Chapter 1

Polynomial Functions

1.1 Graphs of Polynomials

Three of the families of functions studied thus far: constant, linear and quadratic, belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 1.1. A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1. $\ldots a_n$ are real numbers and $n \geq 1$ is a natural number.[a] The domain of a polynomial function is $(-\infty, \infty)$.

[a]Recall this means n is a 'counting number' $n = 1, 2, 3, \ldots$

There are several things about Definition 1.1 that may be off-putting or downright frightening. The best thing to do is look at an example. Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with Definition 1.1, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$, and $a_0 = -5$. In other words, a_5 is the coefficient of x^5, a_4 is the coefficient of x^4, and so forth; the subscript on the a's merely indicates to which power of x the coefficient belongs. The business of restricting n to be a natural number lets us focus on well-behaved algebraic animals.[1]

Example 1.1.1. Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x) = \dfrac{4 + x^3}{x}$
2. $p(x) = \dfrac{4x + x^3}{x}$
3. $q(x) = \dfrac{4x + x^3}{x^2 + 4}$
4. $f(x) = \sqrt[3]{x}$
5. $h(x) = |x|$
6. $z(x) = 0$

Solution.

1. We note directly that the domain of $g(x) = \frac{x^3+4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, g can't be a polynomial.
2. Even though $p(x) = \frac{x^3+4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in Definition 1.1, the domain of p, which, as you may recall, we determine **before** we simplify, excludes 0. Alas, p is not a polynomial function for the same reason g isn't.
3. After what happened with p in the previous part, you may be a little shy about simplifying $q(x) = \frac{x^3+4x}{x^2+4}$ to $q(x) = x$, which certainly fits Definition 1.1. If we look at the domain of

[1]Enjoy this while it lasts. Before we're through with the book, you'll have been exposed to the most terrible of algebraic beasts. We will tame them all, in time.

q before we simplified, we see that it is, indeed, all real numbers. A function which can be written in the form of Definition 1.1 whose domain is all real numbers is, in fact, a polynomial.

4. We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\frac{1}{3}$ is not a natural number, f is not a polynomial.

5. The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of x (even though it can be written as a piecewise function involving polynomials.) As we shall see in this section, graphs of polynomials possess a quality[2] that the graph of h does not.

6. There's nothing in Definition 1.1 which prevents all the coefficients a_n, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

Definition 1.2. Suppose f is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term $a_n x^n$ is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.[a]

[a] Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials in Definition 1.2. Why not just lump them all together and, instead of forcing n to be a natural number, $n = 1, 2, \ldots$, let n be a whole number, $n = 0, 1, 2, \ldots$. We could unify all the cases, since, after all, isn't $a_0x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$ and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not.[3] Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from Definition 1.2 is that we can now think of linear functions as

[2] One which really relies on Calculus to verify.

[3] Technically, 0^0 is an indeterminant form, which is a special case of being undefined. The authors realize this is beyond pedantry, but we wouldn't mention it if we didn't feel it was neccessary.

degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

Example 1.1.2. Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x + x^3$
3. $h(x) = \dfrac{4-x}{5}$
4. $p(x) = (2x-1)^3(x-2)(3x+2)$

Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 1.2, and we see the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

2. The form given in Definition 1.2 has the highest power of x first. To that end, we re-write $g(x) = 12x + x^3 = x^3 + 12x$, and see the degree of g is 3, the leading term is x^3, the leading coefficient is 1 and the constant term is 0.

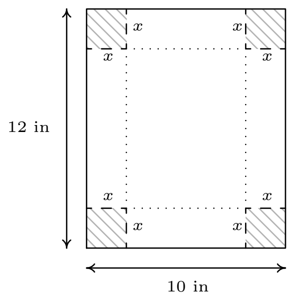
3. We need to rewrite the formula for h so that it resembles the form given in Definition 1.2: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. We see the degree of h is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

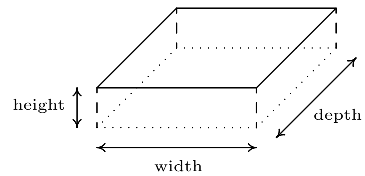
4. It may seem that we have some work ahead of us to get p in the form of Definition 1.2. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2x-1)^3(x-2)(3x+2)$. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

Our next example shows how polynomials of higher degree arise 'naturally'[4] in even the most basic geometric applications.

Example 1.1.3. A box with no top is to be fashioned from a 10 inch × 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.

[4] this is a dangerous word...





1. Find the volume V of the box as a function of x. Include an appropriate applied domain.

2. Use a graphing calculator to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

SOLUTION.

1. From Geometry, we know Volume = width $\times$ height $\times$ depth. The key is to now find each of these quantities in terms of x. From the figure, we see the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10 - 2x$ inches for the width.[5] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12 - 2x$ inches remaining. As a function[6] of x, the volume is

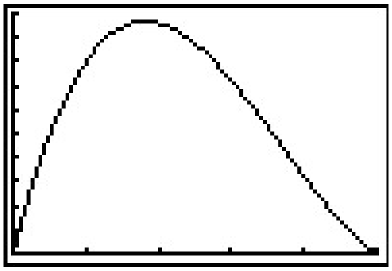
$$V(x) = x(10 - 2x)(12 - 2x) = 4x^3 - 44x^2 + 120x.$$

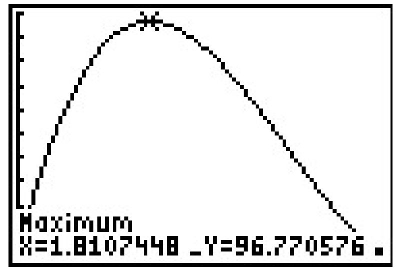
   To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graphing calculator, we see the graph of $y = V(x)$ has a relative maximum. For $0 < x < 5$, this is also the absolute maximum. Using the 'Maximum' feature of the calculator, we get $x \approx 1.81$, $y \approx 96.77$. The height, $x \approx 1.81$ inches, the width, $10 - 2x \approx 6.38$ inches, and the depth $12 - 2x \approx 8.38$ inches. The y-coordinate is the maximum volume, which is approximately 96.77 cubic inches (also written in^3).

