Randomness and Halting Probabilities

Verónica Becher * Santiago Figueira * Serge Grigorieff †
vbecher@dc.uba.ar sfigueir@dc.uba.ar seg@liafa.jussieu.fr
Joseph S. Miller ‡
joseph.miller@math.uconn.edu

February 15, 2006

Contents

*Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina
†LIAFA, Université Paris 7 & CNRS, France
‡Department of Mathematics, University of Connecticut, USA

Abstract

We consider the question of randomness of the probability $\Omega_U[X]$ that an optimal Turing machine U halts and outputs a string in a fixed set X. The main results are as follows:

- $\Omega_U[X]$ is 1-random whenever X is Σ^0_n-complete or Π^0_n-complete for some $n \geq 2$.
- However, for $n \geq 2$, $\Omega_U[X]$ is not n-random when X is Σ^0_n or Π^0_n. Nevertheless, there exists Δ^0_{n+1} sets such that $\Omega_U[X]$ is n-random.
- There are Δ^0_2 sets X such that $\Omega_U[X]$ is rational. Also, for every $n \geq 1$, there exists a set X which is Δ^0_{n+1} and Σ^0_n-hard such that $\Omega_U[X]$ is not random.

We also look at the range of Ω_U as an operator. We prove that the set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals. I follows that for any optimal machine U and any sufficiently small real r, there is a set $X \subseteq 2^{<\omega}$ recursive in $\emptyset' \oplus r$, such that $\Omega_U[X] = r$. The same questions are also considered in the context of infinite computations, and lead to similar results.

1 Introduction

1.1 Notations

We denote by $2^{<\omega}$ the set of all finite words on the alphabet $\{0, 1\}$ and by $2^{\leq n}$ the set of all words up to size n. The empty word is denoted by λ and the length of a word a by $|a|$. We denote by $\#X$ the number of elements of the finite set X. We use $\mu(\mathcal{X})$ to denote the Lebesgue measure of a subset $\mathcal{X}$ of the Cantor space 2^ω of all infinite binary words of length ω.

We commit to prefix Turing machines, which are exactly the partial recursive functions with prefix-free domain. We assume Martin-Löf's definition of randomness (or its equivalent counterpart in terms of program-size complexity). As usual, for $n \geq 1$, n-randomness is randomness relative to oracle $\emptyset^{(n-1)}$ (so 1-randomness is just randomness).

If M is a prefix Turing machine, we define $K_M(x)$ as the length of the shortest description of x using machine M, i.e. $K_M(x) = \min\{|p| : M(p) = x\}$ and $K_M(x) = +\infty$ in case $x \notin \mathit{range}(M)$.

1.2 A conjecture on randomness

Definition 1.1. Let $U : 2^{<\omega} \to 2^{<\omega}$ denote a prefix Turing machine. For $X \subseteq 2^{<\omega}$, let $U^{-1}(X) = \{p \in 2^{<\omega} : U(p) \in X\}$ and define

$$\Omega_U[X] = \sum_{p \in U^{-1}(X)} 2^{-|p|} = \mu(U^{-1}(X)2^\omega)$$

The third author has put forward the following conjecture on randomness, in the spirit of Rice's theorem for computability. It involves the notion of optimal prefix Turing machine as defined in the theory of program-size complexity (cf. Definition 3.1).

Conjecture 1.2. For any nonempty $X \subseteq 2^{<\omega}$, the probability $\Omega_U[X]$ that an optimal prefix Turing machine U on an arbitrary input halts and gives an output in X is random. Moreover, if X is Σ^0_n-hard then this probability is n-random (i.e. random in $\emptyset^{(n-1)}$).

It turns out that the notion of optimality considered is decisive in this conjecture. Also, *the conjecture is false as stated: hardness is not sufficient and n-randomness is too much.* The following two theorems gather known negative and positive results about the conjecture with some of the main results of this paper.

Theorem 1.3 (Negative results)**.**
1. There are optimal machines U for which

i. $\Omega_U[X]$ is rational (hence not 1-random) for any finite set X,

ii. $\Omega_U[X]$ is not normal (hence not 1-random) for some infinite Π^0_1 set X.

Cf.Proposition 4.1 and also [12] 2005, Corollary 4.2 and Remark 4.3.

2. For any optimal machine U,

i. There is a Δ^0_2 set X such that $\Omega_U[X]$ is rational. Cf. Theorems 4.7, 6.2

ii. (Hardness is not sufficient). For any $A \subseteq \mathbb{N}$, there is a $\Delta^{0,A}_2$ set X which is $\Sigma^{0,A}_1$-hard and such that $\Omega_U[X]$ is not normal (hence not 1-random). In particular, if $n \geq 1$ then there is a Δ^0_{n+1} set which is Σ^0_n-hard and such that $\Omega_U[X]$ is not 1-random. Cf. Theorem 4.8.

3. (n-randomness is too much). For any optimal machine U and any $A \subseteq \mathbb{N}$ such that $\emptyset' \leq_T A$, if X is $\Sigma^{0,A}_1$ or $\Pi^{0,A}_1$ then $\Omega_U[X]$ is not random in A.
In particular, if $n \geq 2$ and X is Σ^0_n or Π^0_n then $\Omega_U[X]$ is not n-random. Cf. Theorem 4.9.

Nevertheless, the conjecture holds under some particular or some stronger hypotheses. The first result supporting the conjecture is Chaitin's [5] random real Ω, and corresponds to the case $\Omega_U[X]$ where $X = 2^{<\omega}$. The real Ω depends on U, the optimal machine, but independently of the optimal machine U used in the definition, each Ω_U is random.

Theorem 1.4 (Positive results)**.**
1. Let U be any optimal machine. If $X \subseteq 2^{<\omega}$ is infinite and Σ^0_1 then $\Omega_U[X]$ is 1-random. Cf. Chaitin [4], 1988[1].

2. If U is optimal by adjunction (see Def. 3.1) and X is finite not empty then $\Omega_U[X]$ is 1-random. Cf. [1], 2005.

3. Let U be any optimal machine. If $A \subseteq \mathbb{N}$ is such that $\emptyset' \leq_T A$ and X is $\Sigma^{0,A}_1$-complete or $\Pi^{0,A}_1$-complete then $\Omega_U[X]$ is random.
In particular, if $n \geq 2$ and X is Σ^0_n-complete or Π^0_n-complete then $\Omega_U[X]$ is 1-random. Cf. Theorem 5.2

4. Let U be any optimal machine. If $A \subseteq \mathbb{N}$ is such that $\emptyset' \leq_T A$ then there is a $\Delta^{0,A}_2$ set X such that $\Omega_U[X]$ is random in A. In particular, if $n \geq 1$ then there is a Δ^0_{n+1} set X such that $\Omega_U[X]$ is n-random. Cf. Corollary 6.4.

Open problems

1. If $n \geq 3$ and X is Σ^0_n-complete or Π^0_n-complete, is $\Omega_U[X]$ $(n-1)$-random?
2. Are there Π^0_1 sets X such that $\Omega_U[X]$ is 1-random?

[1]Stated without proof in [4], last assertion of Note p.141.

1.3 Road map

The sense in which $\Omega_U[X]$ is a genuine probability is considered in §2.

§3 is devoted to the notion of optimal machine and introduces some particularizations.

In §4 we study different cases where Conjecture 1.2 fails. Theorem 4.7 proves that there are Δ^0_2 sets that do not lead to randomness, whatever be the optimal machine. Improvement with hardness condition is given in Theorem 4.8. Also, Σ^0_n sets cannot be n-random (Theorem 4.9).

§4 gives positive instances of Conjecture 1.2. Theorem 5.2 proves that the conjecture holds for Σ^0_n-complete sets and Σ^0_n-complete sets with 1-randomness, whatever be the optimal machine.

In §6 and §7 we consider the following question related to the converse of the Conjecture: given a real $r \in [0,1]$, is there some optimal machine U and a set $X \subseteq 2^{<\omega}$ such that $r = \Omega_U[X]$? And if so, what are such pairs (U, X) ?

Theorem 6.2 proves that for any optimal machine U and any sufficiently small real r, there is a set $X \subseteq 2^{<\omega}$ recursive in $\emptyset' \oplus r$, such that $\Omega_U[X] = r$. In particular, this result asserts that for any optimal machine U there are Δ^0_2 sets X such that $\Omega_U[X]$ is a rational number, the farthest to be random that it can be. This also yields that the range $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals.

Theorem 7.1 shows that for any given computably enumerable random real r, and for any given recursively enumerable set X, there is an optimal machine U such that $r = \Omega_U[X]$.

In §8 we study the version of Conjecture 1.2 for infinite computations on monotone machines, a landscape where more positive instances have been obtained.

2 Uniform probability on $(2^{<\omega}, \leq_{prefix})$ and $\Omega_U[X]$

As done in the above Conjecture 1.2, it is usual to consider $\Omega_U[X]$ as the probability that U halts and produces output in X. In which precise sense is this real $\Omega_U[X]$ a probability? The function $u \mapsto 2^{-|u|}$ induces the usual uniform probability on the set 2^n of words of fixed length n, for any n. However, as concerns the whole space of words $2^{<\omega}$, it induces a measure which takes value $+\infty$ on $2^{<\omega}$, hence is not a probability.

There are two ways to look at $\Omega_U[X]$ as a probability. Using the fact that $U^{-1}(X)$ is prefix-free (as is the domain of U), a first simple solution is to embed finite inputs into infinite ones and to consider the usual Lebesgue measure on 2^ω. This amounts to the equality stated in Definition 1.1:

$$\Omega_U[X] = \sum_{p \in U^{-1}(X)} 2^{-|p|} = \mu(U^{-1}(X)2^\omega).$$

Another solution, which keeps within the space of finite words, is to consider a notion of probability on ordered sets for which the additivity axiom $p(A \cup B) = p(A) + p(B)$ is not supposed for general disjoint events $A, B \subseteq 2^{<\omega}$ but only for incompatible ones with respect to the ordering. In case of $(2^{<\omega}, \leq_{prefix})$, this means $A2^{<\omega} \cap B2^{<\omega} = \emptyset$.

