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Effectiveness for embedded spheres and balls

Joseph S. Miller [1]

Department of Mathematics
Cornell University
Ithaca, NY 14853

Abstract

We consider arbitrary dimensional spheres and closed balls embedded in $\mathbb{R}^n$ as Π^0_1 classes. Such a strong restriction on the topology of a Π^0_1 class has computability theoretic repercussions. Algebraic topology plays a crucial role in our exploration of these consequences; the use of homology chains as computational objects allows us to take algorithmic advantage of the topological structure of our Π^0_1 classes. We show that a sphere embedded as a Π^0_1 class is necessarily located, i.e., the distance to the class is a computable function, or equivalently, the class contains a computably enumerable dense set of computable points. Similarly, a ball embedded as a Π^0_1 class has a dense set of computable points, though not necessarily c.e. To prove location for balls, it is sufficient to assume that both it and its boundary sphere are Π^0_1. However, the converse fails, even for arcs; using a priority argument, we prove that there is a located arc in $\mathbb{R}^2$ without computable endpoints. Finally, the requirement that the embedding map itself be computable is shown to be stronger than the other effectiveness criteria considered. A characterization in terms of computable local contractibility is stated; the proof will be the subject of a sequel.

1 Introduction

If $\mathcal{C}$ is a compact space, then a continuous injection $f\colon \mathcal{C} \to \mathbb{R}^n$ induces a homeomorphism between $\mathcal{C}$ and $f[\mathcal{C}]$. We say that f *embeds* $\mathcal{C}$ into $\mathbb{R}^n$. In the present work, we study Π^0_1 classes in $\mathbb{R}^n$ which are embeddings of either spheres or closed balls of any dimension. It should not be surprising that such strong topological assumptions on Π^0_1 classes restrict possible computable behavior. Nor should it be surprising that homology is a useful tool in studying these classes. Homology is too coarse to detect the pathologies that embedded spheres and balls can exhibit. [2] The homology of the complements of such embeddings is well-known to be well-behaved; it is by considering this homology

[1] Email:jmiller@math.cornell.edu
[2] Alexander's horned sphere is the best known example of a "wild" embedding.

that we prove our main theorems. In particular, we make computational use of homology cycles in the proofs of Theorems 2.1 and 3.1.

We begin with a quick review of both computable analysis and homology theory. In Section 2, it is shown that a sphere embedded as a Π^0_1 class is located. This is not true for balls embedded as Π^0_1 classes; two results deal with this case. First, Corollary 2.3 says that if the boundary sphere is also a Π^0_1 class, then the ball is located. Then, in Section 3, it is shown that an arbitrary embedding of a ball as a Π^0_1 class must be *weakly located*, i.e. it has a dense set of computable points. Section 4 shows that the converse to Corollary 2.3 fails by giving an example of an arc in $\mathbb{R}^2$ which is located, but which has non-computable endpoints. Finally, Section 5 considers embeddings of spheres and balls which satisfy a much stronger effectiveness criterion: that the embedding map itself be computable. We state a characterization of such embeddings using computable local contractibility. The proofs, which use techniques beyond the scope of this paper, will be given elsewhere.

Our use of homology in the study of Π^0_1 classes appears to be novel, but the study of Π^0_1 classes in computable analysis is not new. In 1957, Lacombe proved that the nonempty Π^0_1 classes in $I = [0, 1]$ are exactly the sets on which computable functions achieve their maxima [3]. Using the fact that there is a nonempty Π^0_1 class in I which contains no computable points, Lacombe concluded that there is a computable function on I which does not achieve its maximum at a computable point. In 1963, Orevkov constructed an example of a computable function $f\colon I^2 \to I^2$ without a computable fixed point [6]. The set of fixed points of any such function must be a nonempty Π^0_1 class. Furthermore, it can be shown that it must contain a nonempty, connected Π^0_1 component [4]. Cenzer and Remmel have given quite a few results for Π^0_1 classes in analysis [2]. Many of these are direct translations of corresponding results for Π^0_1 classes in $2^{\mathbb{N}}$, while some make use of the topology of real space.

1.1 Computable Analysis

For a thorough introduction to computable analysis see Weihrauch [8], though the framework is considerably more general and the notation different from that used here. Another good introduction is Pour-El and Richards [7].

It is assumed that the reader has a basic understanding of computability on $\mathbb{N}$. The following definitions isolate those subsets of, points in, and continuous functions on $\mathbb{R}^n$ which we consider to be effective. These are the fundamental objects of study in computable analysis. An open set in $\mathbb{R}^n$ is called a Σ^0_1 *class* if it is a c.e. union of open balls with rational centers and radii. The complement of a Σ^0_1 class is called a Π^0_1 *class*; these are the effective closed sets. A *name* for $x \in \mathbb{R}$ is a Cauchy sequence $\lambda\colon \mathbb{N} \to \mathbb{Q}$ for x with convergence regulated by 2^{-N}. A real is *computable* if it has a computable name, and a point in $\mathbb{R}^n$ is *computable* if the coordinates are. Note that $x \in \mathbb{R}^n$ is computable iff $\{x\}$ is Π^0_1.