[5] There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.

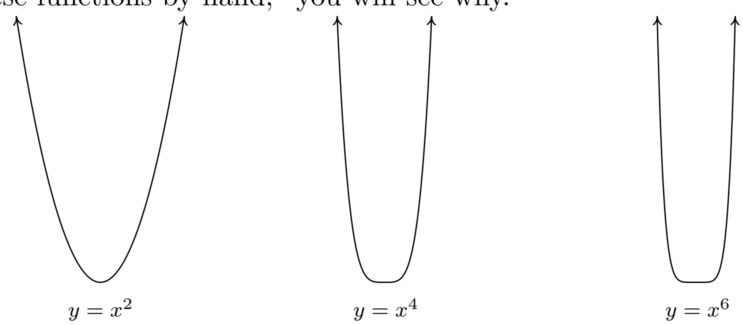
[6] When we write $V(x)$, it is in the context of function notation, not the volume V times the quantity x.





□

In order to solve Example 1.1.3, we made good use of the graph of the polynomial $y = V(x)$. So we ought to turn our attention to graphs of polynomials in general. Below are the graphs of $y = x^2$, $y = x^4$, and $y = x^6$, side-by-side. We have omitted the axes so we can see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand,[7] you will see why.



All of these functions are even, (Do you remember how to show this?) and it is exactly because the exponent is even.[8] One of the most important features of these functions which we can be seen graphically is their **end behavior**. The end behavior of a function is a way to describe what is happening to the function values as the x values approach the 'ends' of the x-axis:[9] that is, as they become small without bound[10] (written $x \to -\infty$) and, on the flip side, as they become large without bound[11] (written $x \to \infty$). For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behavior, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behavior of f as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$. The same can be said for any function of the form $f(x) = x^n$ where n is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the x-axis,[12] we have

[7] Make sure you choose some x-values between -1 and 1.

[8] Herein lies one of the possible origins of the term 'even' when applied to functions.

[9] Of course, there are no ends to the x-axis.

[10] We think of x as becoming a very large **negative** number far to the left of zero.

[11] We think of x as moving far to the right of zero and becoming a very large **positive** number.

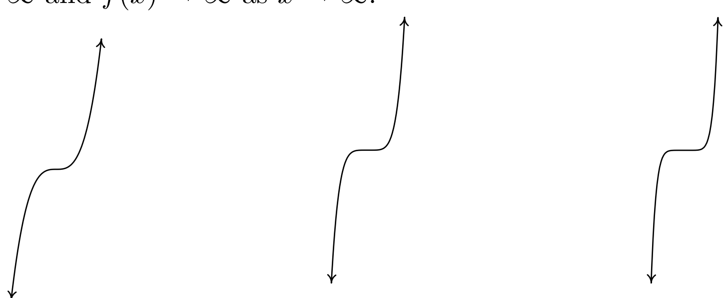
[12] See Theorems **??** and **??** in Section **??**.

End Behavior of functions $f(x) = ax^n$, n even.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

$a > 0$ $\qquad$ $a < 0$

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$ is an odd natural number.[13] Below we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd.[14] The end behavior of these functions is all the same, with $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



$y = x^3$ $\qquad$ $y = x^5$ $\qquad$ $y = x^7$

As with the even degreed functions we studied earlier, we can generalize their end behavior.

End Behavior of functions $f(x) = ax^n$, n odd.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

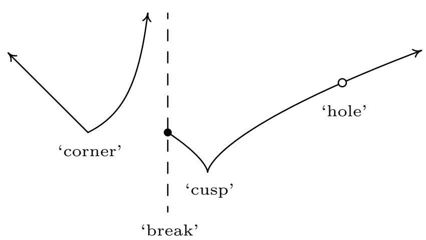
$a > 0$ $\qquad$ $a < 0$

Despite having different end behavior, all functions of the form $f(x) = ax^n$ for natural numbers n share two properties which help distinguish them from other animals in the algebra zoo: they are

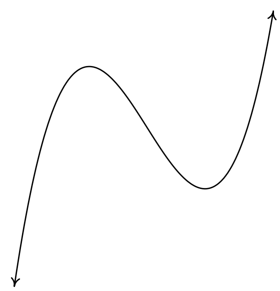
[13] We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.

[14] And are, perhaps, the inspiration for the moniker 'odd function'.

continuous and **smooth**. While these concepts are formally defined using Calculus,[15] informally, graphs of continuous functions have no 'breaks' or 'holes' in their graphs, and smooth functions have no 'sharp turns.' It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. Below we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in graph on the right.



Pathologies not found on graphs of polynomials



The graph of a polynomial

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteric '$\vee$' shape, cannot be a polynomial. The notion of continuity is what allowed us to construct the sign diagram for quadratic inequalities as we did in Section **??**. This last result is formalized in the following theorem.

Theorem 1.1. **The Intermediate Value Theorem (Polynomial Zero Version):** If f is a polynomial where $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most oft used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense, since it says, geometrically, that the graph of a polynomial function cannot be above the x-axis at one point and below the x-axis at another point without crossing the x-axis somewhere in between. The following example uses the Intermediate Value Theorem to establish a fact that that most students take for granted. Many students, and sadly some instructors, will find it silly.

[15] In fact, if you take Calculus, you'll find that smooth functions are automatically continuous, so that saying 'polynomials are continuous and smooth' is redundant.

Example 1.1.4. Use the Intermediate Value Theorem to establish that $\sqrt{2}$ is a real number.

Solution. Consider the polynomial function $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(3) = 7$. Since $f(1)$ and $f(3)$ have different signs, the Intermediate Value Theorem guarantees us a real number c between 1 and 3 with $f(c) = 0$. If $c^2 - 2 = 0$ then $c = \pm\sqrt{2}$. Since c is between 1 and 3, c is positive, so $c = \sqrt{2}$. $\square$

Our primary use of the Intermediate Value Theorem is in the construction of sign diagrams, as in Section **??**, since it guarantees us that polynomial functions are always positive (+) or always negative (−) on intervals which do not contain any of its zeros. The general algorithm for polynomials is given below.

