THE DEGREES OF BI-HYPERHYPERIMMUNE SETS

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Abstract. We study the degrees of bi-hyperhyperimmune (bi-hhi) sets. Our main result characterizes these degrees as those that compute a function that is not dominated by any Δ^0_2 function, and equivalently, those that compute a weak 2-generic. These characterizations imply that the collection of bi-hhi Turing degrees is closed upwards.

1. Introduction

Csima and Kalimullin [3] gave an example of a structure whose degree spectrum is contained in the bi-hyperhyperimmune (bi-hhi) degrees. They suggested that its spectrum *might* be exactly the bi-hhi degrees, but they pointed out that it is not even known if bi-hyperhyperimmunity is closed upwards in the Turing degrees. It was this simple, if esoteric, question that motivated our paper. We prove that the collection of bi-hyperhyperimmune Turing degrees is closed upwards, and in fact, that it is a very natural degree class. Even so, it turns out that Csima and Kalimullin's structure does not capture these degrees (see Corollary 4.3).

To put into context the fact that the collection of bi-hyperhyperimmune degrees is closed upwards, note that the same is true of the bi-immune (Jockusch [8]) and bi-hyperimmune degrees. The latter follows from Kurtz [12], who showed that every hyperimmune degree contains a bi-hyperimmune set (hence as Jockusch noted, the hyperimmune and bi-hyperimmune degrees coincide), and the fact that the collection of hyperimmune degrees is closed upwards [13]. While it is certainly true that the collection of hyperhyperimmune degrees is closed upwards [7, 9], they do not coincide with the bi-hyperhyperimmune degrees.[1] Similarly, Jockusch [6] proved that the immune and bi-immune degrees do not coincide. This might lead us to look for a parallel between the bi-hhi degrees and the bi-immune degrees, but as it turns out, the bi-hyperimmune degrees offer a much better guide to the behavior of the bi-hhi degrees.

Basic Definitions. A *weak array* is a uniformly c.e. sequence $\mathcal{A} = \{F_n\}_{n\in\omega}$ of finite sets. We say that $\mathcal{A}$ is *disjoint* if its members are pairwise disjoint. A set $X \subseteq \omega$ *contains* $F \subseteq \omega$ if $F \subseteq X$. Similarly, X *avoids* $F \subseteq \omega$ if $F \cap X = \emptyset$. We say that X is *bi-hyperhyperimmune* (*bi-hhi*) if it both contains a member and avoids

Date: November 27, 2013.

2010 *Mathematics Subject Classification.* 03D30.

Gerdes would like to thank Johanna Franklin for bringing the problem to his (and ultimately our) attention. Miller is supported by the National Science Foundation under grant DMS-1001847.

[1] Every Δ^0_2 hyperhyperimmune set is strongly hyperhyperimmune (this is claimed by Cooper [1]; see Downey, Jockusch and Schupp [4] for a proof), implying that it is disjoint from an infinite c.e. set. Therefore, no Δ^0_2 set is bi-hyperhyperimmune. But there are Δ^0_2 hyperhyperimmune sets.

a member of every disjoint weak array. A degree $\mathbf{d}$ is *bi-hyperhyperimmune* if it contains a bi-hhi set.

It will be convenient to use a strong variant of bi-hyperhyperimmunity. If $F \subseteq \omega$ is finite, let $\widehat{F} = \{n \in \omega \mid \min(F) \leq n \leq \max(F)\}$. We say that $X \subseteq \omega$ *blockwise contains* a finite set $F \subseteq \omega$ if $\widehat{F} \subseteq X$. Similarly, X *blockwise avoids* a finite set $F \subseteq \omega$ if $\widehat{F} \cap X = \emptyset$. We say that X is *blockwise hyperhyperimmune* if it is infinite and blockwise avoids a member of every disjoint weak array. Call X *blockwise bi-hyperhyperimmune* if it and its complement are both blockwise hyperhyperimmune, i.e., if X both blockwise contains a member and blockwise avoids a member of every disjoint weak array $\mathcal{A}$. A degree $\mathbf{d}$ is *blockwise bi-hhi* if it contains a blockwise bi-hhi set. Blockwise bi-hyperhyperimmunity is somewhat easier to work with than bi-hyperhyperimmunity. In particular, we will be able to show directly that the collection of blockwise bi-hhi Turing degrees is closed upwards. We will see that the blockwise bi-hhi degrees coincide with the bi-hhi degrees (Theorem 1.1), but that not every bi-hhi set is blockwise bi-hhi (Theorem 4.5).