A *computable function on* $\mathbb{R}$ effectively maps Cauchy sequences to Cauchy sequences, preserving equivalence. More formally, it is a computable function $f^\lambda\colon \mathbb{N} \to \mathbb{Q}$, with oracle λ, such that if λ is a name, then f^λ is a name and if λ_1 and λ_2 are both names for $x \in \mathbb{R}$, then f^{λ_1} and f^{λ_2} are names for the same real. We identify f with the function that it induces on $\mathbb{R}$. This definition extends without difficulty to computable functions on $\mathbb{R}^n$ and to computable partial functions. It is well-known that a computable partial function is continuous on its domain and that the Π^0_1 classes are exactly the level sets of total computable functions.

A closed set $X \subseteq \mathbb{R}^n$ is *located* if the distance function $d(x, X)$ is computable. This is equivalent to X being Π^0_1 and containing a dense, c.e. set of computable points. Location is a powerful hypothesis on closed sets, much stronger than merely being a Π^0_1 class. A Π^0_1 class is called *weakly located* if it contains a dense set of computable points (not necessarily c.e.). As has already been indicated, a nonempty Π^0_1 class need not contain any computable points at all.

1.2 Homology

Before we can prove, or even properly state, our main theorems, we must recall some elementary homology theory. We will employ a slight modification of the simple, reduced homology with $\mathbb{Z}_2$ coefficients found in [5]. A *cell* in $\mathbb{R}^n$ is an intersection of closed rational half spaces aligned with the coordinate axes. We will actually work in $\widehat{\mathbb{R}^n}$, the one point compactification of $\mathbb{R}^n$. Cells in $\widehat{\mathbb{R}^n}$ are the closures of the cells from $\mathbb{R}^n$, plus a new 0-cell: ω, the point at infinity. We call k-dimensional cells *k-cells*. A *k-chain* is a finite set of k-cells which intersect only on their boundaries. The *locus* of a chain is the union of the its cells. Because we are working with $\mathbb{Z}_2$ coefficients, we can identify chains which have the same locus. This means that we are free to *refine* a chain by splitting its cells. The sum of k-chains is taken modulo 2, with the understanding that the chains may need to be refined. The sum of two k-chains corresponds to the closure of the symmetric difference of the loci.

The *boundary* of a cell is evident and, for any chain Σ, the boundary $\partial\Sigma$ is the sum of the boundaries of its cells. Note that ∂ takes k-chains to $(k-1)$-chains. Let 0 represent the empty chain. If $k > 0$, a k-chain with boundary 0 is a *k-cycle*. A 0-chain is a *0-cycle* iff it consists of an even number of points. It is this last detail that gives us *reduced* homology; it is a technical trick to ensure that the theorems stated below hold in the 0-dimensional case. A cycle Σ *bounds* if it is the boundary of another chain. We write this as $\Sigma \sim 0$ and say that Σ is *homologous* to 0. If $\Sigma_1 + \Sigma_2 \sim 0$, then Σ_1 is *homologous* to Σ_2, which we write as $\Sigma_1 \sim \Sigma_2$.

Boundaries are always cycles (in particular, $\partial^2 = 0$). Conversely, cycles in $\widehat{\mathbb{R}^n}$ always bound, but this fails if we restrict our chains to a subset $X \subseteq \widehat{\mathbb{R}^n}$. A *chain in X* is a chain in $\widehat{\mathbb{R}^n}$ with locus contained in X. A cycle in X is

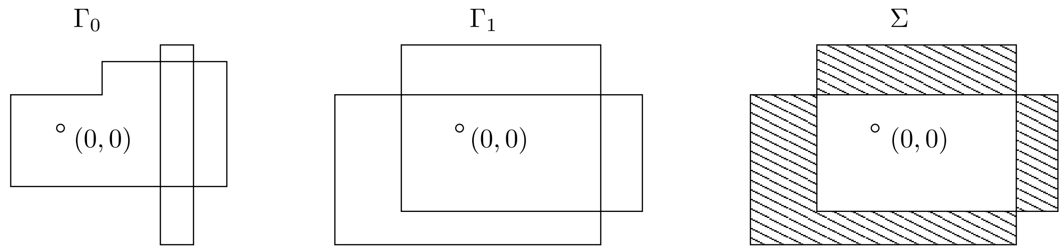

Fig. 1. Both Γ_0 and Γ_1 are 1-cycles in $\mathbb{R}^2 \setminus \{(0,0)\}$, while Σ is a 2-chain in the same space. Clearly, $\Gamma_1 = \partial\Sigma$ is a boundary in $\mathbb{R}^2 \setminus \{(0,0)\}$; Γ_0 is non-bounding.

said to *bound in* X if it is the boundary of a chain in X. Intuitively, cycles are chains that want to be boundaries; the failure of a cycle in X to bound in X detects the presence of a hole. For a simple example, consider fig. 1. Both Γ_0 and Γ_1 are 1-cycles in $\mathbb{R}^2 \setminus \{(0,0)\}$, but only Γ_1 bounds in the punctured plane. It follows from Theorem 1.2 below, with $n = 2$ and $m = 0$, that all non-bounding 1-cycles in $\mathbb{R}^2 \setminus \{(0,0)\}$ are homologous to Γ_0 in $\mathbb{R}^2 \setminus \{(0,0)\}$.