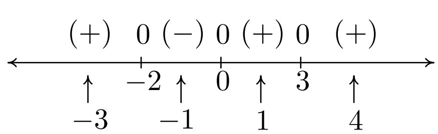
Steps for Constructing a Sign Diagram for a Polynomial Function

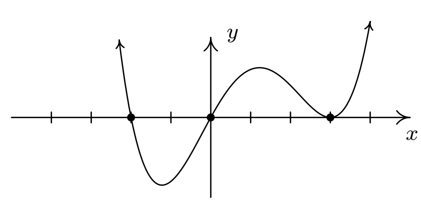
Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine the sign of $f(x)$ for each test value in step 2, and write that sign above the corresponding interval.

Example 1.1.5. Construct a sign diagram for $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$. Use it to give a rough sketch of the graph of $y = f(x)$.

Solution. First, we find the zeros of f by solving $x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$. We get $x = 0$, $x = 3$, and $x = -2$. (The equation $x^2 + 1 = 0$ produces no real solutions.) These three points divide the real number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. We select the test values $x = -3$, $x = -1$, $x = 1$, and $x = 4$. We find $f(-3)$ is (+), $f(-1)$ is (−) and $f(1)$ is (+) as is $f(4)$. Wherever f is (+), its graph is above the x-axis; wherever f is (−), its graph is below the x-axis. The x-intercepts of the graph of f are $(-2, 0)$, $(0, 0)$ and $(3, 0)$. Knowing f is smooth and continuous allows us to sketch its graph.





A sketch of $y = f(x)$

$\square$

A couple of notes about the Example 1.1.5 are in order. First, note that we purposefully did not label the y-axis in the sketch of the graph of $y = f(x)$. This is because the sign diagram gives us the zeros and the relative position of the graph - it doesn't give us any information as to how high or low the graph strays from the x-axis. Furthermore, as we have mentioned earlier in the text, without Calculus, the values of the relative maximum and minimum can only be found approximately using a calculator. If we took the time to find the leading term of f, we would find it to be x^8. Looking at the end behavior of f, we notice it matches the end behavior of $y = x^8$. This is no accident, as we find out in the next theorem.

THEOREM 1.2. **End Behavior for Polynomial Functions:** The end behavior of a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_n x^n$.

To see why Theorem 1.2 is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behavior, we look to see the behavior of f as $x \to \pm\infty$. Since we're concerned with x's far down the x-axis, we are far away from $x = 0$ and so can rewrite $f(x)$ for these values of x as

$$f(x) = 4x^3 \left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

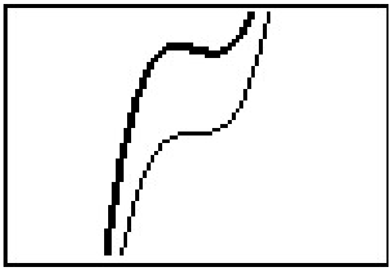
As x becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

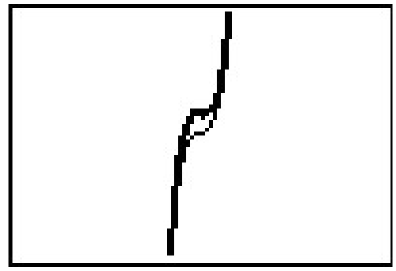
In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3(1 - 0 + 0) = 4x^3$, which is the leading term of f. The formal proof of Theorem 1.2 works in much the same way. Factoring out the leading term leaves

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

As $x \to \pm\infty$, any term with an x in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_n x^n$. Geometrically, Theorem 1.2 says that if we graph $y = f(x)$, say, using a graphing calculator, and continue to 'zoom out,' the graph of it and its leading term become indistinguishable. Below are the graphs of $y = 4x^3 - x + 5$ (the thicker line) and $y = 4x^3$ (the thinner line) in two different windows.

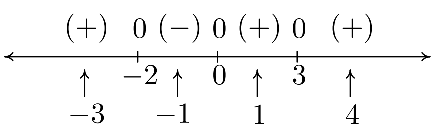


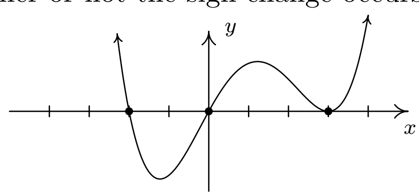
A view 'close' to the origin.



A 'zoomed out' view.

Let's return to the function in Example 1.1.5, $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, whose sign diagram and graph are reproduced below for reference. Theorem 1.2 tells us that the end behavior is the same as that of its leading term, x^8. This tells us that the graph of $y = f(x)$ starts and ends above the x-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate f at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate f at the other test values? What we would really need to know is how the function behaves near its zeros – does it cross through the x-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$. From the sign diagram, the graph of f will cross the x-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.





A sketch of $y = f(x)$

Fortunately, f was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, x^3, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is **squared** kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and x^3, we note they are both odd - so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors change which results in the sign of the function value changing. This is the key to the behavior of the function near the zeros. We need a definition and then a theorem.

DEFINITION 1.3. Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x=c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x)=x^3(x-3)^2(x+2)$ as $f(x)=(x-0)^3(x-3)^2(x-(-2))^1$, we see that $x=0$ is a zero of multiplicity 3, $x=3$ is a zero of multiplicity 2, and $x=-2$ is a zero of multiplicity 1.

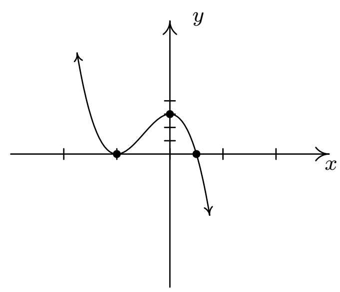
THEOREM 1.3. **The Role of Multiplicity:** Suppose f is a polynomial function and $x=c$ is a zero of multiplicity m.

- If m is even, the graph of $y=f(x)$ touches and rebounds from the x-axis as $(c,0)$.
- If m is odd, the graph of $y=f(x)$ crosses through the x-axis as $(c,0)$.