Proposition 2.1. *For all $A \subseteq 2^{<\omega}$,*

$$\mu(A2^\omega) = \lim_{n\to\infty} \frac{\#(A2^{<\omega} \cap 2^n)}{2^n} = \lim_{n\to\infty} \frac{\#(A2^{<\omega} \cap 2^{\leq n})}{2^{n+1} - 1}$$

Proof. Let $\min A$ be the prefix-free set of minimal elements of A relative to the prefix ordering. Then $\mu(A2^\omega) = \sum_{u \in \min A} 2^{-|u|}$ and, for every n,

$$A2^{<\omega} \cap 2^n = \bigcup_{u \in (\min A) \cap 2^{\leq n}} u2^{n-|u|} \quad , \quad \frac{\#(A2^{<\omega} \cap 2^n)}{2^n} = \sum_{u \in (\min A) \cap 2^{\leq n}} 2^{-|u|}$$

This proves that $\frac{\#(A2^{<\omega} \cap 2^n)}{2^n}$ is monotone increasing in n with limit $\mu(A2^\omega)$.

Also,

$$\frac{\#(A2^{<\omega} \cap 2^{\leq n})}{2^{n+1}} = \frac{\#(A2^{<\omega} \cap 2^{\leq k})}{2^{n+1}} + \sum_{k<m\leq n} \frac{\#(A2^{<\omega} \cap 2^m)}{2^m} 2^{-(n-m+1)} = \alpha + \beta$$

Given $\varepsilon > 0$, fix k such that $0 \leq \mu(A2^\omega) - \frac{\#(A2^{<\omega} \cap 2^m)}{2^m} \leq \varepsilon/3$ for all $m \geq k$. Then, for $n \geq k + \log(3/\varepsilon)$,

$$\alpha \leq \frac{2^{k+1}-1}{2^{n+1}} \leq 2^{-(n-k)} \leq \varepsilon/3$$

and

$$\begin{aligned}
\mu(A2^\omega) - \beta &= \mu(A2^\omega)2^{-(n-k)} + \mu(A2^\omega) \sum_{k<m\leq n} 2^{-(n-m+1)} \\
&\quad - \sum_{k<m\leq n} \frac{\#(A2^{<\omega} \cap 2^m)}{2^m} 2^{-(n-m+1)} \\
|\mu(A2^\omega) - \beta| &\leq 2^{-(n-k)} + \sum_{k<m\leq n} |\mu(A2^\omega) - \frac{\#(A2^{<\omega} \cap 2^m)}{2^m}| \, 2^{-(n-m+1)} \\
&\leq \varepsilon/3 + \varepsilon/3
\end{aligned}$$

Whence $|\mu(A2^\omega) - \frac{\#(A2^{<\omega} \cap 2^{\leq n})}{2^{n+1}}| \leq \alpha + |\mu(A2^\omega) - \beta| \leq \varepsilon$. This proves that $\frac{\#(A2^{<\omega} \cap 2^{\leq n})}{2^{n+1}}$ also tends to $\mu(A2^\omega)$ when $n \to +\infty$. □

Definition 2.2. We let $\pi : P(2^{<\omega}) \to [0,1]$ be the function such that, for all $A \subseteq 2^{<\omega}$,

$$\pi(A) = \lim_{n\to\infty} \frac{\#(A2^{<\omega} \cap 2^n)}{2^n}$$

A straightforward application of Proposition 2.1 shows that π is a probability on the ordered set $(2^{<\omega}, \leq_{prefix})$.

Proposition 2.3. *In the sense of the ordered set* $(2^{<\omega}, \leq_{prefix})$, π *is a probability, i.e.* $\pi(\emptyset) = 0$, $\pi(2^{<\omega}) = 1$ *and, for all* $A, B \subseteq 2^{<\omega}$

$$\begin{aligned}
\pi(A \cup B) &\leq \pi(A) + \pi(B) \\
\pi(A \cup B) = \pi(A) + \pi(B) &\Leftrightarrow A2^{<\omega} \cap B2^{<\omega} = \emptyset
\end{aligned}$$

Also, $\pi(A) = \pi(\min(A)) = \pi(A2^{<\omega})$ *and, if* A *is prefix-free then* $\pi(A) = \sum_{u \in A} 2^{-|u|}$.

Since *domain*(U) is prefix-free, we see that $\Omega_U[X]$ is the probability of $U^{-1}(X)$ relative to π.

Proposition 2.4. *Let* $U : 2^{<\omega} \to 2^{<\omega}$ *be a prefix Turing machine and* $X \subseteq 2^{<\omega}$. *Then,* $\Omega_U[X] = \pi(U^{-1}(X))$.

3 On the notion of optimality

Since for some sets the validity of Conjecture 1.2 depends on the machine U used to define Ω_U, we shall consider the usual notion of optimality and also a refinement that we name *optimality by adjunction*.

Let $(M_e)_{e\in\mathbb{N}}$ be a recursive enumeration of all prefix Turing machines.

Definition 3.1 (Optimality and optimality by adjunction)**.** Let $U : 2^{<\omega} \to 2^{<\omega}$ be a prefix Turing machine.

1. U is *optimal* if and only if

$$\forall e\ \exists c_e\ \forall p\ \exists \sigma_{e,p}\ (U(\sigma_{e,p}) = M_e(p)\ \wedge |\sigma_{e,p|} \le |p| + c_e).$$

   U is *effectively optimal* if there is a total recursive function $c : \mathbb{N} \times 2^{<\omega} \to 2^{<\omega}$ such that we can take $\sigma_{e,p} = c(e,p)$.

2. U is *optimal by adjunction* if and only if

$$\forall e\ \exists \sigma_e\ \forall p\ U(\sigma_e p) = M_e(p).$$

   Hence, in this case, $c_e = |\sigma_e|$ and $\sigma_{e,p} = \sigma_e p$ (concatenation of words σ_e and p).

   U is *effectively optimal by adjunction* if there is a total recursive function $g : \mathbb{N} \to 2^{<\omega}$ such that we can take $\sigma_e = g(e)$.

Clearly, U is optimal if and only if it satisfies the Invariance Theorem (of program-size complexity) which states that for all e there is a constant c_e such that $K_U(y) \le K_{M_e}(y) + c_e$ for all y.

Optimality by adjunction can be obtained from effective optimality plus some extra conditions on the coding function c.

Proposition 3.2. *Let V be effectively optimal such that the associated $c : \mathbb{N} \times 2^{<\omega} \to 2^{<\omega}$ is injective and has recursive range. Then there exists U optimal by adjunction such that*

$$\forall x \in 2^{<\omega} \quad \Omega_U[\{x\}] = \Omega_V[\{x\}] \tag{1}$$

Proof. 1. Since V is optimal, $\Omega_V[2^{<\omega}]$ is random, hence $\neq 1$, so that there exists k such that $\Omega_V[2^{<\omega}] < 1 - 2^{-k}$. Fix such a k. The idea of the proof is as follows. First, define U on a prefix-free subset of $0^{k+1}2^{<\omega}$ in a way that insures that U is optimal by adjunction. Then define U on a prefix-free subset of $0^{\le k}12^{<\omega}$ so that to get condition (1).

2. For (e,p) such that $c(e,p) \in \mathit{domain}(V)$, and $\gamma \in \mathbb{N}$ such that $|c(e,p)| \le |p| + \gamma$, we set

$$U(0^{k+1+\gamma}\ 1^{e+1}\ 0\ p) \quad = \quad V(c(e,p)) \tag{2}$$

Since $V(c(e,p)) = M_e(p)$ we see that $U(0^{k+1+\gamma}1^{e+1}0p) = M_e(p)$ for all $\gamma \ge |c(e,p)| - |p|$. The optimality of V insures that there exists c_e such that $|c(e,p)| \le |p| + c_e$ for all p. Then $U(0^{k+1+c_e}1^{e+1}0p) = M_e(p)$ for all p. This proves that U is optimal by adjunction with $\sigma_e = 0^{k+1+c_e}1^{e+1}0$.

3. Observe that, for given e and p,

$$\begin{aligned}\sum_{\gamma\geq\max(0,|c(e,p)|-|p|)} 2^{-|0^{k+1+\gamma}1^{e+1}0p|} &= 2^{-(k+e+3)} \sum_{\gamma\geq\max(0,|c(e,p)|-|p|)} 2^{-(\gamma+|p|)} \\ &= 2^{-(k+2+e+\max(|p|,|c(e,p)|))}\end{aligned}$$

Let $Q_{e,p}$ be the finite subset of $\mathbb{N}$ such that

$$\sum_{j\in Q_{e,p}} 2^{-j} = 2^{-|c(e,p)|} - 2^{-(k+2+e+\max(|p|,|c(e,p)|))}$$

Let $(v_i)_{i\in\mathbb{N}}$ be a recursive enumeration of $domain(V)$ without repetitions.
To define U on $\bigcup_{\ell\leq k} 0^\gamma 12^{<\omega}$, we introduce the following Kraft-Chaitin set

$$\begin{aligned}KC &= \{(j, V(c(e,p))) : (e,p) \in domain(V \circ c), j \in Q_{e,p}\} \\ &\quad \cup \{(|v|, V(v)) : v \in domain(V) \setminus range(c)\}\end{aligned}$$

Since the range of c is recursive, there is a recursive enumeration $(l_n, y_n)_{n\in\mathbb{N}}$ of KC. Let's show that KC is indeed a Kraft-Chaitin set.

$$\begin{aligned}\sum_{n\in\mathbb{N}} 2^{-l_n} &= \sum_{(e,p)\in domain(V\circ c)} \sum_{j\in Q_{e,p}} 2^{-j} + \sum_{v\in domain(V)\setminus range(c)} 2^{-|v|} \\ &\leq \sum_{(e,p)\in domain(V\circ c)} 2^{-|c(e,p)|} + \sum_{v\notin range(c)} 2^{-|v|} \\ &\leq \sum_{v\in domain(V)\cap range(c)} 2^{-|v|} + \sum_{v\in domain(V)\setminus range(c)} 2^{-|v|} \\ &< 1 - 2^{-k}\end{aligned}$$

A straightforward extension of the Kraft-Chaitin theorem shows that there exists a recursive injective sequence $(r_n)_{n\in\mathbb{N}}$ such that $\{r_n : n \in \mathbb{N}\}$ is a prefix-free subset of $0^{\leq\ell}12^{<\omega}$ and $|r_n| = l_n$ for all n. We complete the definition of U on $0^{\leq\ell}12^{<\omega}$ by setting for all n

$$U(r_n) = y_n \tag{3}$$

Observe that U, as defined by (2) and (3), has prefix-free domain. Also, for $x \in 2^{<\omega}$, we have

$$\begin{aligned}\Omega_V[\{x\}] &= \sum \{2^{-|v|} : v \in domain(V) \cap range(c) \wedge V(v) = x\} && (4) \\ &\quad + \sum \{2^{-|v|} : v \in domain(V) \setminus range(c) \wedge V(v) = x\} && (5)\end{aligned}$$

Since c is injective, for any $v \in range(c)$, there is a unique pair (e,p) such that $v = c(e,p)$. Thus, the sum (4) is exactly $\sum \{2^{-|c(e,p)|} : V(c(e,p)) = x\}$. Due to (2), the definition of $Q_{e,p}$ and (3), this is exactly $\sum \{2^{-|c(e,p)|} : U(c(e,p)) = x\}$. Also, (3) insures that the sum (5) is equal to $\sum \{2^{-|v|} : v \in domain(U) \setminus range(c) \wedge U(v) = x\}$. Thus, $\Omega_U[\{x\}] = \Omega_V[\{x\}]$ □

Remark 3.3. In the theorem above, the condition that c has recursive image is used to see that KC is r.e. This condition can be replaced by the r.e. character of $domain(V) \setminus range(c)$.

