = \frac{2\left(2-\sqrt{x^2-4x+3}\right)}{\left(2-\sqrt{x^2-4x+3}\right)+1}$. For the square root, we need $x^2-4x+3 \geq 0$, which we determined in number 1 to be $(-\infty, 1] \cup [3, \infty)$. Next, we set the denominator to zero and solve: $\left(2-\sqrt{x^2-4x+3}\right)+1 = 0$. We get $\sqrt{x^2-4x+3} = 3$, and, after squaring both sides, we have $x^2-4x+3 = 9$. To solve $x^2-4x-6 = 0$, we use the quadratic formula and get $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $\left(2-\sqrt{x^2-4x+3}\right)+1 = 0$. Hence we must exclude these numbers from the domain of $h \circ (g \circ f)$. Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2-\sqrt{10}) \cup (2-\sqrt{10}, 1] \cup \left[3, 2+\sqrt{10}\right) \cup \left(2+\sqrt{10}, \infty\right)$.

7. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 4, we gave $(h \circ g)(x) = \frac{4-2\sqrt{x+3}}{3-\sqrt{x+3}}$. We now proceed as before.

   - *inside out*: We insert the expression $f(x)$ into $h \circ g$ first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= (h \circ g)\left(x^2-4x\right) \\ &= \frac{4-2\sqrt{(x^2-4x)+3}}{3-\sqrt{(x^2-4x)+3}} \\ &= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}} \end{aligned}$$

   - *outside in*: We use the formula for $(h \circ g)(x)$ first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) \\ &= \frac{4-2\sqrt{(f(x))+3}}{3-\sqrt{f(x))+3}} \\ &= \frac{4-2\sqrt{(x^2-4x)+3}}{3-\sqrt{(x^2-4x)+3}} \\ &= \frac{4-2\sqrt{x^2-4x+3}}{3-\sqrt{x^2-4x+3}} \end{aligned}$$

We note that the formula for $((h \circ g) \circ f)(x)$ before simplification is identical to that of $(h \circ (g \circ f))(x)$ before we simplified it. Hence, the two functions have the same domain, $h \circ (f \circ g)$ is $(-\infty, 2-\sqrt{10}) \cup (2-\sqrt{10}, 1] \cup \left[3, 2+\sqrt{10}\right) \cup \left(2+\sqrt{10}, \infty\right)$.

□

It should be clear from Example 1.1.1 that, in general, when you compose two functions, such as f and g above, the order matters.[2] We found that the functions $f \circ g$ and $g \circ f$ were different as were $g \circ h$ and $h \circ g$. Thinking of functions as processes, this isn't all that surprising. If we think of one process as putting on our socks, and the other as putting on our shoes, the order in which we do these two tasks does matter.[3] Also note the importance of finding the domain of the composite function **before** simplifying. For instance, the domain of $f \circ g$ is much different than its simplified formula would indicate. Composing a function with itself, as in the case of $h \circ h$, may seem odd. Looking at this from a procedural perspective, however, this merely indicates performing a task h and then doing it again - like setting the washing machine to do a 'double rinse'. Composing a function with itself is called 'iterating' the function, and we could easily spend an entire course on just that. The last two problems in Example 1.1.1 serve to demonstrate the **associative** property of functions. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. This property as well as another important property are listed in the theorem below.

THEOREM 1.1. **Properties of Function Composition:** Suppose f, g, and h are functions.

- $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- If I is defined as $I(x) = x$ for all real numbers x, then $I \circ f = f \circ I = f$.

By repeated applications of Definition 1.1, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 1.1. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise **??** in Section **??**. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(x)$. This establishes that we have an identity for function composition much in the same way the real number 1 is an identity for real number multiplication. That is, just as for any real number x, $1 \cdot x = x \cdot 1 = x$, we have for any function f, $I \circ f = f \circ I = f$. We shall see the concept of an identity take on great significance in the next

[2] This shows us function composition isn't **commutative**. An example of an operation we perform on two functions which is commutative is function addition, which we defined in Section **??**. In other words, the functions $f + g$ and $g + f$ are always equal. Which of the remaining operations on functions we have discussed are commutative?

[3] A more mathematical example in which the order of two processes matters can be found in Section **??**. In fact, all of the transformations in that section can be viewed in terms of composing functions with linear functions.

section. Out in the wild, function composition is often used to relate two quantities which may not be directly related, but have a variable in common, as illustrated in our next example.

EXAMPLE 1.1.2. The surface area S of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

SOLUTION. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi \left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'middle man' r. $\square$

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

EXAMPLE 1.1.3. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x) = |3x - 1|$
2. $G(x) = \dfrac{2}{x^2 + 1}$
3. $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

SOLUTION. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 1.1, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$, so we are done.

2. We attack deconstructing G from an operational approach. Given an input x, the first step is to square x, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of three functions: f, g and h. Our first function, f, is the function that squares its input, $f(x) = x^2$. The next function is the function that adds 1 to its input, $g(x) = x + 1$. Our last function takes its input and divides it into 2, $h(x) = \frac{2}{x}$. The claim is that $G = h \circ g \circ f$. We find $(h \circ g \circ f)(x) = h(g(f(x))) = h\left(g\left(x^2\right)\right) = h\left(x^2 + 1\right) = \frac{2}{x^2+1} = G(x)$.