There are slight differences in notation from [5], but the only significant change is our restriction to *rational* half spaces when defining cells. This will not break the homology theory, at least as far as we are concerned, and it will allow us to use homology computationally. In particular, we can enumerate the chains in $\widehat{\mathbb{R}^n}$. The following theorems are proved exactly as in [5].

Theorem 1.1 *If $D \subseteq \mathbb{R}^n$ is an embedded m-dimensional ball, then every k-cycle in $\widehat{\mathbb{R}^n} \setminus D$ bounds in $\widehat{\mathbb{R}^n} \setminus D$.*

Theorem 1.2 *Let $S \subseteq \mathbb{R}^n$ be an embedded m-dimensional sphere.*

- *If $k \neq n - m - 1$, then every k-cycle in $\widehat{\mathbb{R}^n} \setminus S$ bounds in $\widehat{\mathbb{R}^n} \setminus S$.*
- *There exists a non-bounding $(n-m-1)$-cycle in $\widehat{\mathbb{R}^n} \setminus S$.*
- *If Γ_1 and Γ_2 are non-bounding in $\widehat{\mathbb{R}^n} \setminus S$, then $\Gamma_1 \sim \Gamma_2$ in $\widehat{\mathbb{R}^n} \setminus S$*

2 Location for Embedded Spheres and Balls

Theorem 2.1 *Let $S \subseteq \mathbb{R}^n$ be an embedded m-dimensional sphere. If S is Π^0_1, then it is located. The distance function can be computed uniformly from the Π^0_1 index, an $R \in \mathbb{Q}$ such that $S \subseteq B(\vec{0}, R)$, and a non-bounding $(n-m-1)$-cycle Γ in $\widehat{\mathbb{R}^n} \setminus S$.*

Proof. Assume that we are given R and Γ. The existence of an appropriate Γ is guaranteed by Theorem 1.2. Let $\{S_j\}_{j\in\mathbb{N}}$ be the computable sequence of approximations to S and let $\{C_k\}_{k\in\mathbb{N}}$ be an enumeration of all $(n-m)$-chains in $\widehat{\mathbb{R}^n}$.

Given $x \in \mathbb{R}^n$, we will compute $d(x, S)$ by finding upper and lower approximations. The fact that S is a Π^0_1 class gives us an approximation of the distance from below; $b_i = d(x, S_i)$ is a monotonically increasing sequence, computable in x, and $\lim_{i\to\infty} b_i = d(x,S)$. It is to approximate $d(x,S)$ from above that we will employ homology.

Define $\hat{a}_{\langle j,k\rangle}$ to be the maximum distance from x to any point of $C_k \cap S_j \cap B(\vec{0},R)$, provided that $\partial C_k = \Gamma$. Otherwise, define $\hat{a}_{\langle j,k\rangle} = \infty$. Now let $a_i = \min\{\hat{a}_{\langle j,k\rangle} \mid j,k \leq i\}$. It is clear that $\{a_i\}_{i\in\mathbb{N}}$ is a monotonically decreasing sequence. If $\partial C_k = \Gamma$, then $C_k \cap S \neq \emptyset$, because Γ does not bound in $\widehat{\mathbb{R}^n} \smallsetminus S$. Therefore, there is a point of S in $C_k \cap S_j \cap B(\vec{0},R)$, implying that $\hat{a}_{\langle j,k\rangle} \geq d(x,S)$. Thus, $a_i \geq d(x,S)$ for all $i \in \mathbb{N}$.

All that remains is to show that $\lim_{i\to\infty} a_i = d(x,S)$. Let $\varepsilon > 0$. Choose $z \in S$ such that $d(x,S) = d(x,z)$ and $q \in \mathbb{Q}$ such that $d(q,z) < \varepsilon$. Now, any nonempty open subset of S contains an open "cap", the removal of which leaves an embedded ball. In particular, there is an embedded m-ball $D \subseteq S$ such that $S \smallsetminus D \subseteq B(q,\varepsilon)$. By Theorem 1.1, Γ bounds in $\widehat{\mathbb{R}^n} \smallsetminus D$. Take k such that $\partial C_k = \Gamma$ and $C_k \subseteq \widehat{\mathbb{R}^n} \smallsetminus D$. For large enough j, $S_j \cap C_k \cap \overline{B}(\vec{0},R) \subseteq B(q,\varepsilon)$. This follows from the compactness of $C_k \cap \overline{B}(\vec{0},R) \smallsetminus B(q,\varepsilon)$. So, $\hat{a}_{\langle j,k\rangle} \leq d(x,q) + \varepsilon < d(x,S) + 2\varepsilon$. If $i \geq j,k$, then $a_i \leq d(x,q) + 2\varepsilon$. But ε was arbitrary. Therefore, $\lim_{i\to\infty} a_i = d(x,S)$. This was all that needed to be verified, so the proof is complete. □

Unlike in the case of spheres, we cannot expect arbitrary Π^0_1 embeddings of balls to be located, as the following simple example illustrates.

Example 2.2 Not every Π^0_1 arc in $\mathbb{R}^n$ is located (even for $n = 1$).