4 Negative results about the Conjecture

4.1 Failure for finite sets with particular optimal machines

Proposition 4.1. *Every prefix Turing machine M has a restriction M' to some recursively enumerable set such that $K_M = K_{M'}$ (hence M' is optimal whenever M is) and $\Omega_{M'}[X]$ is rational (hence not random), for every finite set $X \subseteq 2^{<\omega}$.*

Proof. Let $(p_i, y_i)_{i\in\mathbb{N}}$ be a recursive enumeration of the graph of M. Define a total recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $f(i)$ is the smallest $j \le i$ satisfying

$$y_j = y_i \ , \ |p_j| = \min\{|p_k| : k \le i, y_k = y_i\}$$

Let M' be the prefix machine with graph $\{(p_{f(i)}, y_{f(i)}) : i \in \mathbb{N}\}$. Clearly, M' is a restriction of M to some recursively enumerable set. Also, for every $x \in 2^{<\omega}$, if j is least such that $x = y_j$ and $|p_j| = K_M(x)$ then $f(i) = j$ for all $i \ge j$ such that $y_i = x$. Therefore, $M'^{-1}(\{x\})$ is finite, hence $\Omega_{M'}[\{x\}] = \sum_{q\in M'^{-1}(x)} 2^{-|q|}$ is a finite sum of rational numbers, hence is rational. The same is true for finite sets $X \subseteq 2^{<\omega}$. □

Applying the above Proposition to an optimal machine U, we get the following straightforward corollary, first obtained in [12] with a different proof.

Corollary 4.2. *There is an optimal Turing machine U such that for every finite set $X \subseteq 2^{<\omega}$ the real $\Omega_U[X]$ is rational, hence not random.*

Remark 4.3. Using Proposition 4.1 it is easy to construct an infinite Π^0_1 set X such that $\Omega_U[X]$ is not normal, hence not random. In fact, in [12] it was proven that there is an infinite Π^0_1 set X such that $\Omega_U[X]$ is neither c.e. nor random.

4.2 Failure for Δ^0_2 sets with all optimal machines

We recall some results of [5] which will be used in the proofs.

Lemma 4.4. *Let U be optimal.*

1. *Coding Theorem:* $\exists c_1\ \forall\sigma\ 2^{-K_U(\sigma)} \le \Omega_U[\{\sigma\}] \le 2^{-K_U(\sigma)+c_1}$
2. *Maximal complexity of finite strings:*

$$\exists c_2\ \forall\sigma\ K_U(\sigma) < |\sigma| + K_U(|\sigma|) + c_2$$

$$\exists c_3\ \#\{\sigma \in 2^m : K_U(\sigma) < m + K_U(m) - k\} \le 2^{m-k+c_3}$$

The next lemma can be found in unpublished work of Solovay [9, IV-20]. We include the proof because Solovay's notes are not universally available.

Lemma 4.5. *If U is optimal then $\exists c_4\ \forall n\ \exists m \le n\ (n \le m + K_U(m) \le n + c_4)$.*

Proof. Choose $c_4 \in \mathbb{N}$ such that $c_4 > K_U(0)$ and $K_U(m+1) \le K_U(m)+c_4-1$, for all $m \in \mathbb{N}$. Given $n \in \mathbb{N}$, let $m \in \mathbb{N}$ be the least number satisfying $n \le m+K_U(m)$, which clearly holds for some $m \le n$. We claim that $m+K_U(m) < n+c_4$. This holds because $0+K_U(0) < c_4 \le n+c_4$ and, since $m-1+K_U(m-1) < n$, then $m+K_U(m) \le m-1+K_U(m-1)+c_4 < n+c_4$. □

Putting these two lemmas together, we get the following result.

Lemma 4.6. *If U is optimal then $\exists d\ \forall n\ \exists\sigma\ (2^{-n-d} \le \Omega_U[\{\sigma\}] \le 2^{-n+d})$. In fact, for some constant d' there are at least $2^n/(d'\,n^2)$ strings $\sigma \in 2^{<\omega}$ satisfying the inequalities.*

Proof. Let c_1, c_2, c_3, c_4 be constants as in Lemma 4.4 and Lemma 4.5. Then $\#\{\sigma \in 2^m : K_U(\sigma) < m + K_U(m) - (c_3+1)\} \le 2^{m-1}$, for all $m \in \mathbb{N}$. For $n + c_3 + 1$, there is an $m \le n$ such that $n + c_3 + 1 \le m + K_U(m) \le n + c_3 + 1 + c_4$. In particular, all strings in $\sigma \in 2^m$ satisfy $K_U(\sigma) \le m + K_U(m) + c_2 \le n + c_2 + c_3 + c_4 + 1$.

Now, there are at least 2^{m-1} strings $\sigma \in 2^m$ such that $K_U(\sigma) \ge m + K_U(m) - (c_3+1)$ hence such that $K_U(\sigma) \ge n + c_3 + 1 - (c_3+1) = n$. For such strings, we then have $n \le K_U(\sigma) \le n + c_2 + c_3 + c_4 + 1$. Therefore, for $d = \max(c_1, c_2 + c_3 + c_4 + 1)$, there are at least 2^{m-1} strings σ such that $2^{-n-d} \le \Omega_U[\{\sigma\}] \le 2^{-n+d}$. Finally, note that

$$m - 1 \ge n + c_3 - K_U(m) \ge n - 2\log(m) - \mathcal{O}(1)$$

Therefore, at least $\mathcal{O}(1)2^n/n^2$ strings $\sigma \in 2^{<\omega}$ satisfy $2^{-n-d} \le \Omega_U[\{\sigma\}] \le 2^{-n+d}$. □

With this lemma we can prove that Conjecture 1.2 fails for Δ^0_2 sets.

Theorem 4.7. *For every optimal U there is a Δ^0_2 set $X \subseteq 2^{<\omega}$ such that $\Omega_U[X]$ is not random.*

Proof. Let $d, d' \in \mathbb{N}$ be the constants from Lemma 4.6 and let k be such that $i < 2^i/(d'\,i^2)$ for $i \ge k$. Letting $c = k + d$, Lemma 4.6 insures the existence of a sequence $(\sigma_i)_{i\in\mathbb{N}}$ of distinct strings such that $2^{-i-c-1} < \Omega_U[\{\sigma_i\}] \le 2^{-i+c}$, for all $i \in \mathbb{N}$. Note that $\emptyset'$ can compute such a sequence (and even compute the set of strings in the sequence). Indeed, denoting by U_s the computable approximation of U obtained with s computation steps, $\Omega_{U_s}[\{\tau\}] = \sum_{U_s(p)=\tau} 2^{-|p|}$ is nondecreasing in s and tends to $\Omega_U[\{\tau\}]$ when $s \to \infty$. Thus, for any rational r, $\Omega_U[\{\tau\}] > r$ iff $\exists s\ \Omega_{U_s}[\{\tau\}] > r$. Hence it is decidable in $\emptyset'$ whether $\Omega_U[\{\tau\}] > r$ or not.

We build a Δ^0_2 set X in stages $\{X_s\}_{s\in\mathbb{N}}$. At stage $s+1$ we determine if σ_s is in X in order to ensure that the block of bits of $\Omega_U[X]$ from $s - c$ to $s + c + 1$ is not all zeros.

Stage 0. Let $X_0 = \emptyset$.

Stage $s + 1$. Using $\emptyset'$, decide if the $2c + 2$ bits of $\Omega_U[X_s]$ from $s - c$ to $s + c + 1$ are all zero. If these bits are all zero, let $X_{s+1} = X_s \cup \{\sigma_s\}$. Otherwise, let $X_{s+1} = X_s$. Consider the first case. Because $\Omega_U[\{\sigma_s\}] > 2^{-s-c-1}$ there exists $j \le s + c + 1$ such that the j-th bit of $\Omega_U[\{\sigma_s\}]$ is 1. On the other hand, because $\Omega_U[\{\sigma_s\}] \le 2^{-s+c}$, we have $\Omega_U[\{\sigma_s\}] \restriction s - c - 1 = 0^{s-c-1}$. Then there is $s - c \le j \le s + c + 1$ such that the j-th bit of $\Omega_U[\{\sigma_s\}]$ is 1. Notice that if bit $s - c$ is 1 then all the bits of positions greater than $s - c$ are 0. Hence, $\Omega_U[X_{s+1}] \restriction s - c - 1 = \Omega_U[X_s] \restriction s - c - 1$. Therefore, the work of earlier stages has been preserved and also $\Omega_U[X_{s+1}]$ is not all zeros on the block of bits from $s - c$ to $s + c + 1$.

It follows inductively that, for every s, the block of bits of $\Omega_U[X]$ from $s - c$ to $s + c + 1$ is not all zeros. Therefore, $\Omega_U[X]$ is not normal and hence not random.

Notice that this construction works independently of the optimal machine chosen U and the binary representation of $\Omega_U[X_s]$ in case such real is a dyadic rational. □

The above result can be dramatically improved: Theorem 6.2 (cf. §6) shows that there are Δ^0_2 sets X such that $\Omega_U[X]$ is a rational number. Another improvement shows that hardness is not enough to get randomness.

Theorem 4.8. *For every optimal U and any $A \subseteq \mathbb{N}$, there is a $\Delta_2^{0,A}$ set $X \subseteq 2^{<\omega}$ which is $\Sigma_1^{0,A}$-hard and such that $\Omega_U[X]$ is not random.*
In particular, if $n \geq 1$ there is a Δ_{n+1}^0 set $X \subseteq 2^{<\omega}$ which is Σ_n^0-hard and such that $\Omega_U[X]$ is not random.

Proof. Modify the proof of Theorem 4.7 as follows.

1. At stage s deal with the digits from $(2c+1)s-c$ to $(2c+1)s+c$ so as the $[(2c+1)s-c,(2c+1)s+c]$'s are disjoint intervals.