3. If we look $H(x) = \frac{\sqrt{x}+1}{\sqrt{x}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{x}$ is a simple piece of the larger function. If we define $f(x) = \sqrt{x}$, we have $H(x) = \frac{f(x)+1}{f(x)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula from $g(x)$ by looking at what is being done to $f(x)$. We find $g(x) = \frac{x+1}{x-1}$. We check $(g \circ f)(x) = g(f(x)) = \frac{f(x)+1}{f(x)-1} = \frac{\sqrt{x}+1}{\sqrt{x}-1} = H(x)$, as required. $\square$

1.1.1 Exercises

1. Let $f(x) = 3x - 6$, $g(x) = |x|$, $h(x) = \sqrt{x}$ and $k(x) = \dfrac{1}{x}$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ k)(x)$
   (h) $(k \circ f)(x)$
   (i) $(h \circ k)(x)$
   (j) $(k \circ h)(x)$
   (k) $(f \circ g \circ h)(x)$
   (l) $(h \circ g \circ k)(x)$
   (m) $(k \circ h \circ f)(x)$
   (n) $(h \circ k \circ g \circ f)(x)$

2. Let $f(x) = 2x + 1$, $g(x) = x^2 - x - 6$ and $h(x) = \dfrac{x+6}{x-6}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(g \circ f)(x)$
   (b) $(h \circ f)(x)$
   (c) $(h \circ g)(x)$
   (d) $(h \circ h)(x)$

3. Let $f(x) = \sqrt{x-3}$, $g(x) = 4x + 3$ and $h(x) = \dfrac{x-2}{x+3}$. Find and simplify the indicated composite functions. Find the domain of each.

   (a) $(f \circ g)(x)$
   (b) $(g \circ f)(x)$
   (c) $(f \circ h)(x)$
   (d) $(h \circ f)(x)$
   (e) $(g \circ h)(x)$
   (f) $(h \circ g)(x)$
   (g) $(f \circ f)(x)$
   (h) $(g \circ g)(x)$
   (i) $(h \circ h)(x)$

4. Let $f(x) = \sqrt{9 - x^2}$ and $g(x) = x^2 - 9$. Find and simplify the indicated composite functions. State the domain of each.

   (a) $(f \circ f)(x)$
   (b) $(g \circ g)(x)$
   (c) $(g \circ f)(x)$
   (d) $(f \circ g)(x)$

5. Let f be the function defined by $f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$ and let g be the function defined $g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$.

   Find the each of the following values if it exists.

   (a) $(f \circ g)(3)$
   (b) $f(g(-1))$
   (c) $(f \circ f)(0)$
   (d) $(f \circ g)(-3)$
   (e) $(g \circ f)(3)$
   (f) $g(f(-3))$
   (g) $(g \circ g)(-2)$
   (h) $(g \circ f)(-2)$
   (i) $g(f(g(0)))$
   (j) $f(f(f(-1)))$
   (k) $f(f(f(f(f(1)))))$
   (l) $(\overbrace{g \circ g \circ \cdots \circ g}^{n \text{ times}})(0)$

6. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2}-4$?

7. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3+1$? What is the proper order of composition?

8. Write the following as a composition of two or more non-identity functions.

   (a) $h(x) = \sqrt{2x-1}$
   (b) $r(x) = \dfrac{2}{5x+1}$
   (c) $F(x) = \left(x^2-1\right)^3$
   (d) $R(x) = \dfrac{2x^3+1}{x^3-1}$

9. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

10. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

11. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2+20t+92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.
    (b) Find and simplify $(R \circ x)(t)$. What does this represent?
    (c) What is the revenue per hour at noon?

12. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

1.1.2 Answers

1. (a) $(f \circ g)(x) = 3|x| - 6$
   Domain: $(-\infty, \infty)$

   (b) $(g \circ f)(x) = |3x - 6|$
   Domain: $(-\infty, \infty)$

   (c) $(f \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$

   (d) $(h \circ f)(x) = \sqrt{3x - 6}$
   Domain: $[2, \infty)$

   (e) $(g \circ h)(x) = \sqrt{x}$
   Domain: $[0, \infty)$

   (f) $(h \circ g)(x) = \sqrt{|x|}$
   Domain: $(-\infty, \infty)$

   (g) $(f \circ k)(x) = \dfrac{3}{x} - 6$
   Domain: $(-\infty, 0) \cup (0, \infty)$

   (h) $(k \circ f)(x) = \dfrac{1}{3x - 6}$
   Domain: $(-\infty, 2) \cup (2, \infty)$

   (i) $(h \circ k)(x) = \sqrt{\dfrac{1}{x}}$
   Domain: $(0, \infty)$

   (j) $(k \circ h)(x) = \dfrac{1}{\sqrt{x}}$
   Domain: $(0, \infty)$

   (k) $(f \circ g \circ h)(x) = 3\sqrt{x} - 6$
   Domain: $[0, \infty)$

   (l) $(h \circ g \circ k)(x) = \sqrt{\left|\dfrac{1}{x}\right|}$
   Domain: $(-\infty, 0) \cup (0, \infty)$

   (m) $(k \circ h \circ f)(x) = \dfrac{1}{\sqrt{3x - 6}}$
   Domain: $(2, \infty)$

   (n) $(h \circ k \circ g \circ f)(x) = \sqrt{\dfrac{1}{|3x - 6|}}$
   Domain: $(-\infty, 2) \cup (2, \infty)$

2. (a) $(g \circ f)\,(x) = 4x^2 + 2x - 6$
   Domain: $(-\infty, \infty)$

   (b) $(h \circ f)\,(x) = \dfrac{2x + 7}{2x - 5}$
   Domain: $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$

   (c) $(h \circ g)\,(x) = \dfrac{x^2 - x}{x^2 - x - 12}$
   Domain: $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