Proof. Let $\gamma \in [0,1]$ be a non-computable c.e. real (i.e. left computable). For example, we could take a non-computable c.e. set $B \subseteq \mathbb{N}$ and let $\gamma = \Sigma_{i\in B} 2^{-(i+1)}$. If we let $A = [\gamma, 1] \subseteq \mathbb{R}$, then A is clearly both a Π^0_1 set and an arc. But $d(0,A) = \gamma$ is not computable, hence A is not located. □

We respond to the limitation presented by Example 2.2 in two ways. In the next section, it will be proved that a ball embedded as a Π^0_1 class must contain a dense set of computable points, a weak alternative to location. First, we give a reasonable hypothesis under which embedded balls *are* located. The following is a corollary to Theorem 2.1.

Corollary 2.3 *Let $D \subseteq \mathbb{R}^n$ be an embedded m-dimensional ball and let S be the image of the boundary sphere. If both D and S are Π^0_1, then D is located. The distance function can be computed uniformly from the Π^0_1 indices, an $R \in \mathbb{Q}$ such that $D \subseteq B(\vec{0},R)$, and a non-bounding $(n-m)$-cycle Γ in $\widehat{\mathbb{R}^n} \smallsetminus S$.*

Proof. Let $L = (D \times \{-1,1\}) \cup (S \times [-1,1])$. Then L is sphere embedded as a Π^0_1 class in $\mathbb{R}^{n+1}$, hence located. For any point $x \in \mathbb{R}^n$, $d(x,D) = d((x,1),L)$. Therefore, D is located. Only the uniformity claim remains.

Assume that we are given an $R \in \mathbb{Q}$ such that $D \subseteq B(\vec{0},R)$, and a non-bounding $(n-m)$-cycle Γ in $\widehat{\mathbb{R}^n} \smallsetminus S$. If $R_0 = R + 1$, then clearly $R_0 \in \mathbb{Q}$ and $L \subseteq B(\vec{0},R_0)$. Finally, we must compute a non-bounding $(n-m)$-cycle Γ_0 in $\widehat{\mathbb{R}^{n+1}} \smallsetminus L$. It should come as no surprise that $\Gamma_0 = \Gamma \times \{0\}$ is such a cycle

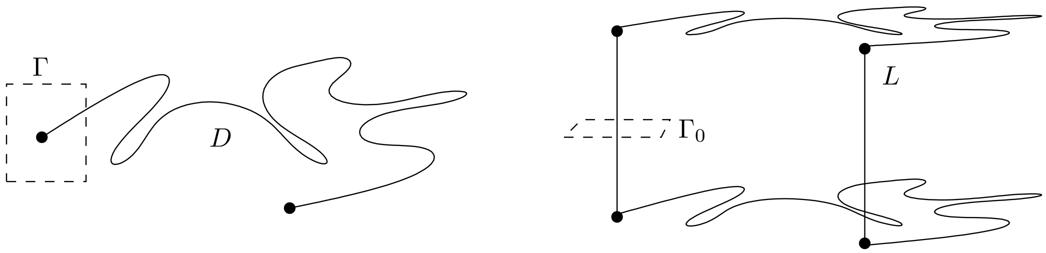

Fig. 2. We construct a non-bounding cycle $\Gamma_0 = \Gamma\times\{0\}$ in $\widehat{\mathbb{R}^{n+1}} \smallsetminus L$, where Γ is a non-bounding cycle in $\widehat{\mathbb{R}^n} \smallsetminus S$.

(fig. 2), but as distressingly self-evident as this is, we shall now make a mess of proving it.

Note that $S_0 = S\times\{0\}$ is the homeomorphic image of an $(m–1)$-dimensional sphere in $\mathbb{R}^{n+1}$. Therefore, there is an $(n–m+1)$-cycle Σ' which does not bound in $\widehat{\mathbb{R}^{n+1}} \smallsetminus S_0$. Let $\Sigma'_+ = \Sigma' \cap (\overline{\mathbb{R}^n\times[0,\infty)})$ and let $\Sigma'_- = \Sigma' + \Sigma'_+$. Let $\Sigma_0 = \partial\Sigma'_+ = \partial\Sigma'_-$. Note that $\Sigma'_- \subseteq \overline{\mathbb{R}^n\times(-\infty,0]}$ and $\Sigma_0 \subseteq \overline{\mathbb{R}^n\times\{0\}} = \widehat{\mathbb{R}^n}\times\{0\}$. Finally, let Σ be the $(n-m)$-cycle in $\widehat{\mathbb{R}^n} \smallsetminus S$ such that $\Sigma_0 = \Sigma\times\{0\}$.

Now assume, for a contradiction, that Σ_0 bounds a complex Θ in $\widehat{\mathbb{R}^{n+1}} \smallsetminus L$. Then $\Theta + \Sigma'_-$ is a cycle—hence bounding by Theorem 1.1—in $\widehat{\mathbb{R}^{n+1}} \smallsetminus (L \cap (\overline{\mathbb{R}^n\times[0,\infty)}))$. So it must bound in the superspace $\widehat{\mathbb{R}^{n+1}} \smallsetminus S_0$. Similarly, $\Theta + \Sigma'_+$ also bounds in $\widehat{\mathbb{R}^{n+1}} \smallsetminus S_0$. Therefore, $\Sigma' = (\Theta + \Sigma'_-) + (\Theta + \Sigma'_+)$ bounds in $\widehat{\mathbb{R}^{n+1}} \smallsetminus S_0$, which is a contradiction. Therefore, Σ_0 is non-bounding in $\widehat{\mathbb{R}^{n+1}} \smallsetminus L$.