2. Let $Z \subset \mathbb{N}^2$ be $\Sigma_1^{0,A}$ universal. Denote by Z_i the section $\{j : (i,j) \in Z\}$. Define a total computable map $f_i : \mathbb{N} \to 2^{<\omega}$ as follows: $f_i(j) = 10^{C(i,j)}$ where $C : \mathbb{N}^2 \to \mathbb{N}$ is Cantor polynomial bijection. In order that f_i be a reduction of Z_i to X, set

$$10^{C(i,j)} \in X \;\Leftrightarrow\; j \in Z_i$$

This uses oracle A' (the jump of A) and makes X a $\Delta_2^{0,A}$ set.

3. Let $D_s = \{10^{C(i,j)} : j \in Z_i \;\wedge\; C(i,j) \in [(2c+1)s-c,(2c+1)s+c]\}$. Observe that
 - two words in D_s have different lengthes,
 - for s large enough, D_s contains at most one word.

The construction is the same as before except that the definition of X_{s+1} contains $X_s \cup D_s$ and a possible σ_s which is not in D_s nor in 10^*. □

4.3 Failure of n-randomness for Σ_n^0 and Π_n^0 sets

Theorem 4.9. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$ (where $\leq_T$ is Turing reducibility). If U is any optimal machine and $X \subseteq 2^{<\omega}$ is $\Sigma_1^{0,A}$ or $\Pi_1^{0,A}$ then $\Omega_U[X]$ is not random in A.*
In particular, if $n \geq 2$ and X is Σ_n^0 or Π_n^0 then $\Omega_U[X]$ is not n-random.

Proof. The case X is finite is trivial since then $\Omega_U[X]$ is Δ_2^0 hence computable in $\emptyset'$.

Case X is infinite $\Sigma_1^{0,A}$. Fix $m \in \mathbb{N}$. With oracle $\emptyset'$, we can (uniformly in m) find a finite subset $Z \subset 2^{<\omega}$ such that $\Omega_U[Z] > \Omega_U - 2^{-m-1}$ and compute $\varepsilon > 0$ such that $\varepsilon < \inf\{\Omega_U[\{z\}] : z \in Z\}$. Then

$$\textstyle\sum_{\sigma:\Omega_U[\{\sigma\}]\leq\varepsilon} \Omega_U[\{\sigma\}] \quad < 2^{-m-1} \tag{6}$$

Let $(x_s)_{s\in\mathbb{N}}$ be an injective A-computable enumeration of X and set $X_s = \{x_t : t < s\}$. We build an A-Martin-Löf test $(T_m)_{m\in\mathbb{N}}$ for $\Omega_U[X]$. The idea is to define a $\Sigma_1^{0,A}$ class T_m by laying down successive intervals right to $\Omega_U[X_s]$. Set $T_m = \bigcup_{s\in\mathbb{N}} I_{m,s}$ where $I_{m,s} =]\Omega_U[X_s], \Omega_U[X_s]+\delta[$ and $\delta = \varepsilon 2^{-m-1}$.
For s big enough, $\Omega_U[X_s] < \Omega_U[X] < \Omega_U[X_s]+\delta$, so that $\Omega_U[X] \in I_{m,s}$. Thus, $\Omega_U[X] \in T_m$.
For $q \in \mathbb{Q}$ and $Z \in P_{<\omega}(2^{<\omega})$, condition $0 < q < \Omega_U[Z]$ is computable with $\emptyset'$ and condition $q > \Omega_U[Z]$ is $\Sigma_1^{0,\emptyset'}$ (express it as $\exists\eta > 0\ \neg(q-\eta < \Omega_U[Z])$). Thus, $I_{m,s}$ and T_m are $\Sigma_1^{0,A}$ (uniformly in m, s and m).
Since $\Omega_U[X_{s+1}] = \Omega_U[X_s] + \Omega_U[\{x_s\}]$, we have

$$\begin{aligned}\Omega_U[\{x_s\}] \geq \delta \quad &\Rightarrow \quad I_{m,s} \text{ and } I_{m,s+1} \text{ are disjoint}\\ &\Rightarrow \quad \mu(\bigcup_{t\leq s+1} I_{m,t}) = \mu(\bigcup_{t\leq s} I_{m,t}) + \delta\end{aligned}$$

Now, for all s, $\mu(\bigcup_{t\leq s+1} I_{m,t}) \leq \mu(\bigcup_{t\leq s} I_{m,t}) + \Omega_U[\{x_s\}]$. Since $\varepsilon \geq \delta$, the above properties yield

$$\begin{aligned}\mu(T_m) &\leq (\sum_{s:\Omega_U[\{x_s\}]\leq\varepsilon} \Omega_U[\{x_s\}]) + \delta\ \sharp\{s : \Omega_U[\{x_s\}] > \varepsilon\} \\ &< 2^{-m-1} + \delta(1/\varepsilon) = 2^{-m}\end{aligned}$$

(use (6) and the fact that $\#\{\sigma : \Omega_U[\{\sigma\}] \geq \varepsilon\} \leq \Omega_U/\varepsilon \leq 1/\varepsilon$ since the $U^{-1}(\sigma)$'s are pairwise disjoint).
Thus, we have constructed an A-Martin-Löf test $(T_m)_{m\in\mathbb{N}}$ such that $\Omega_U[X] \in \bigcap_{m\in\mathbb{N}} T_m$, proving that $\Omega_U[X]$ is not random in A.

Case X is $\Pi^{0,A}_1$. Since $\Omega_U[X] = \Omega_U - \Omega_U[2^{<\omega} \setminus X]$, use the above case and the fact that Ω_U is A-computable. □

5 Positive results about the Conjecture

In this section we give positive instances of Conjecture 1.2; in particular, the random numbers yielded by Theorems 5.2 and 5.3 are not necessarily computably enumerable. The proof method we use broadens the known proof techniques, which relied on the property that the numbers be computably enumerable in their degree of randomness.

5.1 Completeness and computable choice

To prove 1-randomness in Theorems 5.2, 5.3, we use the following technical Lemma 5.1, which insures that some computable reductions associated to complete sets can be used as computable choice functions in a highly non computable environment.

Lemma 5.1. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$. Suppose $X \subseteq \mathbb{N}$ is $\Sigma^{0,A}_1$-complete and $\mathcal{R} \subseteq 2^{<\omega} \times P_{<\omega}(\mathbb{N})$ is $\Sigma^{0,A}_1$ and satisfies*

$$\forall Z \in P_{<\omega}(\mathbb{N}) \quad \{\sigma : \mathcal{R}(\sigma, Z)\} \textit{ has at least } \sharp(Z) + 1 \textit{ elements} \tag{7}$$

(in particular, this is the case if $\{\sigma : \mathcal{R}(\sigma, Z)\}$ is infinite for all Z).
Then there exists $f : 2^{<\omega} \to \mathbb{N}$ injective total computable *such that*

$$\forall\sigma \in 2^{<\omega}\ [(\exists Z \subset X\ \mathcal{R}(\sigma, Z)) \ \Rightarrow\ \exists Z \subset X\ (\mathcal{R}(\sigma, Z) \wedge f(\sigma) \in X \setminus Z)]$$

Moreover, for such an f one can take some computable reduction of $\{\sigma : \exists Z \in P_{<\omega}(X)\ \mathcal{R}(\sigma, Z)\}$ to X. Also, an index for f as a partial computable function can be computed from indexes for X and $\mathcal{R}$ as $\Sigma^{0,A}_1$ set and relation.

Proof. 1. Let $W^{(A)} \subset \mathbb{N}^2$ be universal for $\Sigma^{0,A}_1$ subsets of $\mathbb{N}$, i.e. W is $\Sigma^{0,A}_1$ and every $\Sigma^{0,A}_1$ subset of $\mathbb{N}$ is a section $W^{(A)}_e = \{n : (e, n) \in W^{(A)}\}$ of $W^{(A)}$ for some e. Since X is $\Sigma^{0,A}_1$-complete, there exists a total computable injective reduction $F : \mathbb{N}^2 \to \mathbb{N}$ of $W^{(A)}$ to X, i.e. $W^{(A)} = F^{-1}(X)$. Then, for every e, the map $F_e : \mathbb{N} \to \mathbb{N}$ such that $F_e(n) = F(e, n)$ is a total computable injective reduction of $W^{(A)}_e$ to X.

2. Let S=$\{\sigma : \exists Z \in P_{<\omega}(X)\ \mathcal{R}(\sigma, Z)\}$. Clearly, S is $\Sigma^{0,A}_1$. Property (7) insures that S is

infinite.
Letting e be some integer (to be fixed by the recursion theorem such that $W_e^A = range(\theta_e) = S$), uniformly in e, we inductively define an injective total A-computable map $\theta_e : \mathbb{N} \to S$ (to be an enumerations of S).
Since F_e is computable, its range is computable with oracle $\emptyset'$, so that the set $X \setminus \text{range}(F_e)$ is $\Sigma_1^{0,A}$. Fix some A-computable enumeration ρ of $\mathcal{R}$.

Stage s. Let (σ, Z) be the least pair (relative to ρ) such that

$$\sigma \notin \{\theta_e(t) : t < s\} \ \wedge \ Z \subseteq \{F_e(\theta_e(t)) : t < s\} \cup (X \setminus \text{range}(F_e)) \tag{8}$$

Property (7) insures that there is always such a σ. Set $\theta_e(s) = \sigma$.

3. Let $\xi : \mathbb{N} \to \mathbb{N}$ be total computable such that $\text{range}(\theta_e) = W^A_{\xi(e)}$. The recursion theorem insures that there exists e so that $W_e^A = W^A_{\xi(e)}$.
Since F_e is an injective total computable reduction of W_e^A to X, the last equality insures that F_e is a reduction of $\text{range}(\theta_e)$ to X. In particular,

$$\text{range}(F_e \circ \theta) = F_e(\text{range}(\theta_e)) = F_e(W^A_{\xi(e)}) = F_e(W_e^A) = \text{range}(F_e) \cap X$$

Hence $\text{range}(F_e \circ \theta_e) \cup (X \setminus \text{range}(F_e)) = X$. Using (8), this insures that $\sigma \in S$. This also yields that every finite subset of X is included in $\{F_e(\theta_e(t)) : t < s\} \cup (X \setminus \text{range}(F_e))$ for s large enough. Using (8) again, we see that every $\sigma \in S$ is in the range of θ_e. Thus, $S = \text{range}(\theta_e)$.