   (d) $(h \circ h)\,(x) = -\dfrac{7x - 30}{5x - 42}$
   Domain: $(-\infty, 6) \cup \left(6, \frac{42}{5}\right) \cup \left(\frac{42}{5}, \infty\right)$

3. (a) $(f \circ g)\,(x) = 2\sqrt{x}$
   Domain: $[0, \infty)$

   (b) $(g \circ f)\,(x) = 4\sqrt{x - 3} + 3$
   Domain: $[3, \infty)$

   (c) $(f \circ h)\,(x) = \sqrt{\dfrac{-2x - 11}{x + 3}}$
   Domain: $\left[-\frac{11}{2}, -3\right)$

   (d) $(h \circ f)\,(x) = \dfrac{\sqrt{x - 3} - 2}{\sqrt{x - 3} + 3}$
   Domain: $[3, \infty)$

   (e) $(g \circ h)\,(x) = \dfrac{7x + 1}{x + 3}$
   Domain: $(-\infty, -3) \cup (-3, \infty)$

   (f) $(h \circ g)\,(x) = \dfrac{4x + 1}{4x + 6}$
   Domain: $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$

   (g) $(f \circ f)\,(x) = \sqrt{\sqrt{x - 3} - 3}$
   Domain: $[12, \infty)$

   (h) $(g \circ g)\,(x) = 16x + 15$
   Domain: $(-\infty, \infty)$

   (i) $(h \circ h)\,(x) = \dfrac{-x - 8}{4x + 7}$
   Domain:
   $(-\infty, -3) \cup \left(-3, -\frac{7}{4}\right) \cup \left(-\frac{7}{4}, \infty\right)$

4. (a) $(f \circ f)(x) = |x|$
   Domain: $[-3, 3]$

   (b) $(g \circ g)(x) = x^4 - 18x^2 + 72$
   Domain: $(-\infty, \infty)$

   (c) $(g \circ f)(x) = -x^2$
   Domain: $[-3, 3]$

   (d) $(f \circ g)(x) = \sqrt{-x^4 + 18x^2 - 72}$
   Domain: $[-\sqrt{12}, -\sqrt{6}] \cup [\sqrt{6}, \sqrt{12}]$[4]

5. (a) $(f \circ g)(3) = f(g(3)) = f(2) = 4$

   (b) $f(g(-1)) = f(-4)$ which is undefined

   (c) $(f \circ f)(0) = f(f(0)) = f(1) = 3$

   (d) $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

   (e) $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

   (f) $g(f(-3)) = g(4)$ which is undefined

   (g) $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

   (h) $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

   (i) $g(f(g(0))) = g(f(0)) = g(1) = -3$

   (j) $f(f(f(-1))) = f(f(0)) = f(1) = 3$

   (k) $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$

   (l) $\overbrace{(g \circ g \circ \cdots \circ g)}^{n \text{ times}}(0) = 0$

6. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

7. One possible solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

8. (a) $h(x) = (g \circ f)(x)$ where $f(x) = 2x - 1$ and $g(x) = \sqrt{x}$.

   (b) $r(x) = (g \circ f)(x)$ where $f(x) = 5x + 1$ and $g(x) = \dfrac{2}{x}$.

   (c) $F(x) = (g \circ f)(x)$ where $f(x) = x^2 - 1$ and $g(x) = x^3$.

   (d) $R(x) = (g \circ f)(x)$ where $f(x) = x^3$ and $g(x) = \dfrac{2x+1}{x-1}$.

9. $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \dfrac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

10. $V(x) = x^3$ so $V(x(t)) = (t+1)^3$

11. (a) $R(x) = 2x$

    (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.

    (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

[4] The quantity $-x^4 + 18x^2 - 72$ is a 'quadratic in disguise' which factors nicely.

1.2 Inverse Functions

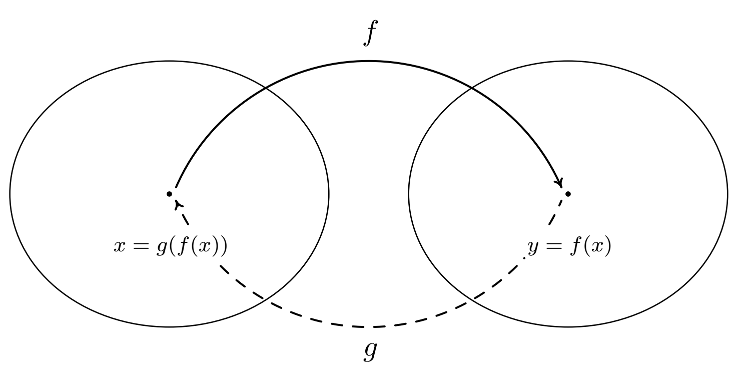
Thinking of a function as a process like we did in Section **??**, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, as we saw in Section **??**

1. multiply by 3
2. add 4

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, the function g should

1. *subtract* 4
2. *divide* by 3

Following this procedure, we get $g(x) = \frac{x-4}{3}$. Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and put that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 1.1, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively. Abstractly, we can visualize the relationship between f and g in the diagram below.



The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

DEFINITION 1.2. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f **and**
2. $(f \circ g)(x) = x$ for all x in the domain of g.

Then f and g are said to be **inverses** of each other. The functions f and g are said to be **invertible**.

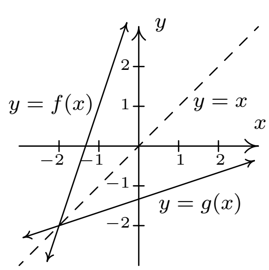
Our first result of the section formalizes the concepts that inverse functions exchange inputs and outputs and is a consequence of Definition 1.2 and the Fundamental Graphing Principle for Functions.