We say that $S \subseteq 2^{<\omega}$ is *dense* if every $\sigma \in 2^{<\omega}$ has an extension in S. A sequence $X \in 2^\omega$ is *weakly n-generic* if for every dense Σ^0_n set $S \subseteq 2^{<\omega}$, there is a prefix of X in S. We will primarily be concerned with weak 2-genericity. Finally, a function $f\colon \omega \to \omega$ is Δ^0_2 *escaping* if it is not dominated by any Δ^0_2 function.

The main theorem. We are ready to state the various characterizations of the bi-hyperhyperimmune degrees.

Theorem 1.1. *The following are equivalent for a Turing degree* $\mathbf{d}$:

(1) $\mathbf{d}$ *computes a* Δ^0_2 *escaping function.*
(2) $\mathbf{d}$ *computes a weakly* 2*-generic sequence.*
(3) $\mathbf{d}$ *contains a blockwise bi-hyperhyperimmune set.*
(4) $\mathbf{d}$ *contains a blockwise hyperhyperimmune set.*
(5) $\mathbf{d}$ *contains a bi-hyperhyperimmune set.*

Proof. (1) ⇒ (2) is Theorem 3.3. (2) ⇒ (3) follows from the fact that the collection of blockwise bi-hhi Turing degrees is closed upwards (Lemma 2.2), and the easy fact that every weak 2-generic is a blockwise bi-hhi set (Lemma 2.3). (3) ⇒ (4) and (3) ⇒ (5) are trivial. Finally, (4) ⇒ (1) and (5) ⇒ (1) are Lemmas 3.1 and 3.2, respectively. □

This result answers the question that motivated this work.

Corollary 1.2. *The collection of bi-hyperhyperimmune Turing degrees is closed upwards.*

Note that we have also shown that the blockwise hyperhyperimmune and blockwise bi-hyperhyperimmune degrees coincide.

The parallel between the bi-hyperimmune and bi-hhi degrees. Compare our main theorem to the following result.

Theorem 1.3 (Kurtz [11]). *The following are equivalent for a Turing degree* $\mathbf{d}$:

(1) $\mathbf{d}$ *computes a function that is not dominated by any computable function (i.e.,* $\mathbf{d}$ *has hyperimmune degree).*
(2) $\mathbf{d}$ *contains a weakly* 1*-generic sequence.*
(3) $\mathbf{d}$ *contains a bi-hyperimmune set.*

Some differences are clear. The collection of weakly 2-generic degrees is not closed upwards, so "computes" in part (2) of Theorem 1.1 cannot be replaced with "contains". More importantly, hyperhyperimmunity is not strong enough to guarantee that a degree is Δ^0_2 escaping, a fact that is closely related to the non-coincidence of the hyperhyperimmune and bi-hyperhyperimmune degrees.

Despite these differences, the similarity between the results is clear and it leads to a natural question: what is the right common generalization? No one has had the audacity to suggest a definition for bi-hyperhyperhyperimmunity, but perhaps we could prove that a degree is Δ^0_n escaping if and only if it computes a weak n-generic. This turns out not to be the case. In Theorem 4.6, we show that not every Δ^0_3 escaping function computes a weak 3-generic. In fact, no amount of "non-domination strength" is enough to guarantee that a function computes a weak 3-generic.