4. Let $f = F_e$. Then f is injective total computable. Also, if $\sigma = \theta_e(s)$ and Z is as in property (8), then $\mathcal{R}(\sigma, Z)$ holds and, since $F_e \circ \theta_e$ is injective, $f(\sigma) = F_e(\theta_e(s)) \notin \{F_e(\theta_e(t)) : t < s\}$, hence $f(\sigma) \notin Z$. □

5.2 Randomness of $\Omega_U[X]$ when X is Σ_n^0 or Π_n^0 complete, $n \geq 2$

The above Lemma 5.1 allows to extend Chaitin's argument to prove randomness of Ω_U to $\Omega_U[X]$.

Theorem 5.2. *Let U be optimal. If $X \subseteq 2^{<\omega}$ is $\Sigma_1^{0,A}$-complete for some $A \subset \mathbb{N}$ such that $\emptyset' \leq_T A$ then $\Omega_U[X]$ is 1-random.*
In particular, if $n \geq 2$ and X is Σ_n^0 complete then $\Omega_U[X]$ is 1-random.

Proof. 1. *Last assertion of the Theorem.* Set $A = \emptyset^{(n-1)}$.

2. *The relation $\mathcal{R} \subset 2^{<\omega} \times P_{<\omega}(\mathbb{N})$.* In order to apply Lemma 5.1, we set

$$\mathcal{R} = \{(\lambda, \emptyset)\} \ \cup \ \{(\sigma, Z) : \sigma \in \text{domain}(U) \ \wedge \ \Omega_U[Z] > U(\sigma)\}$$

where $U(\sigma)$ is identified with a dyadic rational number. Observe that digits of $\Omega_U[Z]$ can be computed from the finite set Z using oracle $\emptyset'$, and the strict inequality $\Omega_U[Z] > U(\sigma)$ can be decided using oracle $\emptyset'$. Since $\emptyset' \leq_T A$, this insures that $\mathcal{R}$ is A-computable. One easily checks that $\mathcal{R}$ satisfies property (7) of Lemma 5.1 (in fact $\{\sigma : \mathcal{R}(\sigma, Z)\}$ is even infinite when $Z \neq \emptyset$).

3. *A constant from the invariance theorem.*
Let f be given by Lemma 5.1. Consider the restriction of f to domain(U). This is a partial

computable function with prefix-free domain. Hence there exists a constant c such that, for all $\sigma \in \text{domain}(U)$,

$$K_U(f(\sigma)) \leq K_{f\restriction domain(U)}(f(\sigma)) + c \leq |\sigma| + c$$

4. *Chaitin's argument pushed up to $\Omega_U[X]$.*
Consider the infinite binary expansion of $\Omega_U[X]$ which, in case it is dyadic (which is not the case, in fact), does end with 1^ω. For $m \in \mathbb{N}$, let σ be such that $U(\sigma) = \Omega_U[X] \restriction m$. Since $\Omega_U[X] > \Omega_U[X] \restriction m$, we see that there exists a finite subset Z of X such that $\Omega_U[Z] > \Omega_U[X] \restriction m$, i.e. such that $\mathcal{R}(\sigma, Z)$.
Clearly, Z must contain all elements $a \in X$ such that $\Omega_U[\{a\}] > 2^{-m}$.
Using Lemma 5.1, we see that $f(\sigma) \in X \setminus Z$. Thus, $\Omega_U[\{f(\sigma)\}] \leq 2^{-m}$. In particular, $K_U(f(\sigma)) \geq m$. Now, since $\sigma \in \text{domain}(U)$, Point 2 yields $K_U(f(\sigma)) \leq |\sigma| + c$. Hence $|\sigma| \geq m - c$.
Thus, every program σ such that $U(\sigma) = \Omega_U[X] \restriction m$ has length $\geq m - c$. This proves that $K_U(\Omega_U[X] \restriction m) \geq m - c$ and hence that $\Omega_U[X]$ is random.

□

The case of $\Pi^{0,A}_1$-complete sets X is obtained with a similar argument.

Theorem 5.3. *Let $A \subset \mathbb{N}$ be such that $\emptyset' \leq_T A$.*
If X is $\Pi^{0,A}_1$-complete then $\Omega_U[X]$ is random.
In particular, if $n \geq 2$ and X is Π^0_n complete then $\Omega_U[X]$ is random.

Proof. 1. *The relation $\mathcal{R}$.* We now let

$$\mathcal{R} = \{(\sigma, Z) : \sigma \in \text{domain}(U) \ \wedge \ \Omega_U - \Omega_U[Z] < U(\sigma) + 2^{-|U(\sigma)|+1}\}$$

Now, $\mathcal{R}$ is $\Sigma^{0,A}_1$ (express $\Omega_U - \Omega_U[Z] < \ldots$ as $\exists m \ \Omega_U \restriction m - \Omega_U[Z] \restriction m + 2^{-m+1} < (\ldots) \restriction m$) and satisfies property (7) from Lemma 5.1.

2. *Chaitin's argument pushed up to $\Omega_U[X]$.*
For $m \in \mathbb{N}$, let σ be such that $U(\sigma) = \Omega_U[X] \restriction m$. Observe that,

$$\Omega - \Omega_U[\mathbb{N} \setminus X] = \Omega_U[X] < \Omega_U[X] \restriction m + 2^{-m+1}$$

so that there exists a finite subset Z of $\mathbb{N} \setminus X$ such that

$$\Omega - \Omega_U[Z] < \Omega_U[X] \restriction m + 2^{-m+1}$$

i.e. such that $\mathcal{R}(\sigma, Z)$.
Observe that if $z \in \mathbb{N} \setminus (Z \cup X)$ then

$$\begin{array}{rcll} \Omega & \geq & \Omega_U[Z] + \Omega_U[X] + \Omega_U[\{z\}] & \\ \Omega_U[\{z\}] & \leq & \Omega - \Omega_U[Z] - \Omega_U[X] & \\ & \leq & \Omega_U[X] \restriction m + 2^{-m+1} - \Omega_U[X] & \\ & \leq & 2^{-m+1} & \text{since } \Omega_U[X] \restriction m - \Omega_U[X] \leq 0 \end{array}$$

Let f be as in Lemma 5.1. Then $f(\sigma) \in (\mathbb{N} \setminus X) \setminus Z$. Therefore $\Omega_U[\{f(\sigma)\}] \leq 2^{-m+1}$. In particular, $K_U(f(\sigma)) \geq m - 1$. Now, since $\sigma \in \text{domain}(U)$, Point 2 of the proof of Theorem 5.2 yields $K_U(f(\sigma)) \leq |\sigma| + c$. Hence $|\sigma| \geq m - 1 - c$.
Thus, every program σ such that $U(\sigma) = \Omega_U[X] \restriction m$ has length $\geq m - 1 - c$. Which proves that $\Omega_U[X]$ is random. □

6 The set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$

6.1 A lemma about sums of subseries

Lemma 6.1. *Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of strictly positive real numbers satisfying*

1. $\lim_{i\to+\infty} a_i = 0$;
2. $a_i \leq \sum_{j>i} a_j$ *for all* i.

Let $\alpha = \sum_{i\in\mathbb{N}} a_i$ (which may be $+\infty$). Then

$$\{\sum_{i\in I} a_i : I \subseteq \mathbb{N}\} = [0,\alpha]$$

Furthermore, for every $r \in [0,\alpha]$ there exists $I(r) \subseteq \mathbb{N}$ such that $\sum_{i\in I(r)} a_i = r$ and which is computable (non uniformly) from r and $(a_i)_{i\in\mathbb{N}}$.

Proof. Take $r \in [0,\alpha]$. We define a monotone increasing sequence $(I_t(r))_{t\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ by the following induction:

$$I_0(r) = \emptyset \quad , \quad I_{t+1}(r) = \begin{cases} I_t(r) \cup \{t\} & \text{if } a_t + \sum_{i\in I_t(r)} a_i \leq r; \\ I_t(r) & \text{otherwise.} \end{cases}$$

Let $I(r) = \bigcup_{t\in\mathbb{N}} I_t(r)$. Since inequality $\sum_{i\in I_t(r)} a_i \leq r$ is true for all t, we get $\sum_{i\in I(r)} a_i \leq r$. We show that $r = \sum_{i\in I(r)} a_i$.

Case $r = \alpha$. Then $I(r) = \mathbb{N}$ and the equality is trivial.

Case $r < \alpha$ and there are infinitely many t's such that $I_{t+1}(r) = I_t(r)$. For such t's we have $\sum_{i\in I_t(r)} a_i \leq r < a_t + \sum_{i\in I_t(r)} a_i$. Taking limits over such t's and using condition 1, we get equality $\sum_{i\in I(r)} a_i = r$.

Case $r < \alpha$ and there are finitely many t's such that $I_{t+1}(r) = I_t(r)$. We show that this case does not occur. Since $r < \alpha$ we have $I(r) \neq \mathbb{N}$ so that there is at least one t such that $I_{t+1}(r) = I_t(r)$. Let u be the largest such t. Then, $\sum_{i\in I_u(r)} a_i \leq r < a_u + \sum_{i\in I_u(r)} a_i$ and, for all $v > u$, $I_{v+1} = I_v \cup \{v\}$. Therefore, $I(r) = I_u(r) \cup \{i : i > u\}$. Since condition 2 insures $a_u \leq \sum_{i>u} a_i$, we get $r < \sum_{i>u} a_i + \sum_{i\in I_u(r)} a_i = \sum_{i\in I(r)} a_i$, which contradicts inequality $\sum_{i\in I(r)} a_i \leq r$.

The last assertion of the Lemma about the relative computability of $I(r)$ is trivial if $I(r)$ is finite. Since the a_t's are strictly positive, if $I(r)$ is infinite then $r \neq a_t + \sum_{i\in I_t(r)} a_i$ for all t. Thus, enumerating the digits of r and $a_t + \sum_{i\in I_t(r)} a_i$, we get at some finite time either $r < a_t + \sum_{i\in I_t(r)} a_i$ or $r > a_t + \sum_{i\in I_t(r)} a_i$, which proves that the test in the definition of $I_{t+1}(r)$ can be done recursively in r and $(a_i)_{i\in\mathbb{N}}$. □

6.2 $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is a finite union of closed intervals

Point 2 of the following theorem gives an alternative proof of Theorem 4.7 above.

Theorem 6.2. *Let U be optimal.*
1. The set $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is the union of finitely many pairwise disjoint closed intervals with positive lengths, i.e.

$$\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = [a_1, b_1] \cup\ [a_2, b_2] \cup ... \cup\ [a_n, b_n]$$

where $0 = a_1 < b_1 < ... < a_n < b_n = \Omega_U$.