THEOREM 1.2. **Properties of Inverse Functions:** Suppose f and g are inverse functions.

- The range[a] of f is the domain of g and the domain of f is the range of g
- $f(a) = b$ if and only if $g(b) = a$
- (a, b) is on the graph of f if and only if (b, a) is on the graph of g

[a]Recall this is the set of all outputs of a function.

The third property in Theorem 1.2 tells us that the graphs of inverse functions are reflections about the line $y = x$. For a proof of this, we refer the reader to Example **??** in Section **??**. A plot of the inverse functions $f(x) = 3x + 4$ and $g(x) = \frac{x-4}{3}$ confirms this to be the case.



If we abstract one step further, we can express the sentiment in Definition 1.2 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[1] to the domain of f and I_2 is the identity function restricted to the domain of g. In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 1.1, we can show that function inverses are unique.[2] Suppose g and h are both inverses of a function f. By Theorem 1.2, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.[3] We summarize the discussion of the last two paragraphs in the following theorem.[4]

Theorem 1.3. **Uniqueness of Inverse Functions and Their Graphs :** Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read f-inverse)
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely **not** the case since, for instance, $f(x) = 3x+4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$. Why does this confusing notation persist? As we mentioned in Section 1.1, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

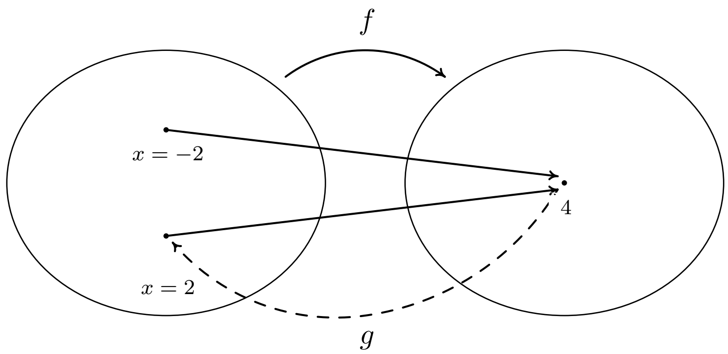
Let's turn our attention to the function $f(x) = x^2$. Is f invertible? A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. Checking the composition yields $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x for all x in the domain $(-\infty, \infty)$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$, which means g failed to return the input -2 from its output 4. What g did, however, is match the output 4 to a **different** input, namely 2, which satisfies $f(2) = 4$. This issue is presented schematically in the picture below.

[1]The identity function I, which was introduced in Section **??** and mentioned in Theorem 1.1, has a domain of all real numbers. In general, the domains of f and g are not all real numbers, which necessitates the restrictions listed here.

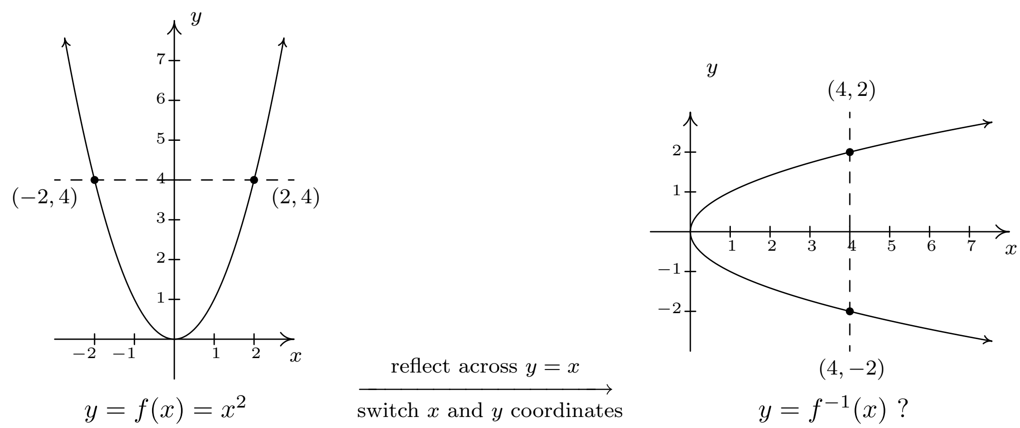
[2]In other words, invertible functions have exactly one inverse.

[3]It is an excellent exercise to explain each step in this string of equalities.

[4]In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.



We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a **function** which takes 4 back to **both** $x = 2$ and $x = -2$. (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if $y = f^{-1}(x)$ exists, its graph can be obtained by reflecting $y = x^2$ about the line $y = x$, in accordance with Theorem 1.3. Doing so produces



$y = f(x) = x^2$ $\qquad$ $y = f^{-1}(x)$?

We see that the line $x = 4$ intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, Theorem **??**, and as such, does not represent y as a function of x. The vertical line $x = 4$ on the graph on the right corresponds to the *horizontal line* $y = 4$ on the graph of $y = f(x)$. The fact that the horizontal line $y = 4$ intersects the graph of f twice means two **different** inputs, namely $x = -2$ and $x = 2$, are matched with the **same** output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, **different** inputs must go to **different** outputs, or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

DEFINITION 1.3. A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

THEOREM 1.4. **The Horizontal Line Test:** A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if f is invertible, then f must be one-to-one, otherwise the graph given by reflecting the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function, then f is invertible and we have found f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y. We summarize these results below.

THEOREM 1.5. **Equivalent Conditions for Invertibility:** Suppose f is a function. The following statements are equivalent.

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

We put this result to work in the next example.