It is easy to see that Σ does not bound in $\widehat{\mathbb{R}^n} \smallsetminus S$, for if bounds a complex Θ, then Σ_0 bounds $\Theta\times\{0\} \subseteq (\widehat{\mathbb{R}^{n+1}} \smallsetminus L)$, which we have just shown is impossible. Since the $(n-m)$-cycles Σ and Γ are both non-bounding in $\widehat{\mathbb{R}^n} \smallsetminus S$, they are homologous. But this implies that Σ_0 and Γ_0 are homologous in $\widehat{\mathbb{R}^{n+1}} \smallsetminus L$, proving that Γ_0 is also non-bounding in this space. This completes the proof.□

In particular, we have the following simple corollary.

Corollary 2.4 *A Π^0_1 arc $A \subseteq \mathbb{R}^n$ with computable endpoints is located. The distance function can be computed from the Π^0_1 index for A and the indices for the endpoints.*

It is worth noting that Corollary 2.3 does not characterize located embeddings of disks. In particular, there is a located arc $A \subseteq \mathbb{R}^2$ with non-computable endpoints. An example will be constructed in Section 4.

3 Weak Location for Embedded Balls

Recall that a Π^0_1 class is called *weakly located* if it contains a dense set of computable points.

Theorem 3.1 *If $D \subseteq \mathbb{R}^n$ is an embedding of an m-dimensional ball as a Π^0_1 class, then it is weakly located.*

To prove this theorem we will need another fact from homology theory: the homology of the complement of an annulus is the same as the homology of the complement of a circle. Stated in more generality:

Lemma 3.2 *Let $A = S^m \times I$ and let $f\colon A \hookrightarrow \mathbb{R}^n$ be a continuous injection. For all $x \in I$, the inclusion of $\widehat{R^n} \smallsetminus f[A]$ into $\widehat{R^n} \smallsetminus f[S^m \times \{x\}]$ induces an isomorphism on homology.*

The homology of $\widehat{R^n} \smallsetminus f[A]$ can be computed by induction on m using the Mayer-Vietoris sequence, as is usually done with the homology of the complement of an embedded sphere. The fact that inclusion induces an isomorphism follows from the naturality of the Mayer-Vietoris sequence.

Proof of Theorem 3.1. Let $S \subseteq \mathbb{R}^n$ be the boundary sphere of D. Assume that we are given $\vec{x} \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$ such that $B(\vec{x}, r) \cap D \neq \emptyset$ but $B(\vec{x}, r) \cap S$ is empty. We will show that D must have a computable point in $B(\vec{x}, r)$; the theorem clearly follows from this observation.

In order to construct a computable point in $B(\vec{x}, r) \cap D$, we will construct a computable sequence $\{\vec{x}_s\}_{s\in\mathbb{N}}$ of points in $\mathbb{Q}^n$ and a sequence $\{\vec{r}_s\}_{s\in\mathbb{N}}$ of rational radii, such that $\overline{B}(\vec{x}_0, r_0) \subseteq B(\vec{x}, r)$, and for all s, $B(\vec{x}_{s+1}, r_{s+1}) \subseteq B(\vec{x}_s, r_s)$, $r_{s+1} \leq r_s/2$, and $B(\vec{x}_s, r_s) \cap D \neq \emptyset$. Then $\lim_{s\to\infty} \vec{x}_s \in B(\vec{x}, r) \cap D$ is computable. It is to ensure that $B(\vec{x}_s, r_s) \cap D \neq \emptyset$ that we employ homology.

Now, any nonempty open subset of D contains an open "cap", the removal of which leaves an embedding of $S^m \times I$. In particular, there is a continuous injection $g\colon S^m \times I \to \mathbb{R}^n$ such that $\text{image}(g) \subseteq D$ and $D \smallsetminus \text{image}(g) \subseteq B(\vec{x}, r)$. By Lemma 3.2 and Theorem 1.2, there is an $(n-m-1)$-cycle Γ in $\widehat{\mathbb{R}^n} \smallsetminus \text{image}(g) \subseteq (\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}, r)$ which does not bound in $\widehat{\mathbb{R}^n} \smallsetminus S$.

Now choose $\vec{x}_0 \in \mathbb{Q}^n$ and $r_0 \in \mathbb{Q}$ such that $\overline{B}(\vec{x}_0, r_0) \subseteq B(\vec{x}, r)$, $B(\vec{x}_0, r_0) \cap \text{image}(g)$ is empty and $B(\vec{x}_0, r_0) \cap D \neq \emptyset$. We can choose such a $\vec{x}_0$ and r_0 because $\text{image}(g)$ is closed and $B(\vec{x}, r) \cap D \smallsetminus \text{image}(g)$ is nonempty. We now have everything that we need to construct a computable point in $B(\vec{x}, r) \cap D$.

Consider the following algorithm: at stage $s+1$, search for $\vec{x}_{s+1} \in \mathbb{Q}^n$, $r_{s+1} \in \mathbb{Q}$ and an $(n-m-1)$-cycle Γ_{s+1} in $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}_{s+1}, r_{s+1})$ such that $B(\vec{x}_{s+1}, r_{s+1}) \subseteq B(\vec{x}_s, r_s)$, $r_{s+1} \leq r_s/2$ and $\Gamma_{s+1} \sim \Gamma$ in $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}, r)$. Two things must be verified. First, that the algorithm succeeds; in other words, the search is successful at each stage. Second, that if the algorithm does succeeds, then $\lim_{s\to\infty} \vec{x}_s$ is a computable point in $D \cap B(\vec{x}, r)$.