Our last example shows how end behavior and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 1.1.6. Sketch the graph of $f(x) = -3(2x-1)(x+1)^2$ using end behavior and the multiplicity of its zeros.

Solution. The end behavior of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2 = -6x^3$. This tells us the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV. Next, we find the zeros of f. Fortunately for us, f is factored.[16] Setting each factor equal to zero gives is $x = \frac{1}{2}$ and $x = -1$ as zeros. To find the multiplicity of $x = \frac{1}{2}$ we note that it corresponds to the factor $(2x-1)$. This isn't strictly in the form required in Definition 1.3. If we factor out the 2, however, we get $(2x-1) = 2\left(x - \frac{1}{2}\right)$, and we see the multiplicity of $x = \frac{1}{2}$ is 1. Since 1 is an odd number, we know from Theorem 1.3 that the graph of f will cross through the x-axis at $\left(\frac{1}{2}, 0\right)$. Since the zero $x = -1$ corresponds to the factor $(x+1)^2 = (x-(-1))^2$, we see its multiplicity is 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1, 0)$. Though we're not asked to, we can find the y-intercept by finding $f(0) = -3(2(0)-1)(0+1)^2 = 3$. Thus $(0, 3)$ is an additional point on the graph. Putting this together gives us the graph below.



□

[16] Obtaining the factored form of a polynomial is the main focus of the next few sections.

1.1.1 Exercises

1. For each polynomial given below, find the degree, the leading term, the leading coefficient, the constant term and the end behavior.

   (a) $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$

   (b) $p(t) = -t^2(3 - 5t)(t^2 + t + 4)$

   (c) $Z(b) = 42b - b^3$

   (d) $s(t) = -4.9t^2 + v_0 t + s_0$

   (e) $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)$

   (f) $q(r) = 1 - 16r^4$

2. For each polynomial given below, find its real zeros and their corresponding multiplicities. Use this information along with a sign chart to provide a rough sketch of the graph of the polynomial.

   (a) $a(x) = x(x + 2)^2$

   (b) $F(x) = x^3(x + 2)^2$

   (c) $P(x) = (x - 1)(x - 2)(x - 3)(x - 4)$

   (d) $Z(b) = b(42 - b^2)$

   (e) $Q(x) = (x + 5)^2(x - 3)^4$

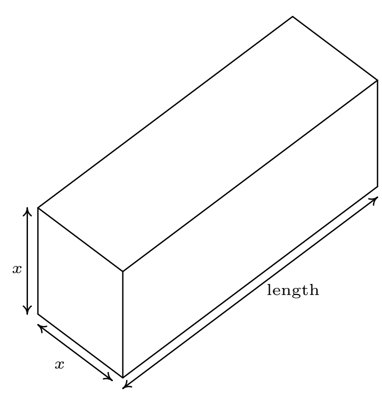
   (f) $g(x) = x(x + 2)^3$

3. According to US Postal regulations, a rectangular shipping box must satisfy the inequality "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.[17] Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

   (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth = 130 inches, express the length of the box in terms of x and then express the volume, V, of the box in terms of x.

   (b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

   (c) Repeat parts 3a and 3b if the box is shipped using "other services".



[17] See here for details.

4. Use transformations to sketch the graphs of the following polynomials.

   (a) $f(x) = (x+2)^3 + 1$

   (b) $g(x) = (x+2)^4 + 1$

   (c) $h(x) = -x^5 - 3$

   (d) $j(x) = 2 - 3(x-1)^4$

5. Use the Intermediate Value Theorem to find intervals of length 1 which contain the real zeros of $f(x) = x^3 - 9x + 5$.

6. The original function used to model the cost of producing x PortaBoys Game Systems given in Example **??** was $C(x) = 80x + 150$. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy revised their cost function using a cubic polynomial. The new cost of producing x PortaBoys is given by $C(x) = .03x^3 - 4.5x^2 + 225x + 250$. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Find the production level x that maximizes the profit made by producing and selling x PortaBoys.

7. Here is a chart of the number of hours of daylight they get on the 21$^{\text{st}}$ of each month in Fairbanks, Alaska based on the 2009 sunrise and sunset data found on the U.S. Naval Observatory website. We let $x = 1$ represent January 21, 2009, $x = 2$ represent February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

   Find cubic (third degree) and quartic (fourth degree) polynomials which model this data and comment on the goodness of fit for each. What can we say about using either model to make predictions about the year 2020? (Hint: Think about the end behavior of polynomials.) Use the models to see how many hours of daylight they got on your birthday and then check the website to see how accurate the models are. Knowing that Sasquatch are largely nocturnal, what days of the year according to your models are going to allow for at least 14 hours of darkness for field research on the elusive creatures?

8. An electric circuit is built with a variable resistor installed. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in the table below. [18]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

   (a) Make a scatter diagram of the data using the Resistance as the independent variable and Power as the dependent variable.

[18] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

(b) Use your calculator to find quadratic (2nd degree), cubic (3rd degree) and quartic (4th degree) regression models for the data and judge the reasonableness of each.

(c) For each of the models found above, find the predicted maximum power that can be delivered to the load. What is the corresponding resistance value?

(d) Discuss with your classmates the limitations of these models - in particular, discuss the end behavior of each.

9. Show that the end behavior of a linear function $f(x) = mx + b$ is as it should be according to the results we've established in the section for polynomials of odd degree. (That is, show that the graph of a linear function is "up on one side and down on the other" just like the graph of $y = a_n x^n$ for odd numbers n.)