EXAMPLE 1.2.1. Determine if the following functions are one-to-one in two ways: (a) analytically using Definition 1.3 and (b) graphically using the Horizontal Line Test.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$
3. $h(x) = x^2 - 2x + 4$
4. $F = \{(-1,1), (0,2), (2,1)\}$

SOLUTION.

1. (a) To determine if f is one-to-one analytically, we assume $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

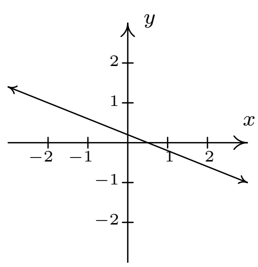
(b) To check if f is one-to-one graphically, we look to see if the graph of $y = f(x)$ passes the Horizontal Line Test. We have that f is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of f passes the Horizontal Line Test as seen below.

2. (a) We begin with the assumption that $g(c) = g(d)$ and try to show $c = d$.

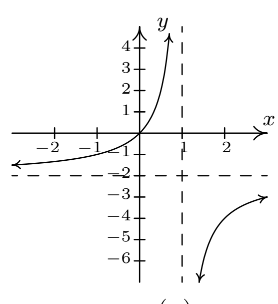
$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c-2cd &= 2d-2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that g is one-to-one.

(b) We can graph g using the six step procedure outlined in Section **??**. We get the sole intercept at $(0, 0)$, a vertical asymptote $x = 1$ and a horizontal asymptote (which the graph never crosses) $y = -2$. We see from that the graph of g passes the Horizontal Line Test.



$y = f(x)$



$y = g(x)$

3. (a) We begin with $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

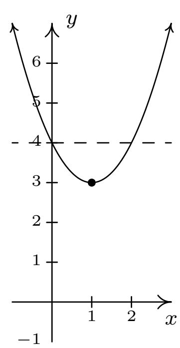
$$\begin{aligned}
h(c) &= h(d) \\
c^2 - 2c + 4 &= d^2 - 2d + 4 \\
c^2 - 2c &= d^2 - 2d \\
c^2 - d^2 - 2c + 2d &= 0 \\
(c+d)(c-d) - 2(c-d) &= 0 \\
(c-d)((c+d)-2) &= 0 \qquad \text{factor by grouping} \\
c - d = 0 \quad &\text{or} \quad c + d - 2 = 0 \\
c = d \quad &\text{or} \quad c = 2 - d
\end{aligned}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that f may not be one-to-one. Taking $d = 0$, we get $c = 0$ or $c = 2$. With $f(0) = 4$ and $f(2) = 4$, we have produced two different inputs with the same output meaning f is not one-to-one.

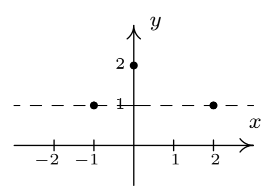
(b) We note that h is a quadratic function and we graph $y = h(x)$ using the techniques presented in Section **??**. The vertex is $(1, 3)$ and the parabola opens upwards. We see immediately from the graph that h is not one-to-one, since there are several horizontal lines which cross the graph more than once.

4. (a) The function F is given to us as a set of ordered pairs. The condition $F(c) = F(d)$ means the outputs from the function (the y-coordinates of the ordered pairs) are the same. We see that the points $(-1, 1)$ and $(2, 1)$ are both elements of F with $F(-1) = 1$ and $F(2) = 1$. Since $-1 \neq 2$, we have established that F is **not** one-to-one.

   (b) Graphically, we see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.



$y = h(x)$



$y = F(x)$

□

We have shown that the functions f and g in Example 1.2.1 are one-to-one. This means they are invertible, so it is natural to wonder what $f^{-1}(x)$ and $g^{-1}(x)$ would be. For $f(x) = \frac{1-2x}{5}$, we can think our way through the inverse since there is only one occurrence of x. We can track step-by-step what is done to x and reverse those steps as we did at the beginning of the chapter. The function $g(x) = \frac{2x}{1-x}$ is a bit trickier since x occurs in two places. When one evaluates $g(x)$ for a specific value of x, which is first, the $2x$ or the $1 - x$? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem 1.2 tells us equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

Steps for finding the Inverse of a One-to-one Function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and outputs.

EXAMPLE 1.2.2. Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$

SOLUTION.

1. As we mentioned earlier, it is possible to think our way through the inverse of f by recording the steps we apply to x and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write $y = f(x)$ and proceed to switch x and y

$$\begin{aligned}
y &= f(x) \\
y &= \frac{1-2x}{5} \\
x &= \frac{1-2y}{5} && \text{switch } x \text{ and } y \\
5x &= 1-2y \\
5x-1 &= -2y \\
\frac{5x-1}{-2} &= y \\
y &= -\tfrac{5}{2}x + \tfrac{1}{2}
\end{aligned}$$

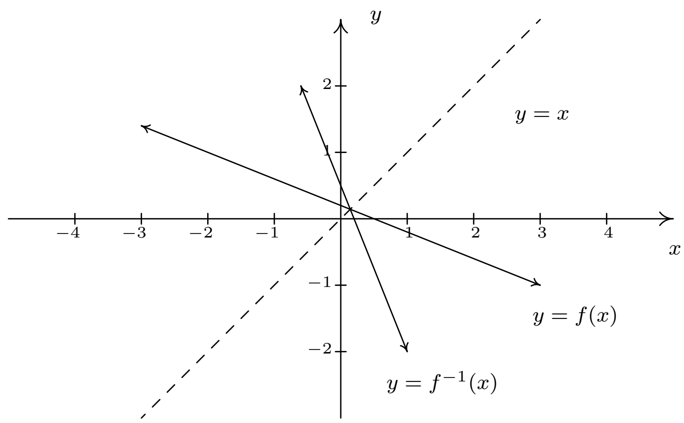
We have $f^{-1}(x) = -\frac{5}{2}x+\frac{1}{2}$. To check this answer analytically, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= -\tfrac{5}{2} f(x) + \tfrac{1}{2} \\ &= -\tfrac{5}{2}\left(\frac{1-2x}{5}\right) + \tfrac{1}{2} \\ &= -\tfrac{1}{2}(1-2x) + \tfrac{1}{2} \\ &= -\tfrac{1}{2} + x + \tfrac{1}{2} \\ &= x \checkmark\end{aligned}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the range of f which is also all real numbers. (Recall that the domain of f^{-1}) is the range of f.)