Let us consider the second goal. Assume that the search is successful at each stage. It is clear that $z = \lim_{s\to\infty} \vec{x}_s$ is a computable point and that $z \in B(\vec{x}, r)$. All that is left to show is that $z \in D$. For that it suffices to prove that, for each s, $B(\vec{x}_s, r_s) \cap D \neq \emptyset$. Assume not for some s; then Γ_s is actually a cycle in $\widehat{\mathbb{R}^n} \smallsetminus D = (\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}_s, r_s)$. But by Theorem 1.1, Γ_s bounds in $\widehat{\mathbb{R}^n} \smallsetminus D$, hence also in $\widehat{\mathbb{R}^n} \smallsetminus S$. But $\Gamma_s \sim \Gamma$ in $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}, r)$, hence also in $\widehat{\mathbb{R}^n} \smallsetminus S$. Therefore, Γ bounds in $\widehat{\mathbb{R}^n} \smallsetminus S$, but this is a contradiction. Therefore, $B(\vec{x}_s, r_s) \cap D \neq \emptyset$ for all s, hence $z \in D$.

Finally, we must prove that the search for $\vec{x}_{s+1}$, r_{s+1} and Γ_{s+1} at stage $s+1$ is successful. First note that if there exist $\vec{x}_{s+1}$, r_{s+1} and Γ_{s+1} which satisfy the requirements of the algorithm, then they will eventually be found by an exhaustive search. In particular, if $\Gamma_{s+1} \sim \Gamma$ in the Σ^0_1 class $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}, r)$, then there is some stage in the enumeration of this class at which the two cycles are homologous.

All that remains is to prove, assuming that the algorithm has reached stage $s+1$, that suitable $\vec{x}_{s+1}$, r_{s+1} and Γ_{s+1} exist. From above, we know that $B(\vec{x}_s, r_s) \cap D \neq \emptyset$. Let $r_{s+1} = r_s/2$ and choose $\vec{x}_{s+1}$ such that $B(\vec{x}_{s+1}, r_{s+1}) \subseteq B(\vec{x}_s, r_s)$ and $B(\vec{x}_{s+1}, r_{s+1}) \cap D \neq \emptyset$. By the same argument used to prove the existence of Γ, there is an $(n-m-1)$-cycle Γ_{s+1} in $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}_{s+1}, r_{s+1})$ which does not bound in $\widehat{\mathbb{R}^n} \smallsetminus S$. But $B(\vec{x}_{s+1}, r_{s+1}) \subseteq B(\vec{x}_0, r_0)$ is disjoint from image(g), hence Γ_{s+1} is also a cycle in $\widehat{\mathbb{R}^n} \smallsetminus \text{image}(g)$. So both Γ and Γ_{s+1} are cycles in $\widehat{\mathbb{R}^n} \smallsetminus \text{image}(g)$ which do not bound in $\widehat{\mathbb{R}^n} \smallsetminus S$; by Lemma 3.2 and Theorem 1.2, $\Gamma \sim \Gamma_{s+1}$ in $\widehat{\mathbb{R}^n} \smallsetminus \text{image}(g)$, hence in $(\widehat{\mathbb{R}^n} \smallsetminus D) \cup B(\vec{x}, r)$. Therefore, we have found $\vec{x}_{s+1}$, r_{s+1} and Γ_{s+1} to satisfy the requirements of the algorithm. Thus the algorithm is successful and the proof is complete. □

4 A Pathological Located Arc

In Corollary 2.3 it was shown that an embedded ball is located if both it and its boundary sphere are Π^0_1 classes. As the following example shows, the converse fails, even in the simplest nontrivial case. The proof can be thought of as a simple priority construction; the requirements to be met being that the ternary digits in the expansion of one of the coordinates of one of the endpoints encode membership in a non-computable c.e. set B.

Example 4.1 There is a located arc in $\mathbb{R}^2$ with non-computable endpoints.

Proof. Let $\rho\colon 2^{\mathbb{N}} \to I$ be the standard homeomorphism between $2^{\mathbb{N}}$ and the middle thirds Cantor set. To be explicit, for $M \subseteq \mathbb{N}$, $\rho(M) = 2\,\Sigma_{i\in M}(1/3)^{i+1}$. Let $B \subseteq \mathbb{N}$ be a non-computable c.e. set and let $\gamma = \rho(B)$. We will build a located arc $A \subseteq \mathbb{R}^2$ so that the first coordinate of one of the endpoints is γ, hence that endpoint will be non-computable. For simplicity, the other endpoint of A will be $(1, 0)$. It could be handled in the same way.

Let B_s be the enumeration of B at stage s; we may assume that $|B_s| = s$. Denote the element enumerated at stage s by n_s. Let $\gamma^{\min}_s = \rho(B_s)$ and $\gamma^{\max}_s = \rho(B_s \cup \{i \mid i \geq n_s\})$. In other words, $\gamma^{\min}_s$ and $\gamma^{\max}_s$ are the lower and upper bounds for γ assuming that, after stage s, nothing is enumerated into B below n_s.