10. There is one subtlety about the role of multiplicity that we need to discuss further; specifically we need to see 'how' the graph crosses the x-axis at a zero of odd multiplicity. In the section, we deliberately excluded the function $f(x) = x$ from the discussion of the end behavior of $f(x) = x^n$ for odd numbers n and we said at the time that it was due to the fact that $f(x) = x$ didn't fit the pattern we were trying to establish. You just showed in the previous exercise that the end behavior of a linear function behaves like every other polynomial of odd degree, so what doesn't $f(x) = x$ do that $g(x) = x^3$ does? It's the 'flattening' for values of x near zero. It is this local behavior that will distinguish between a zero of multiplicity 1 and one of higher odd multiplicity. Look again closely at the graphs of $a(x) = x(x+2)^2$ and $F(x) = x^3(x+2)^2$ from Exercise 2. Discuss with your classmates how the graphs are fundamentally different at the origin. It might help to use a graphing calculator to zoom in on the origin to see the different crossing behavior. Also compare the behavior of $a(x) = x(x+2)^2$ to that of $g(x) = x(x+2)^3$ near the point $(-2, 0)$. What do you predict will happen at the zeros of $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$?

11. Here are a few other questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree n have? How few local extrema can it have?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?

(h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

1.1.2 Answers

1. (a) $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
Degree 17
Leading term $\sqrt{3}x^{17}$
Leading coefficient $\sqrt{3}$
Constant term $\frac{1}{3}$
As $x \to -\infty,\ f(x) \to -\infty$
As $x \to \infty,\ f(x) \to \infty$

(b) $p(t) = -t^2(3-5t)(t^2+t+4)$
Degree 5
Leading term $5t^5$
Leading coefficient 5
Constant term 0
As $t \to -\infty,\ p(t) \to -\infty$
As $t \to \infty,\ p(t) \to \infty$

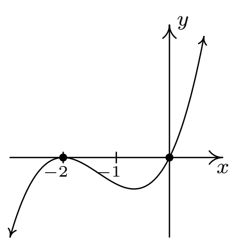
(c) $Z(b) = 42b - b^3$
Degree 3
Leading term $-b^3$
Leading coefficient -1
Constant term 0
As $b \to -\infty,\ Z(b) \to \infty$
As $b \to \infty,\ Z(b) \to -\infty$

(d) $s(t) = -4.9t^2 + v_0t + s_0$
Degree 2
Leading term $-4.9t^2$
Leading coefficient -4.9
Constant term s_0
As $t \to -\infty,\ s(t) \to -\infty$
As $t \to \infty,\ s(t) \to -\infty$

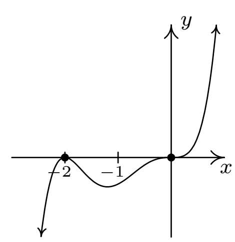
(e) $P(x) = (x-1)(x-2)(x-3)(x-4)$
Degree 4
Leading term x^4
Leading coefficient 1
Constant term 24
As $x \to -\infty,\ P(x) \to \infty$
As $x \to \infty,\ P(x) \to \infty$

(f) $q(r) = 1 - 16r^4$
Degree 4
Leading term $-16r^4$
Leading coefficient -16
Constant term 1
As $r \to -\infty,\ q(r) \to -\infty$
As $r \to \infty,\ q(r) \to -\infty$

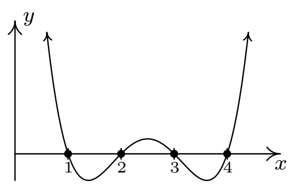
2. (a) $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



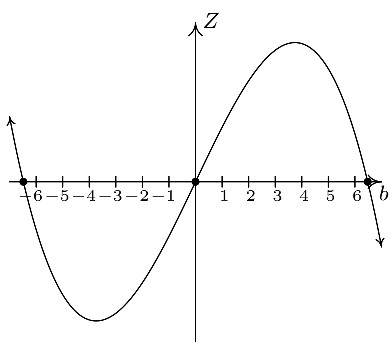
(b) $F(x) = x^3(x+2)^2$
$x = 0$ multiplicity 3
$x = -2$ multiplicity 2



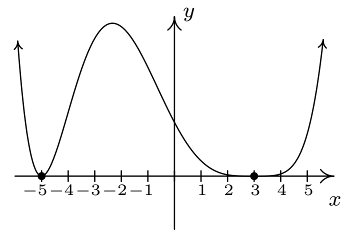
(c) $P(x) = (x-1)(x-2)(x-3)(x-4)$
$x = 1$ multiplicity 1
$x = 2$ multiplicity 1
$x = 3$ multiplicity 1
$x = 4$ multiplicity 1



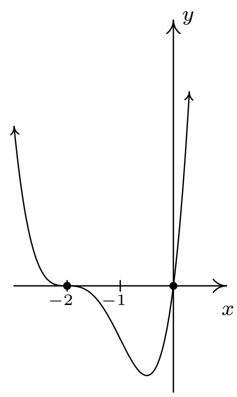
(d) $Z(b) = b(42 - b^2)$
$b = -\sqrt{42}$ multiplicity 1
$b = 0$ multiplicity 1
$b = \sqrt{42}$ multiplicity 1



(e) $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



(f) $g(x) = x(x+2)^3$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 3

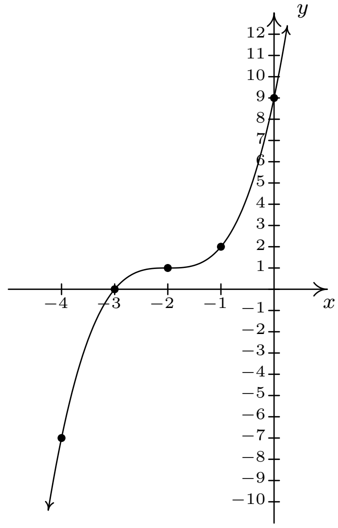


3. (a) Our ultimate goal is to maximize the volume, so we'll start with the maximum Length + Girth of 130. This means the length is $130 - 4x$. The volume of a rectangular box is always length $\times$ width $\times$ height so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

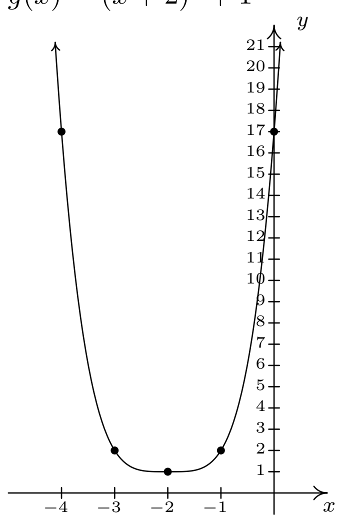
   (b) Graphing $y = V(x)$ on $[0, 33] \times [0, 21000]$ shows a maximum at $(21.67, 20342.59)$ so the dimensions of the box with maximum volume are 21.67in. $\times$ 21.67in. $\times$ 43.32in. for a volume of 20342.59in.^{3}.