2. Every real $s \in \{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is of the form $\Omega_U[Y]$ for some Y which is recursive in $s \oplus \emptyset'$. In particular, there exists some Δ^0_2 set $X \subseteq 2^{<\omega}$ such that $\Omega_U[X]$ is rational, hence not random.

Proof. 1i. First, we get $\alpha > 0$ such that $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} \supseteq [0, \alpha]$.
Let $d, d' \in \mathbb{N}$ be the constants of Lemma 4.6 and let k be such that $2^{2d+1}(i+1) \leq 2^i/(d'i^2)$ for $i \geq k$. Using this inequality and Lemma 4.6, one can inductively define a sequence of *pairwise disjoint* sets of strings $(S_i)_{i\geq k}$ such that $\#S_i = 2^{2d+1}$ and $2^{-i-d-1} < \Omega_U[\{\sigma\}] \leq 2^{-i+d}$ for every $\sigma \in S_i$. Notice that, as in Theorem 4.7, the sequence $(S_i)_{i\geq k}$ is computable in $\emptyset'$.

We define an enumeration ψ of $S = \bigcup_{i\in\mathbb{N}} S_i$: for $i, m \in \mathbb{N}$ and $m < 2^{2d+1}$, let $\psi(2^{2d+1}i+m)$ be the m-th element of S_{k+i}.

Set $a_i = \Omega_U[\{\psi(i)\}]$, it is clearly positive and $\lim_{i\to+\infty} a_i = 0$. Observe that for any $m \in [0, 2^{2d+1})$, $2^{-(k+j)-d-1} < a_{2^{2d+1}j+m} \leq 2^{-(k+j)+d}$ and it is computable in $\emptyset'$. Then, for any such m we have

$$\begin{aligned}\sum_{j>2^{2d+1}q+m} a_j &\geq \sum_{j>q}\ \sum_{s<2^{2d+1}} a_{2^{2d+1}j+s} \\ &> \textstyle\sum_{j>q} 2^{2d+1}2^{-(k+j)-d-1} = 2^{-(k+q)+d} \geq a_{2^{2d+1}q+m}\end{aligned}$$

Thus, the conditions of Lemma 6.1 are satisfied: $\{\Omega_U[Y] : Y \subseteq S\} = [0, \alpha]$ where $\sum_{i\in\mathbb{N}} a_i = \alpha > 0$.

1ii. Now,

$$\begin{aligned}\{\Omega_U[X] : X \subseteq 2^{<\omega}\} &= \{\Omega_U[Y] + \Omega_U[Z] : Y \subseteq S,\ Z \cap S = \emptyset\} \\ &= [0, \alpha] + \{\Omega_U[Z] : Z \cap S = \emptyset\} \\ &= \bigcup_{r\in\mathcal{R}} [r, r + \alpha]\end{aligned}$$

where $\mathcal{R} = \{\Omega_U[Z] : Z \cap S = \emptyset\}$ and $0 \in \mathcal{R}$.
Let $\mathcal{R}_i = \mathcal{R} \cap [i\alpha, (i + 1)\alpha[$. Observe that if $r, r' \in \mathcal{R}_i$ then $[r, r + \alpha]$ and $[r', r' + \alpha]$ have non empty intersection. Hence the union $\bigcup_{r\in\mathcal{R}_i} [r, r + \alpha]$ is an interval J_i (a priori not necessarily closed). Since $\mathcal{R}_i = \emptyset$ for $i\alpha > 1$, we see that $\mathcal{R} = \mathcal{R}_1 \cup ... \cup \mathcal{R}_\ell$ where $\ell \leq \lceil \frac{1}{\alpha} \rceil$. Thus, $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = J_1 \cup ... \cup J_\ell$. Grouping successive intervals J_i's having non empty intersection, we get the representation $\{\Omega_U[X] : X \subseteq 2^{<\omega}\} = I_1 \cup ... \cup I_n$ where the I_i's are pairwise disjoint intervals in $[0, 1]$.

1iii. Since the map $X \mapsto \Omega_U[X]$ is continuous from the compact space $P(2^{<\omega})$ (with the Cantor topology) to $[0, 1]$, its range $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$ is compact. In particular, the intervals I_i's may be taken closed. This proves Point 1 of the Theorem.

2. First, observe that if $I \subseteq \mathbb{N}$ is recursive in $\emptyset'$ then so is $\{\psi(n) : n \in I\}$. Given $\sigma \in 2^{<\omega}$, using

$\emptyset'$, one can check whether $2^{-j} < \Omega_U[\{\sigma\}]$. Hence one can compute i and m such that such that σ is the m-th element of S_{k+i}, i.e. such that $\sigma = \psi(2^{2d+1}i+m)$. Then $\sigma \in \{\psi(n)\colon n \in I\}$ if and only if $2^{2d+1}i + m \in I$

Case $s \in [0, \alpha]$. Lemma 6.1 insures that there is a set $I(s) \subseteq \mathbb{N}$, computable from $s \oplus \emptyset'$, such that $\sum_{i\in I(s)} a_i = s$. Let $X = \{\psi(n)\colon n \in I(s)\}$. Then X is computable from $s \oplus \emptyset'$ and $\Omega_U[X] = s$.

Case $s \in [r, r + \alpha)$ *for some* $r \in \mathcal{R}$. Let $s = \Omega_U[Z] + \beta$ where $r = \Omega_U[Z]$ and $Z \cap S = \emptyset$ and $\beta < \alpha$. Let Z' be a finite subset of Z such that $\Omega_U[Z \setminus Z'] < \alpha - \beta$. Then the real $\Omega_U[Z']$ is computable in $\emptyset'$ and $\Omega_U[Z \setminus Z'] + \beta = s - \Omega_U[Z']$ is computable in $s \oplus \emptyset'$. Since $\Omega_U[Z \setminus Z'] + \beta < \alpha$, Lemma 6.1 yields $X \subseteq S$ which is computable in $s \oplus \emptyset'$ such that $\Omega_U[Z\setminus Z'] + \beta = \Omega_U[X]$. Since Z' is finite, we see that $X \cup Z'$ is computable in $s \oplus \emptyset'$. Finally, $s = \Omega_U[X \cup Z']$.

Case $s \in [a_j, b_j)$ *with* $1 \le j \le n$. Observe that $\bigcup_{r\in\mathcal{R}_i}[r, r + \alpha)$ is equal to J_i with the right endpoint removed. Suppose $I_j = J_i \cup ... \cup J_{i+m}$. Then

$$[a_j, b_j) = \bigcup_{i\le p\le i+m} \; \bigcup_{r\in\mathcal{R}_p} [r, r + \alpha)$$

Thus, $s \in [r, r + \alpha)$ for some $r \in \mathcal{R}$ and the previous case applies.

Case $s = b_j$ *with* $1 \le j \le n$. Let $b_j = \Omega_U[X]$. If $\sigma \notin X$ then $\Omega_U[X \cup \{\sigma\}] > b_j$ hence $\Omega_U[X \cup \{\sigma\}] \ge a_{j+1}$. In particular, $\Omega_U[\{\sigma\}] \ge a_{j+1} - b_i$. Which proves that the complement of X contains at most $\lceil \frac{1}{a_{j+1}-b_j} \rceil$ elements. Thus, X is cofinite, hence recursive. □

In relation with Theorem 6.2, we consider the following question: how much disconnected is $\{\Omega_U[X] : X \subseteq 2^{<\omega}\}$?

Proposition 6.3. *Let U be optimal. For each $n \ge 1$, there exists a finite modification V of U which is still optimal and such that the set $\{\Omega_V[X] : X \subseteq 2^{<\omega}\}$ is not the union of less than n intervals.*

Proof. Let $(p_i)_{i\in\mathbb{N}}$ be an enumeration of *domain*(U) and inductively define integers $i_0 < i_1 < ... < i_n$ such that $i_0 = 0$ and, for $k = 0, ..., n - 1$, letting $H_k = \sum_{i_k\le i<i_{k+1}} 2^{-|p_i|}$ and $T_k = \sum_{i\ge i_k} 2^{-|p_i|}$,

$$H_k \quad > \quad \frac{T_k}{2} \tag{9}$$

We define a first finite modification $\widehat{V}$ of U as follows:

$$\widehat{V}(p_i) = \begin{cases} 0^k & \text{if } i_k \le i < i_{k+1} \text{ and } 0 \le k < n \\ U(p_i) & \text{if } i \ge i_n \end{cases}$$

Clearly, for $0 \le k < n$,

$$\{\Omega_{\widehat{V}}[X] : 0^k \in X \ \wedge \ \forall \ell < k \ 0^\ell \notin X\} \quad \subseteq \quad [H_k, T_k] \tag{10}$$

$$\{\Omega_{\widehat{V}}[X] : \forall \ell < n \ 0^\ell \notin X\} \quad \subseteq \quad [0, T_n] \tag{11}$$

Now, $\Omega_U = T_0$ and inequalities (9) insure that $T_{k+1} = T_k - H_k < H_k$ for $0 \le k < n$. Thus, the intervals $[0, T_n]$, $[H_{n-1}, T_{n-1}]$,..., $[H_0, T_0]$ are pairwise disjoint. Since the sets on the left

in (10), (11) are non empty, we see that $\{\Omega_{\widehat{V}}[X] : X \subseteq 2^{<\omega}\}$ is not the union of less than $n+1$ intervals.
However, if universal functions take each value infinitely many times, an optimal function, such as U is, may take some values only once. Therefore, $\widehat{V}$ may be no more surjective, hence non optimal. We have to insure that $U(p_0), ..., U(p_{i_n-1})$ are indeed values of V. In that purpose, observe that there are infinitely many x's such that $U^{-1}(x)$ has at least two elements. Else, for x large enough, $K_U(x)$ would be $2^{-|p|}$ where p is the unique element such that $U(p) = x$, which would make K_U computable, contradicting optimality of U.
Now, compute i_n many distinct indexes j, all $\geq i_n$, such that the $U(j)$'s are distinct and the $U^{-1}(U(j))$'s have at least two elements. Let $j_0, ..., j_{i_n-1}$ be such indexes and set

$$V(p_i) = \begin{cases} \widehat{V}(p_i) & \text{if } i < i_n \\ U(p_\ell) & \text{if } \ell < i_n \text{ and } i = j_\ell \\ U(p_i) & \text{if } i \geq i_n \text{ and } i \text{ is not among } j_0, ..., j_{i_n-1} \end{cases}$$

V is still a finite modification of U but has the same range as U, hence is surjective. Being surjective and equal to the optimal U almost everywhere, V is also optimal. Finally, observe that inclusions (10), (11) are still true for V since V and $\widehat{V}$ coincide on the p_i's for $i < i_n$. □

6.3 $\Omega_U[X]$ is n-random for some Δ^0_{n+1} sets

As a corollary of Theorem 6.2, we get the following result which is in contrast with Theorems 4.7, 4.8 and 4.9.