We build $A = \bigcup_{s\in\mathbb{N}} A_s$ in stages. Let A_0 be the line segment from $(0, 0)$ to

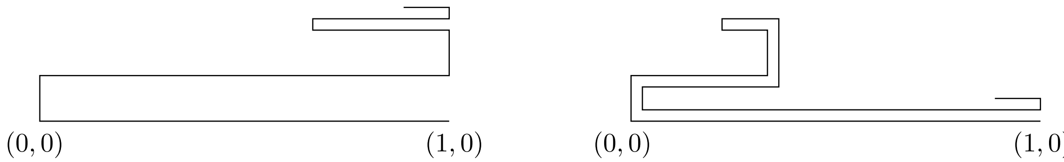

Fig. 3. Both possibilities for A_2 in the case that $B_2 = \{0, 1\}$. The left corresponds to $B_1 = \{0\}$; the right to $B_1 = \{1\}$.

$(1, 0)$. We call the endpoint which starts at $(0, 0)$, and which will be moved around during the construction, the *active* endpoint. For each s, A_{s+1} will extend A_s from the active endpoint, with the first coordinate of that endpoint ending up at $\gamma_{s+1}^{\min}$. In particular, construct A_{s+1} from A_s as follows: backtrack along A_s from the active endpoint until reaching a point with first coordinate $\gamma_{s+1}^{\max}$. Then turn around and create a line segment which stretches until the first coordinate is $\gamma_{s+1}^{\min}$. See fig. 3 for an example. One very important detail has so far been omitted from this algorithm; we must ensure that A is located. This is done by requiring that A_{s+1} be contained in the radius 2^{-s} neighborhood around A_s; we must stay very close to A_s when backtracking to form A_{s+1}. This ensures that, at stage s, the distance function $d(x, A)$ can be approximated to within 2^{-s+1}, hence A is located.

To prove that A is an arc, it suffices to show that the active endpoint converges. There are infinitely many *true* stages s, such that nothing is enumerated into B below n_s after stage s. If s is a true stage, all future extensions to A_s occur between $\gamma_s^{\min}$ and $\gamma_s^{\max}$. But $lim_{s\to\infty}\gamma_s^{\max} - \gamma_s^{\min} = 0$. Also, after a true stage s, the second coordinate of the extensions is restricted to an interval of radius 2^{-s+1}. So, both coordinates of the active endpoint are converging, which proves that A is an arc. Note that the first coordinate of this endpoint is $\gamma = lim_{s\to\infty}\gamma_s^{\min} = \rho(B)$. Therefore, the endpoint is non-computable, as was required.

It is easy to see that we can prevent both endpoints from being computable. Let A' be the union of A, the reflection of A across the line $x = 3/2$, and the line segment joining $(1, 0)$ and $(2, 0)$. Then A' is a located arc with both endpoints non-computable. □

5 Computable Local Connection and Contractibility

Though we have been discussing effectively presented embeddings of spheres and balls, we have not yet considered the most natural notion of effectiveness: that the embedding map itself be computable. As the following example illustrates, this is a stronger notion than we have thus far investigated. Though the example given is an arc, it could easily be modified to show that a located embedded circle need not be computably embedded either.

Example 5.1 There is a located arc $A \subseteq \mathbb{R}^2$, with computable endpoints, which is not the image of the interval under a computable injection.

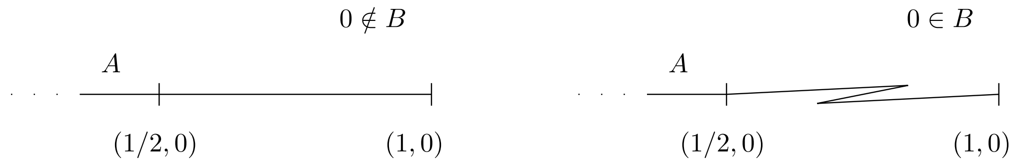

Fig. 4. There is a "switchback" in A between $(1/2, 0)$ and $(1, 0)$ iff $0 \in B$.

Proof. Let $B \in \mathbb{N}$ be a non-computable c.e. set. We will build a located arc $A \subseteq \mathbb{R}^2$ with endpoints at $(0, 0)$ and $(1, 0)$ such that the segment from $(1/2^i, 0)$ to $(1/2^{i+1}, 0)$ encodes whether or not $i \in B$. We restrict our attention to the segment from $(1/2, 0)$ to $(1, 0)$; all other segments are handled similarly and independently. If $0 \notin B$, then this portion of A will simply be the line from $(1/2, 0)$ to $(1, 0)$. On the other hand, if 0 is enumerated into B at stage s, then this portion of A will be the polygonal arc joining $(1/2, 0)$, $(5/6, 1/s)$, $(2/3, -1/s)$ and $(1, 0)$ (see fig. 4). Building A in this manner clearly produces an arc. At stage s we can approximate the distance function $d(x, A)$ to within $1/s$, hence A is also located.