   (c) If we start with Length + Girth = 108 then the length is $108 - 4x$ and the volume is $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ on $[0, 27] \times [0, 11700]$ shows a maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. $\times$ 18.00in. $\times$ 36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all calculator results as approximations and list them as such.)

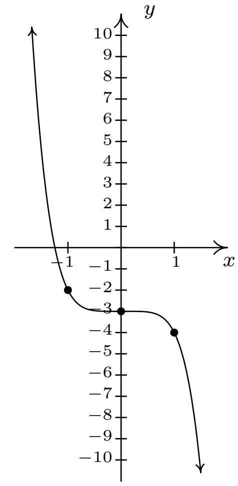
4. (a) $f(x) = (x+2)^3 + 1$



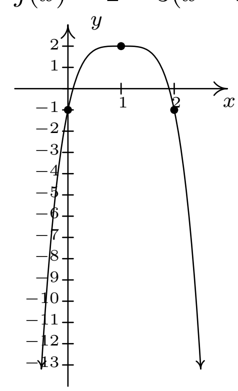
(b) $g(x) = (x+2)^4 + 1$



(c) $h(x) = -x^5 - 3$



(d) $j(x) = 2 - 3(x-1)^4$



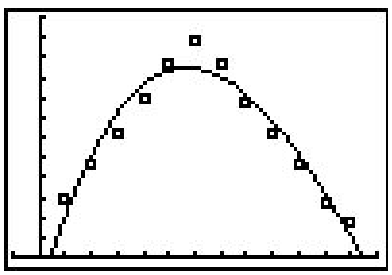
5. We have $f(-4) = -23$, $f(-3) = 5$, $f(0) = 5$, $f(1) = -3$, $f(2) = -5$ and $f(3) = 5$ so the Intermediate Value Theorem tells us that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

6. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

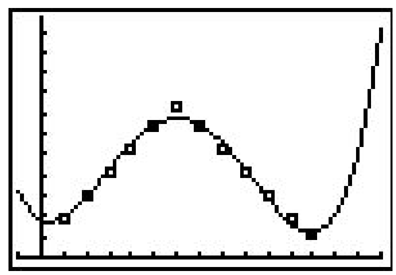
7. The cubic regression model is $p_3(x) = 0.0226x^3 - 0.9508x^2 + 8.615x - 3.446$. It has $R^2 = 0.93765$ which isn't bad. The graph of $y = p_3(x)$ in the viewing window $[-1, 13] \times [0, 24]$ along with the scatter plot is shown below on the left. Notice that p_3 hits the x-axis at about $x = 12.45$ making this a bad model for future predictions. To use the model to approximate the number of hours of sunlight on your birthday, you'll have to figure out what decimal value

of x is close enough to your birthday and then plug it into the model. My (Jeff's) birthday is July 31 which is 10 days after July 21 ($x = 7$). Assuming 30 days in a month, I think $x = 7.33$ should work for my birthday and $p_3(7.33) \approx 17.5$. The website says there will be about 18.25 hours of daylight that day. To have 14 hours of darkness we need 10 hours of daylight. We see that $p_3(1.96) \approx 10$ and $p_3(10.05) \approx 10$ so it seems reasonable to say that we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to February 21, 2009 ($x = 2$) and then again from October 21,2009 ($x = 10$) to December 21, 2009 ($x = 12$).

The quartic regression model is $p_4(x) = 0.0144x^4 - 0.3507x^3 + 2.259x^2 - 1.571x + 5.513$. It has $R^2 = 0.98594$ which is good. The graph of $y = p_4(x)$ in the viewing window $[-1, 15] \times [0, 35]$ along with the scatter plot is shown below on the right. Notice that $p_4(15)$ is above 24 making this a bad model as well for future predictions. However, $p_4(7.33) \approx 18.71$ making it much better at predicting the hours of daylight on July 31 (my birthday). This model says we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to about March 1, 2009 ($x = 2.30$) and then again from October 10, 2009 ($x = 9.667$) to December 21, 2009 ($x = 12$).

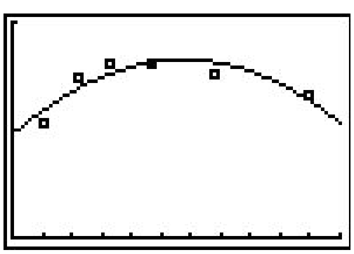


$y = p_3(x)$

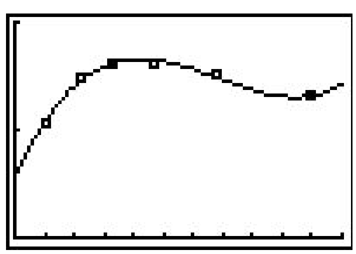


$y = p_4(x)$

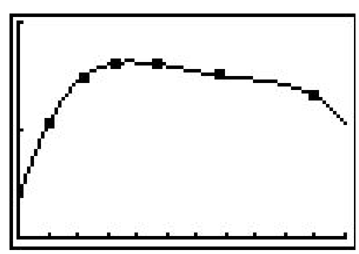
8. (a) The scatter plot is shown below with each of the three regression models.
   (b) The quadratic model is $P_2(x) = -0.02x^2 + 0.241x + 0.956$ with $R^2 = 0.77708$.
   The cubic model is $P_3(x) = 0.005x^3 - 0.103x^2 + 0.602x + 0.573$ with $R^2 = 0.98153$.
   The quartic model is $P_4(x) = -0.000969x^4 + 0.0253x^3 - 0.240x^2 + 0.944x + 0.330$ with $R^2 = 0.99929$.
   (c) The maximums predicted by the three models are $P_2(5.737) \approx 1.648$, $P_3(4.232) \approx 1.657$ and $P_4(3.784) \approx 1.630$, respectively.