$$\begin{aligned}\left(f \circ f^{-1}\right)(x) &= f(f^{-1}(x)) \\ &= \frac{1-2f^{-1}(x)}{5} \\ &= \frac{1-2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} \\ &= \frac{1+5x-1}{5} \\ &= \frac{5x}{5} \\ &= x \checkmark\end{aligned}$$

To check our answer graphically, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.[5] They appear to be reflections across the line $y = x$.



2. To find $g^{-1}(x)$, we start with $y = g(x)$. We note that the domain of g is $(-\infty, 1) \cup (1, \infty)$.

[5] Note that if you perform your check on a calculator for more sophisticated functions, you'll need to take advantage of the 'ZoomSquare' feature to get the correct geometric perspective.

$$
\begin{aligned}
y &= g(x) \\
y &= \frac{2x}{1-x} \\
x &= \frac{2y}{1-y} && \text{switch } x \text{ and } y \\
x(1-y) &= 2y \\
x - xy &= 2y \\
x &= xy + 2y \\
x &= y(x+2) && \text{factor} \\
y &= \frac{x}{x+2}
\end{aligned}
$$

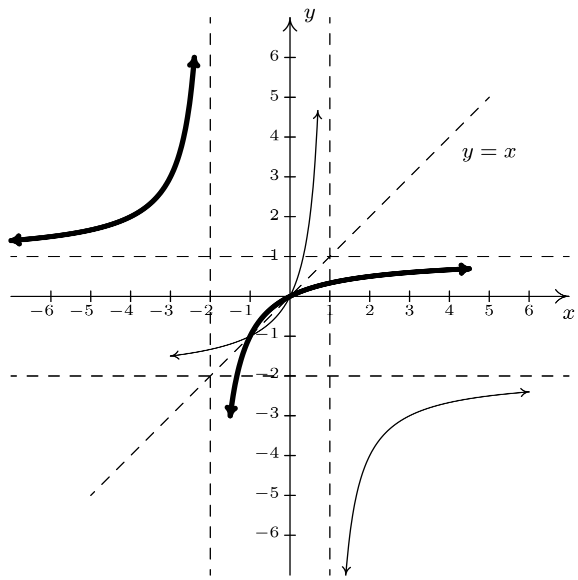
We obtain $g^{-1}(x) = \frac{x}{x+2}$. To check this analytically, we first check $\left(g^{-1} \circ g\right)(x) = x$ for all x in the domain of g, that is, for all $x \neq 1$.

$$
\begin{aligned}
\left(g^{-1} \circ g\right)(x) &= g^{-1}(g(x)) \\
&= g^{-1}\left(\frac{2x}{1-x}\right) \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right) + 2} \\
&= \frac{\left(\frac{2x}{1-x}\right)}{\left(\frac{2x}{1-x}\right) + 2} \cdot \frac{(1-x)}{(1-x)} && \text{clear denominators} \\
&= \frac{2x}{2x + 2(1-x)} \\
&= \frac{2x}{2x + 2 - 2x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}
$$

Next, we check $g\left(g^{-1}(x)\right) = x$ for all x in the range of g. From the graph of g in Example 1.2.1, we have that the range of g is $(-\infty, -2) \cup (-2, \infty)$. This matches the domain we get from the formula $g^{-1}(x) = \frac{x}{x+2}$, as it should.

$$
\begin{aligned}
\left(g \circ g^{-1}\right)(x) &= g\left(g^{-1}(x)\right) \\
&= g\left(\frac{x}{x+2}\right) \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \\
&= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} \quad \text{clear denominators} \\
&= \frac{2x}{(x+2)-x} \\
&= \frac{2x}{2} \\
&= x \checkmark
\end{aligned}
$$

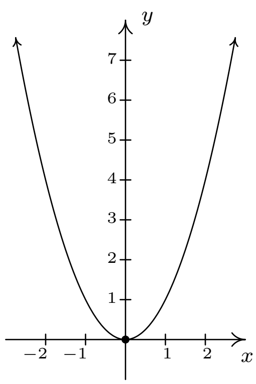
Graphing $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for $y = g^{-1}(x)$. Note that the vertical asymptote $x = 1$ of the graph of g corresponds to the horizontal asymptote $y = 1$ of the graph of g^{-1}, as it should since x and y are switched. Similarly, the horizontal asymptote $y = -2$ of the graph of g corresponds to the vertical asymptote $x = -2$ of the graph of g^{-1}.