Assume, for a contradiction, that A is the image of a computable injection $f : I \hookrightarrow \mathbb{R}^2$. Without loss of generality, assume that $f(0) = (0, 0)$ and $f(1) = (1, 0)$. For $q \in \mathbb{Q}$, consider the Π^0_1 class $C_q = f^{-1}[\{q\} \times \mathbb{R}]$. Then, $0 \notin B$ iff every member of $C_{2/3}$ is less than every member of $C_{5/6}$. But if this is true, then we will be able to detect it at some stage in the enumeration of the complements of $C_{2/3}$ and $C_{5/6}$. In other words, if $0 \notin B$, then we will eventually find out. Similarly for $i \notin B$; hence we can enumerate $\mathbb{N} \setminus B$. But this contradicts the non-computability of B. Therefore, A is not the image of I under a computable injection. □

It is interesting to note that, for the example given, there is a computable $f : I \to \mathbb{R}^2$ such that $f[I] = A$; such an f could commit to "switchbacking" on every segment, failing to be injective on the segment associated with i iff $i \notin B$. This proves, in particular, that there is a difference between effective *pathwise* and effective *arcwise* connection for Π^0_1 classes. As it turns out, the construction in Example 5.1 can easily be strengthened to ensure that A is not the the image of the interval under any computable map. If $i \in \mathbb{N}$ is enumerated into B at stage s, then we put s "switchbacks" into the segment associated to i. It is not difficult to verify that this new construction satisfies the stronger conclusion. With further modification to the construction, we can produce a located arc $A \subseteq \mathbb{R}^2$, with computable endpoints, such that every computable function $f : I \to A$ is constant.

There is an important effectiveness condition for arcs which precludes both Examples 4.1 and 5.1: *computable local connection*.

Definition 5.2 Let $\mathcal{P}$ be a property of subsets of $\mathbb{R}^n$. We say that a compact set $C \subseteq \mathbb{R}^n$ is *computably locally* $\mathcal{P}$ if there is a computable function $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ such that for any $\vec{x} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{Q}_+$, if $D = B(\vec{x}, f(q)) \cap C$ is nonempty, then there is a $\mathcal{P}$-subset of $B(\vec{x}, q) \cap C$ containing D.

Claim 5.3 *Let $A \subseteq \mathbb{R}^n$ be an arc. The following are equivalent:*

- *A is computably homeomorphic to the unit interval.*
- *A is located and computably locally connected.*
- *A is Π^0_1, has computable endpoints and is computably locally connected.*

The problem of characterizing the arcs which are computably homeomorphic to the unit interval was considered in [1], where a complicated set of conditions was given. Those conditions, in fact, form a proper superset of the conditions given here. We defer the proof of Claim 5.3, which is not particularly difficult, but which would be hard to generalize to higher dimensions. In order to generalize, we will need stronger connectivity assumptions; it is easy, for example, to produce a located, computably locally connected embedding of I^3 into $\mathbb{R}^3$ such that the boundary is not embedded as a Π^0_1 class. Under the assumption of *computable local contractibility*, all such pathologies disappear. Of course, for arcs, computable local contractibility is equivalent to computable local connection, so no additional assumption is being made.

Claim 5.4 *Let $D \subseteq \mathbb{R}^n$ be an embedded m-dimensional ball and let S be the image of the boundary sphere. If D is located and computably locally contractible, then S is a Π^0_1 class (hence located as well).*

Claim 5.5 *Let $D \subseteq \mathbb{R}^n$ be an embedded m-dimensional ball and let S be the image of the boundary sphere. The following are equivalent:*

- *D is computably homeomorphic to I^m.*
- *D is located and computably locally contractible.*
- *Both D and S are Π^0_1 classes and D is computably locally contractible.*

The corresponding claim for embedded spheres also holds.

Claim 5.6 *Let $S \subseteq \mathbb{R}^n$ be an embedded m-dimensional sphere. Then S is computably homeomorphic to the standard presentation of the m-dimensional sphere iff S is located and computably locally contractible.*

The methods of this paper are not up to the task of proving Claims 5.4–5.6. For that, we need more information than can be gleaned from the topology of the complement of a Π^0_1 class; we need to be able to work directly with the topology of the class itself. Under the assumption that the class is computably locally contractible, we can do exactly that. The method, which involves a careful analysis of approximations by finite open covers, will be described elsewhere.

References

[1] Brady, S. S., "Recursive topology in Euclidean space," Ph.D. dissertation, Cornell University (1984).

[2] Cenzer, D. and J. B. Remmel, *Index sets in computable analysis*, Theoret. Comput. Sci. **219** (1999), pp. 111–150, computability and complexity in analysis

(Castle Dagstuhl, 1997).

[3] Lacombe, D., *Les ensembles récursivement ouverts ou fermés, et leurs applications à l'analyse récursive*, C. R. Acad. Sci. Paris 245 (1957), 1040–1043 **246** (1957), pp. 28–31.

[4] Miller, J. S., "Co-c.e. classes in computable analysis and topology," Ph.D. dissertation, Cornell University (2002).

[5] Newman, M. H. A., "Elements of the topology of plane sets of points," Cambridge University Press, New York, 1961, vii+214 pp.

[6] Orevkov, V. P., *A constructive map of the square into itself, which moves every constructive point*, Dokl. Akad. Nauk SSSR **152** (1963), pp. 55–58.

[7] Pour-El, M. B. and J. I. Richards, "Computability in analysis and physics," Springer-Verlag, Berlin, 1989, xii+206 pp.

[8] Weihrauch, K., "Computable analysis, an introduction," Springer-Verlag, Berlin, 2000, x+285 pp.