$y = P_2(x)$



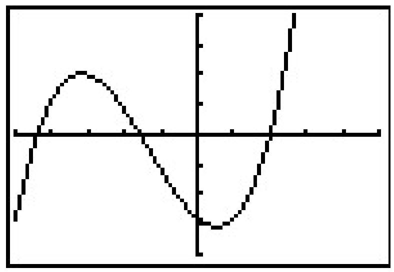
$y = P_3(x)$



$y = P_4(x)$

1.2 The Factor Theorem and The Remainder Theorem

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned[1] in Intermediate Algebra, this equation foils[2] us at every turn. If we graph f using the graphing calculator, we get



The graph suggests that $x = 2$ is a zero, and we can verify $f(2) = 0$. The other two zeros seem to be less friendly, and, even though we could use the 'Zero' command to find decimal approximations for these, we seek a method to find the remaining zeros exactly. Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, it seems reasonable to expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$\begin{array}{rllll}
 & & x^2 & +\ 6x & +\ 7 \\ \cline{2-5}
x-2 & \big) \ x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\
 & -\underline{(x^3} & \underline{-\ 2x^2)} & & \\
 & & 6x^2 & -\ 5x & \\
 & & -\underline{(6x^2} & \underline{-\ 12x)} & \\
 & & & 7x & -\ 14 \\
 & & & -\underline{(7x} & \underline{-\ 14)} \\
 & & & & 0
\end{array}$$

As you may recall, this means $x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$, and so to find the zeros of f, we now can solve $(x - 2)\left(x^2 + 6x + 7\right) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

[1] and probably forgot

[2] pun intended

Theorem 1.4. **Polynomial Division:** Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 1.4 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The proof of Theorem 1.4 is usually relegated to a course in Abstract Algebra,[3] but we will use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 1.5. **The Remainder Theorem:** Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

The proof of Theorem 1.5 is a direct consequence of Theorem 1.4. When a polynomial is divided by $x - c$, the remainder is either 0 or has degree less than the degree of $x - c$. Since $x - c$ is degree 1, this means the degree of the remainder must be 0, which means the remainder is a constant. Hence, in either case, $p(x) = (x - c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, and so we get $r = p(c)$, as required. There is one last 'low hanging fruit'[4] to collect - it is an immediate consequence of The Remainder Theorem.

Theorem 1.6. **The Factor Theorem:** Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

The proof of The Factor Theorem is a consequence of what we already know. If $(x - c)$ is a factor of $p(x)$, this means $p(x) = (x - c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c - c)\,q(c) = 0$, and so c is a zero of p. Conversely, if c is a zero of p, then $p(c) = 0$. In this case, The Remainder Theorem tells us the remainder when $p(x)$ is divided by $(x - c)$, namely $p(c)$, is 0, which means $(x - c)$ is a factor of p. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomials by quantities of the form $x - c$. Fortunately, people like Ruffini and Horner have already blazed this trail. Let's take a closer look at the long division we

[3] Yes, Virginia, there are algebra courses more abstract than this one.
[4] Jeff hates this expression and Carl included it just to annoy him.

performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & +\,6x & +\,7 & \\ \cline{2-5}
x-2 & x^3 & +\,4x^2 & -\,5x & -14 \\
 & -x^3 & +\,2x^2 & & \\ \cline{2-3}
 & & 6x^2 & -\,5x & \\
 & & -6x^2 & +\,12x & \\ \cline{3-4}
 & & & 7x & -14 \\
 & & & -7x & +14 \\ \cline{4-5}
 & & & & 0
\end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, and so we omit them, too.

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & +\,6x & +\,7 & \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & & \\ \cline{2-3}
 & & 6x^2 & & \\
 & & & 12x & \\ \cline{3-4}
 & & & 7x & \\
 & & & & 14 \\ \cline{4-5}
 & & & & 0
\end{array}$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & +\,6x & +\,7 & \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
 & x^3 & 6x^2 & 7x & 0
\end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr} -2 & x^3 & +4x^2 & -\ 5x & -14 \\ & & 2x^2 & 12x & 14 \\ \hline & x^3 & 6x^2 & 7x & 0 \end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$, and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x - 2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, and so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed is the **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we write 2 in the place of the divisor and the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works **only** for these kinds of divisors.[5] Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Example 1.2.1. Use synthetic division to perform the following polynomial divisions. Find the quotient and the remainder polynomials, then write the dividend, quotient and remainder in the form given in Theorem 1.4.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(x^3 + 8\right) \div (x + 2)$
3. $\dfrac{4 - 8x - 12x^2}{2x - 3}$

Solution.

1. When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

   Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. According to Theorem 1.4, we have $5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$.

2. For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

   We get the quotient $q(x) = x^2 - 2x + 4$ and the remainder $r(x) = 0$. Relating the dividend, quotient and remainder gives $x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$.

[5] You'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1.

3. To divide $4 - 8x - 12x^2$ by $2x - 3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2 - 8x + 4$. Second, since synthetic division works only for factors of the form $x - c$, we factor $2x - 3$ as $2\left(x - \frac{3}{2}\right)$. Our strategy is to first divide $-12x^2 - 8x + 4$ by 2, to get $-6x^2 - 4x + 2$. Next, we divide by $\left(x - \frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

From this, we get $-6x^2 - 4x + 2 = \left(x - \frac{3}{2}\right)(-6x - 13) - \frac{35}{2}$. Multiplying both sides by 2 and distributing gives $-12x^2 - 8x + 4 = (2x - 3)(-6x - 13) - 35$. At this stage, we have written $-12x^2 - 8x + 4$ in the **form** $(2x-3)q(x) + r(x)$, but how can we be sure the quotient polynomial is $-6x - 13$ and the remainder is -35? The answer is the word 'unique' in Theorem 1.4. The theorem states that there is only one way to decompose $-12x^2 - 8x + 4$ into a multiple of $(2x - 3)$ plus a constant term. Since we have found such a way, we can be sure it is the only way. $\square$

The next example pulls together all of the concepts discussed in this section.