Corollary 6.4. *1. For any optimal machine U and any $A \subseteq \mathbb{N}$ such that $\emptyset' \leq_T A$, there is a $\Delta^{0,A}_2$ set X such that $\Omega_U[X]$ is random in A.*

2. For every $n \geq 2$ there is a Δ^0_{n+1} set X such that $\Omega_U[X]$ is n-random. For $n = 1$, there is a computable such X.

Proof. 1. Let α be as in Point 2 of Theorem 6.2, let r be Chaitin real $\Omega^{(A)} = \Omega_{U^{(A)}}[2^{<\omega}]$ associated to some optimal machine $U^{(A)}$ with oracle A and $k \in \mathbb{N}$ be such $r2^{-k} < \alpha$. Then r and $r2^{-k}$ are $\Delta^{0,A}_2$ and random in A. Theorem 6.2 insures that there exists some set X which is computable in $r2^{-k} \oplus \emptyset'$ such that $r2^{-k} = \Omega_U[X]$. Since $\emptyset' \leq_T A$, such an X is $\Delta^{0,A}_2$.

2. If $n = 1$, set $X = 2^{<\omega}$ and apply Chaitin's celebrated result. If $n \geq 2$, apply Point 1. □

7 Varying U and X in $\Omega_U[X]$

¿From point 2 of Theorem 6.2, it follows that, for any given optimal machine U, every c.e. random real is $\Omega_U[X]$ for some $X \subseteq 2^{<\omega}$ which is Δ^0_2. We now show that X can be any Σ^0_1 set if we pick an appropriate optimal machine U.

To prove this, we need some well-known facts. In [3] Calude et al. showed that for any c.e. real a there exists a prefix-free set $R \subseteq 2^{<\omega}$ such that $a = \mu(R2^\omega)$.

Let us recall the definition of Solovay's domination between c.e. reals: Let a and b be c.e. reals. We say that a *dominates* b, and write $b \leq_S a$ iff there is a constant c and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that for each rational $q < a$, $f(q)$ is defined and $f(q) < b$ and $b - f(q) \leq c(a - q)$.

In [7], Downey et al. proved that if a and b are c.e. reals such that $b \leq_S a$, then there is a c.e. real d and constant c such that $ca = b + d$.

Using these results, we can prove the following:

Theorem 7.1. *Let $X \subseteq 2^{<\omega}$ be Σ^0_1, $X \neq \emptyset$, and let $a \in (0,1)$ be c.e. random. There is V optimal machine such that $a = \Omega_V[X]$.*

Proof. Let U be the usual optimal by adjunction machine such that

$$U(0^{e-1}1p) = M_e(p)$$

By Chaitin's Theorem (cf.Point 1 of Theorem 1.4), $\Omega_U[X]$ is a c.e. random real and following [8] we know that $a \equiv_S \Omega_U[X]$. Hence, from [7] there is a c such that $2^{-c}\Omega_U[X] < 1 - a$ and $a - 2^{-c}\Omega_U[X]$ is a c.e. real in $(0,1)$. From [3] there is an r.e. prefix-free set R such that $a - 2^{-c}\Omega_U[X] = \mu(R2^\omega)$.
We define the Kraft-Chaitin list for V with the axioms $\{(|r|, y) : r \in R\}$ and $\{(|p| + c, U(p)) : U(p)\downarrow\}$, where $y \in X$. Since for any p, if $U(p)\downarrow$ then $U(p) = V(q)$, for some q with $|q| = |p| + c$, we conclude that V is optimal by adjunction. By construction, we have $\Omega_V[X] = \mu(R2^\omega) + 2^{-c}\Omega_U[X] = a$. □

8 Conjecture for infinite computations

Considering possibly non halting computations, one can associate to any monotone Turing machine (the machine can not erase nor overwrite its current output) a total map $U^\infty : 2^\omega \to 2^{\leq\omega}$ (cf. [1]) where $2^{\leq\omega}$ is the set of finite or infinite binary sequences. For $\mathcal{X} \subseteq 2^{\leq\omega}$ we define

$$\Omega^\infty_U[\mathcal{X}] = \mu((U^\infty)^{-1}(\mathcal{X})).$$

i.e. $\Omega^\infty_U[\mathcal{X}]$ is the probability that U^∞ gives an output in $\mathcal{X}$.

An analog of Conjecture 1.2 can be stated for infinite computations on optimal monotone machines.

Conjecture 8.1. For any proper subset $\mathcal{X}$ of $2^{\leq\omega}$, the probability $\Omega^\infty_U[\mathcal{X}]$ that an arbitrary infinite input to an optimal monotone machine performing infinite computations gives an output in $\mathcal{X}$ is random.

Relatively to monotone Turing machines which are optimal by adjunction, this conjecture has been proved in [1, 2] for many $\mathcal{X} \subseteq 2^{\leq\omega}$, considering the effective levels of the Borel hierarchy on $2^{\leq\omega}$ with a spectral topology (for which the basic open sets are of the form $s2^{\leq\omega}$, for $s \in 2^{<\omega}$).

Theorem 8.2 ([1, 2])**.** *Let $\mathcal{X} \subseteq 2^{\leq\omega}$ be Σ^0_n (spectral) and hard for the class $\Sigma^0_n(2^\omega)$ with respect to effective Wadge reductions, for any $n \geq 1$. Then, $\Omega^\infty_U[\mathcal{X}]$ is random in $\emptyset^{(n-1)}$.*

We now prove that the conjecture fails in about the same way as Conjecture 1.2. The key fact is that Lemma 4.6 can be transferred to infinite computations.

Lemma 8.3. *Let U be a monotone prefix Turing machine which is optimal by adjunction (cf. Def.3.1). Then $\exists d\ \forall n\ \exists\sigma\ 2^{-n-d} \leq \Omega^\infty_U[\{\sigma\}] \leq 2^{-n+d}$. In fact, for some constant d', there are at least $2^n/(d'\,n^2)$ strings $\sigma \in 2^{<\omega}$ satisfying the inequalities.*

Proof. Fix some total recursive injective function $\theta : 2^{<\omega} \to 2^{<\omega}$ with recursive prefix-free range. Thanks to Lemma 4.6, it suffices to prove that there exists k such that for any $\sigma \in 2^{<\omega}$,

$$2^{-k}\Omega_U[\{\sigma\}] \leq \Omega_U^\infty[\{\theta(\sigma)\}] \leq 2^k \Omega_U[\{\sigma\}],$$

Consider the relation $R \subset 2^{<\omega} \times 2^{<\omega}$ such that $(p, u) \in R$ if and only if the computation of U^∞ on any infinite extension of p has current output u. Let $M : 2^{<\omega} \to 2^{<\omega}$ be the machine such that $M(p)$ halts and outputs σ if and only if $(p, \theta(\sigma)) \in R$ but $(q, \theta(\sigma)) \notin R$ for any proper prefix of p. Clearly, M is partial recursive and has prefix-free domain.
Using optimality by adjunction, let $\tau \in 2^{<\omega}$ be such that $M(p) = U(\tau p)$ for all p. Thus, for any $Z \in 2^\omega$ if $U^\infty(Z) = \theta(\sigma)$, then there exists n such that $U(\tau(Z \upharpoonright n))$ halts and $U(\tau(Z \upharpoonright n)) = \sigma$. Hence,

$$\begin{aligned} \Omega_U^\infty[\{\theta(\sigma)\}] &\leq \mu(\{Z \in 2^\omega : \exists n\, U(\tau(Z \upharpoonright n)) = \sigma\}) \\ &= \sum_{U(\tau p) = \sigma} 2^{-|p|} \\ &\leq 2^{|\tau|}\, \Omega_U[\{\sigma\}]. \end{aligned}$$

For the other inequality, let $N : 2^{<\omega} \to 2^{<\omega}$ be the machine such that $N(p) = \theta(U(p))$ and let ρ be such that $U(\rho p) = N(p) = \theta(U(p))$. Then

$$\begin{aligned} \Omega_U[\{\theta(\sigma)\}] &\geq 2^{-|\rho|} \sum_{U(\rho p) = \theta(\sigma)} 2^{-|p|} \\ &= 2^{-|\rho|} \sum_{U(p) = \sigma} 2^{-|p|} \\ &= 2^{-|\rho|}\, \Omega_U[\{\sigma\}] \end{aligned}$$

To conclude, observe that $\Omega_U^\infty[\{\theta(\sigma)\}] \geq \Omega_U[\{\theta(\sigma)\}]$ and take $k = \max(|\tau|, |\rho|)$. □

From Lemma 8.3, the proofs of Theorems 4.7 and 4.8 adapt easily to Ω_U^∞, giving counterexamples which are included in the subset $2^{<\omega}$ of $2^{\leq\omega}$. However, oracle $\emptyset''$ is needed to check inequalities $\Omega_U^\infty[\{\sigma\}] > \tau$ and check if a given bit of $\Omega_U^\infty[X]$ is zero for finite subsets X of $2^{<\omega}$. Which gives a shift to Δ_3^0. We state the analog of Theorem 4.8.

Theorem 8.4. *For every optimal U and any $A \subseteq \mathbb{N}$, there is a $\Delta_3^{0,A}$ set $X \subseteq 2^{<\omega}$ which is $\Sigma_1^{0,A}$-hard and such that $\Omega_U^\infty[X]$ is not random.*
In particular, if $n \geq 1$ there is a Δ_{n+2}^0 set $X \subseteq 2^{<\omega}$ which is Σ_n^0-hard and such that $\Omega_U^\infty[X]$ is not random.

The counterparts of Theorem 6.2 and Proposition 6.3 are as follows.