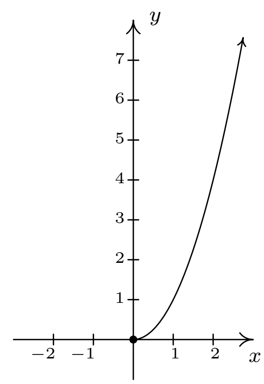


$y = g(x)$ and $\boldsymbol{y = g^{-1}(x)}$

$\square$

We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible. However, if we restrict the domain of f, we can produce a new function g which is one-to-one. If we define $g(x) = x^2$, $x \geq 0$, then we have



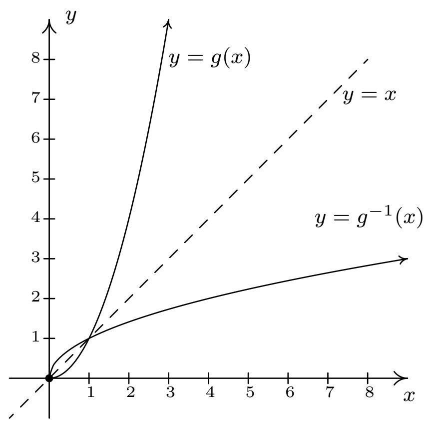


$y = f(x) = x^2$ $\xrightarrow{\text{restrict domain to } x \geq 0}$ $y = g(x) = x^2,\ x \geq 0$

The graph of g passes the Horizontal Line Test. To find an inverse of g, we proceed as usual

$$\begin{array}{rcll} y & = & g(x) & \\ y & = & x^2,\ \ x \geq 0 & \\ x & = & y^2,\ \ y \geq 0 & \text{switch } x \text{ and } y \\ y & = & \pm\sqrt{x} & \\ y & = & \sqrt{x} & \text{since } y \geq 0 \end{array}$$

We get $g^{-1}(x) = \sqrt{x}$. At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on g saves the day. We get $\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) = g^{-1}\left(x^2\right) = \sqrt{x^2} = |x| = x$, since $x \geq 0$. Checking $\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$. Graphing[6] g and g^{-1} on the same set of axes shows that they are reflections about the line $y = x$.



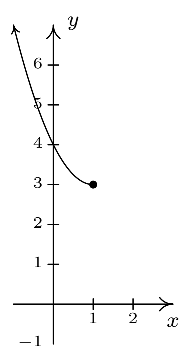
Our next example continues the theme of domain restriction.

Example 1.2.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(x) = \sqrt{x+2} - 1$

Solution.

1. The function j is a restriction of the function h from Example 1.2.1. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. We see that the graph of j passes the Horizontal Line Test and thus j is invertible.



$y = j(x)$

[6]We graphed $y = \sqrt{x}$ in Section **??**.

We now use our algorithm to find $j^{-1}(x)$.

$$\begin{aligned}
y &= j(x) \\
y &= x^2 - 2x + 4, \;\; x \leq 1 \\
x &= y^2 - 2y + 4, \;\; y \leq 1 && \text{switch } x \text{ and } y \\
0 &= y^2 - 2y + 4 - x \\
y &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} && \text{quadratic formula, } c = 4 - x \\
y &= \frac{2 \pm \sqrt{4x - 12}}{2} \\
y &= \frac{2 \pm \sqrt{4(x-3)}}{2} \\
y &= \frac{2 \pm 2\sqrt{x-3}}{2} \\
y &= \frac{2\left(1 \pm \sqrt{x-3}\right)}{2} \\
y &= 1 \pm \sqrt{x-3} \\
y &= 1 - \sqrt{x-3} && \text{since } y \leq 1.
\end{aligned}$$

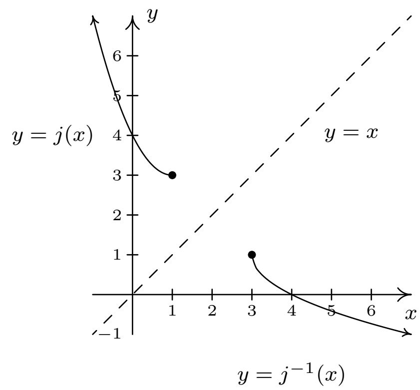
We have $j^{-1}(x) = 1 - \sqrt{x-3}$. When we simplify $\left(j^{-1} \circ j\right)(x)$, we need to remember that the domain of j is $x \leq 1$.

$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \;\; x \leq 1 \\
&= 1 - \sqrt{(x^2 - 2x + 4) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) && \text{since } x \leq 1 \\
&= x \; \checkmark
\end{aligned}$$

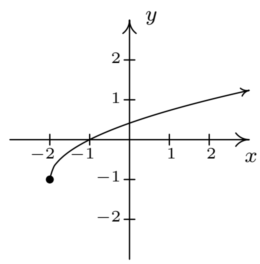
Checking $j \circ j^{-1}$, we get

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 + 2\sqrt{x-3} + 4 \\
&= 3 + x - 3 \\
&= x \; \checkmark
\end{aligned}$$

We can use what we know from Section **??** to graph $y = j^{-1}(x)$ on the same axes as $y = j(x)$ to get



2. We graph $y = k(x) = \sqrt{x+2} - 1$ using what we learned in Section **??** and see k is one-to-one.



$y = k(x)$

We now try to find k^{-1}.

$$
\begin{array}{rcll}
y & = & k(x) & \\
y & = & \sqrt{x+2} - 1 & \\
x & = & \sqrt{y+2} - 1 & \text{switch } x \text{ and } y \\
x + 1 & = & \sqrt{y+2} & \\
(x+1)^2 & = & \left(\sqrt{y+2}\right)^2 & \\
x^2 + 2x + 1 & = & y + 2 & \\
y & = & x^2 + 2x - 1 &
\end{array}
$$

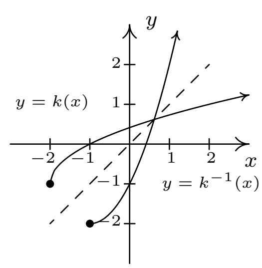
We have $k^{-1}(x) = x^2 + 2x - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem 1.2 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. We now check that this works in our compositions.