Example 1.2.2. Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.
2. Use the fact that $x = 1$ is a zero of p to factor $p(x)$ and find all of the real zeros of p.

Solution.

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x - (-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0, and proceed as above.

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. The Factor Theorem tells us that since $x = 1$ is a zero of p, $x - 1$ is a factor of $p(x)$. To factor $p(x)$, we divide

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

We get a remainder of 0 which verifies that, indeed, $p(1) = 0$. Our quotient polynomial is a second degree polynomial with coefficients 2, 2, and -3. So $q(x) = 2x^2 + 2x - 3$. Theorem 1.4 tells us $p(x) = (x-1)\left(2x^2 + 2x - 3\right)$. To find the remaining real zeros of p, we need to solve $2x^2 + 2x - 3 = 0$ for x. Since this doesn't factor nicely, we use the quadratic formula to find that the remaining zeros are $x = \frac{-1 \pm \sqrt{7}}{2}$. $\square$

In Section 1.1, we discussed the notion of the multiplicity of a zero. Roughly speaking, a zero with multiplicity 2 can be divided twice into a polynomial; multiplicity 3, three times and so on. This is illustrated in the next example.

Example 1.2.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Given that $x = \frac{1}{2}$ is a zero of multiplicity 2, find all of the real zeros of p.

Solution. We set up for synthetic division. Since we are told the multiplicity of $\frac{1}{2}$ is two, we continue our tableau and divide $\frac{1}{2}$ into the quotient polynomial

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
 & 4 & 0 & -12 & \boxed{0} &
\end{array}
$$

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division tells us $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$. Combining these results, we have $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 - 12\right)$. To find the remaining zeros of p, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$. $\square$

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common site in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula. Secondly, we found $x = \pm\sqrt{3}$ are zeros of p. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of p. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
\sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\
 & \downarrow & 4\sqrt{3} & 12 & & \\
\hline
-\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\
 & \downarrow & -4\sqrt{3} & & & \\
\hline
 & 4 & \boxed{0} & & &
\end{array}
$$

This gives us $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right) \left(x - \left(-\sqrt{3}\right)\right)(4)$, or, when written with the constant in front

$$p(x) = 4\left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right) \left(x - \left(-\sqrt{3}\right)\right)$$

We have shown that p is a product of its leading term times linear factors of the form $(x - c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section **??**, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

> THEOREM 1.7. Suppose f is a polynomial of degree n, $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Theorem 1.7 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x - c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

1.2.1 Exercises

1. (An Intermediate Algebra review exercise) Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

   (a) $(5x^4 - 3x^3 + 2x^2 - 1) \div (x^2 + 4)$

   (b) $(-x^5 + 7x^3 - x) \div (x^3 - x^2 + 1)$

   (c) $(9x^3 + 5) \div (2x - 3)$

   (d) $(4x^2 - x - 23) \div (x^2 - 1)$

2. Use synthetic division and the Remainder Theorem to test whether or not the given number is a zero of the polynomial $p(x) = 15x^5 - 121x^4 + 17x^3 - 73x^2 + 2x + 48$.

   (a) $c = -1$

   (b) $c = 8$

   (c) $c = \frac{1}{2}$

   (d) $c = \frac{2}{3}$

   (e) $c = 0$

   (f) $c = -\frac{3}{5}$

3. For each polynomial given below, you are given one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

   (a) $x^3 - 6x^2 + 11x - 6, \quad c = 1$

   (b) $x^3 - 24x^2 + 192x - 512, \quad c = 8$

   (c) $4x^4 - 28x^3 + 61x^2 - 42x + 9, \quad c = \frac{1}{2}$

   (d) $3x^3 + 4x^2 - x - 2, \quad c = \frac{2}{3}$

   (e) $x^4 - x^2, \quad c = 0$

   (f) $x^2 - 2x - 2, \quad c = 1 - \sqrt{3}$

   (g) $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324, \; c = -\frac{3}{5}$

4. Create a polynomial p with the following attributes.

   - As $x \to -\infty$, $p(x) \to \infty$.
   - The point $(-2, 0)$ yields a local maximum.
   - The degree of p is 5.
   - The point $(3, 0)$ is one of the x-intercepts of the graph of p.

5. Find a quadratic polynomial with integer coefficients which has $x = \dfrac{3}{5} \pm \dfrac{\sqrt{29}}{5}$ as its real zeros.

1.2.2 Answers

1. (a) $5x^4 - 3x^3 + 2x^2 - 1 = (x^2 + 4)(5x^2 - 3x - 18) + (12x + 71)$
   (b) $-x^5 + 7x^3 - x = (x^3 - x^2 + 1)(-x^2 - x + 6) + (7x^2 - 6)$
   (c) $9x^3 + 5 = (2x - 3)(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}) + \frac{283}{8}$
   (d) $4x^2 - x - 23 = (x^2 - 1)(4) + (-x - 19)$

2. (a) $p(-1) - 180$
   (b) $p(8) = 0$
   (c) $p(\frac{1}{2}) = \frac{825}{32}$
   (d) $p(\frac{2}{3}) = 0$
   (e) $p(0) = 48$
   (f) $p(-\frac{3}{5}) = 0$

3. (a) $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$
   (b) $x^3 - 24x^2 + 192x - 512 = (x - 8)^3$
   (c) $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4(x - \frac{1}{2})^2(x - 3)^2$
   (d) $3x^3 + 4x^2 - x - 2 = 3(x - \frac{2}{3})(x + 1)^2$
   (e) $x^4 - x^2 = x^2(x - 1)(x + 1)$
   (f) $x^2 - 2x - 2 = (x - (1 - \sqrt{3}))(x - (1 + \sqrt{3}))$
   (g) $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324 = 125(x + \frac{3}{5})^3(x - 6)(x + 2)$

4. Something like $p(x) = -(x + 2)^2(x - 3)(x + 3)(x - 4)$ will work.

5. $q(x) = 5x^2 - 6x - 4$