Theorem 8.5. *Let U be a monotone Turing machine optimal by adjunction.*

1. *The set $\{\Omega_U^\infty[X] : X \subseteq 2^{<\omega}\}$ is the union of finitely many pairwise disjoint closed intervals.*
   For every real s in the above set there exists $X \subseteq 2^{<\omega}$ recursive in $s \oplus \emptyset''$ such that $s = \Omega_U^\infty[X]$.
2. *The set $\{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega} \wedge (U^\infty)^{-1}(\mathcal{X})$ is measurable$\}$ is the union of finitely many pairwise disjoint closed intervals.*

Proof. 1. Points 1 and 2i, 2ii of the proof of Theorem 6.2 adapt easily. To adapt point 2iii, we show that Ω_U^∞ yields a continuous map $P(2^{<\omega}) \to [0,1]$ where $P(2^{<\omega})$ is endowed with the compact Cantor topology.
Observe that, for all $s \in 2^{<\omega}$, the set $(U^\infty)^{-1}(\{s\})$ is a Borel subset of 2^ω. In fact it is the difference of two open sets since $U^\infty(\alpha) = s$ if and only if at all times t, the current output is a prefix of s and at some time it is s. Now, equality $\Omega_U^\infty[X] = \sum_{x\in X} \mu((U^\infty)^{-1}(\{x\})$ proves that $X \mapsto \Omega_U^\infty[X]$ yields a continuous map $P(2^{<\omega}) \to [0,1]$.

2. The proof uses an argument in the spirit of Radon-Nykodim theorem.
First, observe that for every $\xi \in 2^{\leq\omega}$, the set $(U^\infty)^{-1}(\{\xi\})$ is a Borel subset of 2^ω. The case $\xi \in 2^{<\omega}$ has been checked above. If $\xi \in 2^\omega$ then this set is G_δ since $U^\infty(\alpha) = \xi$ if and only if at all times t there is an n such that the current output is a prefix of $\xi \upharpoonright n$ and for all n there is some time at which the current output is $\xi \upharpoonright n$. Now, let

$$\begin{array}{rcl} \mathcal{A} & = & 2^{<\omega} \ \cup \ \{\alpha \in 2^\omega : \mu((U^\infty)^{-1}(\{\alpha\})) > 0\} \\ \mathcal{B} & = & \{\alpha \in 2^\omega : \mu((U^\infty)^{-1}(\{\alpha\})) = 0\} \end{array}$$

We prove that, for some $c \geq 0$ and some finite sequence $0 = a_1 < b_1 < ... < a_n < b_n$,

$$\begin{array}{ll} (*) & \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq \mathcal{A}\} = [a_1, b_1] \cup ... \cup [a_n, b_n] \\ (**) & \{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq \mathcal{B} \ \wedge \ (U^\infty)^{-1}(\mathcal{X}) \text{ is measurable}\} = [0, c] \end{array}$$

Point 2 then follows since every $\Omega_U^\infty[\mathcal{X}]$, with $\mathcal{X} \subseteq 2^{\leq\omega}$ such that $(U^\infty)^{-1}(\mathcal{X})$ is measurable, is the sum $\Omega_U^\infty[\mathcal{X} \cap \mathcal{A}] + \Omega_U^\infty[\mathcal{X} \cap \mathcal{B}]$.

Clearly, $\mathcal{A}$ is countable. As above, Ω_U^∞ yields a continuous map $P(\mathcal{A}) \to [0,1]$ with the compact Cantor topology on $P(\mathcal{A})$. So that the proof of Theorem 6.2 adapts easily, proving $(*)$.

Consider the lexicographic ordering $\prec$ on 2^ω, which is a total ordering, and let $f : 2^\omega \to [0,1]$ be the map such that

$$f(\alpha) = \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq \alpha\}))$$

Let's see that this map is well defined. Observe that, since $\mathcal{A}$ and $\mathcal{B}$ partition 2^ω, we have $(U^\infty)^{-1}(\mathcal{B}) = (U^\infty)^{-1}(2^\omega) \setminus (U^\infty)^{-1}(\mathcal{A})$. Since $\mathcal{A}$ is countable, this set is Borel. Also, $\{\beta : \beta \preceq \alpha\}$ is Borel an so is $(U^\infty)^{-1}(\{\beta : \beta \preceq \alpha\})$. Thus, $f(\alpha) = \mu((U^\infty)^{-1}(\mathcal{B}) \cap (U^\infty)^{-1}(\{\beta : \beta \preceq \alpha\}))$ is the measure of a Borel set.
The two following facts prove that the range of f is a closed interval $[0, c]$. This yields $(**)$ since $f(1^\omega) = \Omega_U^\infty[\mathcal{B}]$ is the maximum value of the $\Omega_U^\infty[\mathcal{X}]$'s for $\mathcal{X} \subseteq \mathcal{B}$.

Fact 1. *1. f is monotone increasing with respect to $\prec$.*

2. $f(\alpha)$ is also equal to $\mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\}))$.

3. $f(1^\omega) = \mu((U^\infty)^{-1}(\mathcal{B}))$ and $f(0^\omega) = 0$ and $f(u01^\omega) = f(u10^\omega)$ for all $u \in 2^{<\omega}$.

4. f is continuous.

Fact 2. *Suppose $g : 2^\omega \to [0,1]$ is a continuous map such that*

$$g(u01^\omega) = g(u10^\omega) \text{ for all } u \in 2^{<\omega} \tag{12}$$

Then the range of g is a closed interval.

Proof of Fact 1. Point 1 is obvious.
2. Observe that $f(\alpha) - \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\})) = \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\alpha\}))$. Now, 2 is obvious if $\alpha \notin \mathcal{B}$. Else, use the definition of $\mathcal{B}$.
3. The assertion about $f(1^\omega)$ is obvious. For $f(0^\omega)$, use 2. Finally, using 2 again, and the fact that $u01^\omega$ is the predecessor of $u10^\omega$, we get

$$\begin{aligned} f(u10^\omega) &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec u10^\omega\})) \\ &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq u01^\omega\})) \\ &= f(u01^\omega) \end{aligned}$$

4. It is sufficient to show that if $(\alpha_n)_{n\in\mathbb{N}}$ is a monotone increasing or decreasing sequence in 2^ω with limit α and all α_n's are different from α then $f(\alpha)$ is the limit of the $f(\alpha_n)$'s. In case the α_n's are increasing with limit α, we have $\beta \prec \alpha$ if and only if $\beta \preceq \alpha_n$ for some n. Thus,

$$\begin{aligned} f(\alpha) &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \prec \alpha\})) \\ &= \mu((U^\infty)^{-1}(\mathcal{B} \cap \bigcup_{n\in\mathbb{N}} \{\beta : \beta \preceq \alpha_n\})) \\ &= \sup_{n\in\mathbb{N}} \mu((U^\infty)^{-1}(\mathcal{B} \cap \{\beta : \beta \preceq \alpha_n\})) \\ &= \sup_{n\in\mathbb{N}} f(\alpha_n) \end{aligned}$$

In case the α_n's are decreasing, we argue similarly, using the fact that $\beta \preceq \alpha$ if and only if $\beta \preceq \alpha_n$ for all n. □

Proof of Fact 2. Let $\theta : [0,1] \to (2^\omega \setminus 2^{<\omega}0^\omega) \cup \{0^\omega\}$ be the bijective map such that $\theta(0) = 0^\omega$ and, for $0 < r \leq 1$, $\theta(r)$ is the sequence of dyadic digits of r which lies in $2^\omega \setminus 2^{<\omega}0^\omega$.
Using (12), we see that $range(g) = range(g \circ \theta)$. Since the range of a continuous map $[0,1] \to [0,1]$ is always a closed interval, it suffices to prove that $g \circ \theta$ is continuous. I.e. to prove that if $(r_n)_{n\in\mathbb{N}}$ is a monotone increasing (resp. decreasing) sequence of reals in $[0,1]$ with limit $r > 0$ (resp. $r < 1$) and such that r_n's all different from r then $g(\theta(r))$ is the limit of the $g(\theta(r_n))$'s.
Case r is not dyadic rational. Then $\theta(r)$ is the limit of the $\theta(r_n)$'s and we can apply continuity of g.
Case r is dyadic rational and the r_n's are increasing. Then the limit of the $\theta(r_n)$'s is the dyadic expansion of r of the form $u01^\omega$ where $u \in 2^{<\omega}$, which is exactly $\theta(r)$. Again, we apply continuity of g.
Case r is dyadic rational and the r_n's are decreasing. Then the limit of the $\theta(r_n)$'s is the dyadic expansion of r of the form $u10^\omega$ where $u \in 2^{<\omega}$. Applying continuity of g, we see that the limit of the $g(\theta(r_n))$'s is $g(u10^\omega)$. To conclude, observe that $u01^\omega = \theta(r)$ and that (12) insures $g(u10^\omega) = g(u01^\omega)$. □

Proposition 8.6. *Let U be optimal. For each $n \geq 1$, there exists a finite modification V of U which is still optimal and such that none of the sets $\{\Omega_U^\infty[X] : X \subseteq 2^{<\omega}\}$ and $\{\Omega_U^\infty[\mathcal{X}] : \mathcal{X} \subseteq 2^{\leq\omega}\}$ is the union of less than n intervals.*

References

[1] V. Becher & S. Grigorieff, *Random reals and possibly infinite computations. Part I: randomness in $\emptyset'$.* J. of Symbolic Logic, vol. 70:3, p.891–913, 2005.

[2] V. Becher & S. Grigorieff, *Random reals and possibly infinite computations. Part II: From index sets to higher order randomness.* In preparation.

[3] C. Calude, P. Hertlund & B. Khoussainov *Recursively enumerable reals and Chaitin Omega numbers.* Theoretical Computer Science, vol. 255 (1-2), p.125–149, 2001.

[4] G. J. Chaitin. *Algorithmic information theory.* Cambridge University Press, 1988.

[5] G. J. Chaitin. *A theory of program size formally identical to information theory. J. ACM*, vol.22, 329–340, 1975.

[6] R. Downey, D. Hirschfeldt. *Algorithmic Randomness and Complexity.* To appear, 2005.

[7] R. Downey, D. Hirschfeldt, A. Nies. *Randomness, computability and density.* SIAM J. on Computing, vol 31 (2002), 1169-1183.

[8] A. Kucera and T. A. Slaman. *Randomness and recursive enumerability*, SIAM J. on Computing, vol. 31 (2001) 199-211.

[9] R. M. Solovay, *Draft of paper (or series of papers) on Chaitin's work, done for the most part during the period of Sept.-Dec. 1974.* Unpublished manuscript, IBM Thomas Watson Research Center, Yorktown Heights, NY, 215 pages.

[10] P. Odifreddi, *Classical Recursion Theory*, North- Holland, 1990.

[11] H. Rogers Jnr., *Theory of Recursive Functions and Effective Computability*, McGraw Hill, New York, 1968.

[12] S. Figueira, F. Stephan and G. Wu. *Randomness and Universal Machines*, Proceedings of CCA 2005 (Second International Conference on Computability and Complexity in Analysis), Kyoto, 2005.