$$
\begin{aligned}
\left(k^{-1} \circ k\right)(x) &= k^{-1}(k(x)) \\
&= k^{-1}\left(\sqrt{x+2}-1\right), \quad x \geq -2 \\
&= \left(\sqrt{x+2}-1\right)^2 + 2\left(\sqrt{x+2}-1\right) - 1 \\
&= \left(\sqrt{x+2}\right)^2 - 2\sqrt{x+2} + 1 + 2\sqrt{x+2} - 2 - 1 \\
&= x + 2 - 2 \\
&= x \checkmark
\end{aligned}
$$

and

$$
\begin{aligned}
\left(k \circ k^{-1}\right)(x) &= k\left(x^2+2x-1\right) \quad x \geq -1 \\
&= \sqrt{(x^2+2x-1)+2} - 1 \\
&= \sqrt{x^2+2x+1} - 1 \\
&= \sqrt{(x+1)^2} - 1 \\
&= |x+1| - 1 \\
&= x + 1 - 1 && \text{since } x \geq -1 \\
&= x \checkmark
\end{aligned}
$$

Graphically, everything checks out as well, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.



□

Our last example of the section gives an application of inverse functions.

Example 1.2.4. Recall from Section **??** that the price-demand equation for the PortaBoy game system is $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$, where x represents the number of systems sold weekly and p is the price per system in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section **??** that the weekly profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.

4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example **??**.

Solution.

1. We leave to the reader to show the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.

2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the weekly sales and furnished the price per system as the output, p^{-1} takes the price per system and returns the weekly sales as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in a week if the price is set at \$220 per system.

3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra,[7] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$. To understand what this means, recall that the original profit function P gave us the weekly profit as a function of the weekly sales. The function p^{-1} gives us the weekly sales as a function of the price. Hence, $P \circ p^{-1}$ takes as its input a price. The function p^{-1} returns the weekly sales, which in turn is fed into P to return the weekly profit. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the weekly profit (in dollars) as a function of the price per system, x, using the weekly sales $p^{-1}(x)$ as the 'middle man'.

4. We know from Section **??** that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation **??**, $x = -\frac{b}{2a} = 165$. Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example **??**, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. $\square$

[7] It is good review to actually do this!

1.2.1 Exercises

1. Show that the following functions are one-to-one and find the inverse. Check your answers algebraically and graphically. Verify the range of f is the domain of f^{-1} and vice-versa.

   (a) $f(x) = 6x - 2$
   (b) $f(x) = 5x - 3$
   (c) $f(x) = 1 - \dfrac{4+3x}{5}$
   (d) $f(x) = -\sqrt{x-5} + 2$
   (e) $f(x) = \sqrt{3x-1} + 5$
   (f) $f(x) = \sqrt[5]{3x-1}$
   (g) $f(x) = x^2 - 10x,\ x \geq 5$
   (h) $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
   (i) $f(x) = x^2 - 6x + 5,\ x \leq 3$
   (j) $f(x) = 4x^2 + 4x + 1,\ x < -1$
   (k) $f(x) = \dfrac{3}{4-x}$
   (l) $f(x) = \dfrac{x}{1-3x}$
   (m) $f(x) = \dfrac{2x-1}{3x+4}$
   (n) $f(x) = \dfrac{4x+2}{3x-6}$
   (o) $f(x) = \dfrac{-3x-2}{x+3}$

2. Show that the Fahrenheit to Celsius conversion function found in Exercise **??** in Section **??** is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

3. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem 1.2 to help you compute $f^{-1}(1),\ f^{-1}(5),$ and $f^{-1}(-3)$.

4. With the help of your classmates, find a formula for the inverse of the following.

   (a) $f(x) = ax + b,\ a \neq 0$
   (b) $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$
   (c) $f(x) = \frac{ax+b}{cx+d},\ a \neq 0,\ b \neq 0,\ c \neq 0,\ d \neq 0$
   (d) $f(x) = ax^2 + bx + c$ where $a \neq 0,\ x \geq -\frac{b}{2a}$.

5. Let $f(x) = \frac{2x}{x^2-1}$. Using the techniques in Section **??**, graph $y = f(x)$. Verify f is one-to-one on the interval $(-1, 1)$. Use the procedure outlined on Page 23 and your graphing calculator to find the formula for $f^{-1}(x)$. Note that since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

6. Suppose f is an invertible function. Prove that if graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at all, they do so on the line $y = x$.

7. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

8. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

9. What graphical feature must a function f possess for it to be its own inverse?

1.2.2 Answers

1. (a) $f^{-1}(x) = \dfrac{x+2}{6}$
   (b) $f^{-1}(x) = \dfrac{x+3}{5}$
   (c) $f^{-1}(x) = -\frac{5}{3}x + \frac{1}{3}$
   (d) $f^{-1}(x) = (x-2)^2 + 5,\ x \leq 2$
   (e) $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3},\ x \geq 5$
   (f) $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
   (g) $f^{-1}(x) = 5 + \sqrt{x+25}$
   (h) $f^{-1}(x) = -\sqrt{\frac{x+5}{3}} - 4$
   (i) $f^{-1}(x) = 3 - \sqrt{x+4}$
   (j) $f^{-1}(x) = -\frac{\sqrt{x}+1}{2},\ x > 1$
   (k) $f^{-1}(x) = \dfrac{4x-3}{x}$
   (l) $f^{-1}(x) = \dfrac{x}{3x+1}$
   (m) $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
   (n) $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
   (o) $f^{-1}(x) = \dfrac{-3x-2}{x+3}$

3. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$