Structure of the paper. The proof of the main theorem is contained in Sections 2 and 3. The former focuses on the blockwise versions of hhi and bi-hhi, while the latter focuses on Δ^0_2 escaping functions. Section 4 starts with a collection of facts about the bi-hhi degrees. For example, we show that every bi-hyperhyperimmune degree is array non-computable and every Δ^0_3 degree strictly above $\emptyset'$ is bi-hhi. We finish the paper with three counterexamples: we prove that there is a bi-hhi degree that does not compute a presentation of Csima and Kalimullin's structure (mentioned above); we show that there is a bi-hhi set that is not blockwise bi-hhi; and we prove that there is a Δ^0_3 escaping function that does not compute a weak 3-generic. Related to the last example, in Theorem 3.4 we prove that every Δ^0_3 escaping function computes a 2-generic.

2. Blockwise (bi-)hyperhyperimmune sets

The definition of blockwise bi-hyperhyperimmunity might seem contrived, but in fact, it appears to be a fairly natural and robust notion. The following lemma, which gives two nice characterizations of blockwise hyperhyperimmune sets, helps make this case. We say that an array $\mathcal{A}$ is *finitely intersecting* if every $n \in \omega$ appears in at most finitely many members of $\mathcal{A}$.

Lemma 2.1. *The following are equivalent for an infinite set $X \subseteq \omega$:*

1. *X is blockwise hyperhyperimmune.*
2. *X avoids a member of every finitely intersecting weak array.*
3. *If $f\colon \omega \to \omega$ is any Δ^0_2 function, then $(\exists n)\ [n, f(n)] \cap X = \emptyset$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{A} = \{F_n\}_{n\in\omega}$ be a finitely intersecting weak array. We construct a disjoint weak array $\mathcal{B} = \{G_n\}_{n\in\omega}$ as follows. For each $n \in \omega$, we define G_n in stages. Let $G_{n,0} = \{\langle n, 0\rangle\}$. At a stage $s \in \omega$, take $m \in \omega$ to be least such that $F_{m,s}$ contains no element less than $\langle n, 0\rangle$. If $F_{m,s} \subseteq \widehat{G_{n,s}}$, then do nothing. Otherwise, put the least $\langle n, k\rangle \geq \max(F_{m,s})$ into $G_{n,s+1}$.

Note that, for each n, there are only finitely many $m \in \omega$ such that F_m contains an element less than $\langle n, 0\rangle$. Hence, the choice of m eventually stabilizes and G_n is finite. Thus $\mathcal{B}$ is a weak array. The definition of $\mathcal{B}$ ensures that its members are pairwise disjoint. By assumption, X blockwise avoids a member of $\mathcal{B}$. But for every $G \in \mathcal{B}$, there is an $F \in \mathcal{A}$ such that $F \subseteq \widehat{G}$. So X avoids a member of $\mathcal{A}$.

(2) $\Rightarrow$ (3) Let $f\colon \omega \to \omega$ be a Δ^0_2 function. Any total Δ^0_2 function is majorized by a function that is computably approximable from below, so we may assume

that f is itself computably approximable from below. Consider the weak array $\mathcal{A} = \{[n, f(n)]\}_{n\in\omega}$. Note that $\mathcal{A}$ is finitely intersecting because each $n \in \omega$ is in at most $n+1$ members of $\mathcal{A}$. So by assumption, there is an interval $[n, f(n)] \in \mathcal{A}$ such that $[n, f(n)] \cap X = \emptyset$.

(3) $\Rightarrow$ (1) Let $\mathcal{A} = \{F_n\}_{n\in\omega}$ be a disjoint weak array. Define a Δ^0_2 function $f\colon \omega \to \omega$ by $f(n) = \max\left(\bigcup_{m\leq n} F_m\right)$. By assumption, there is an $n \in \omega$ such that $[n, f(n)] \cap X = \emptyset$. It cannot be the case that $F_0, F_1, \ldots, F_n$ each contain an element less than n, so there is an $m \leq n$ such that $F_m \subseteq [n, f(n)]$, hence $F_m \cap X = \emptyset$. $\square$

Lemma 2.2. *The collection of blockwise bi-hyperhyperimmune Turing degrees is closed upwards.*

Proof. Assume that $X \subseteq \omega$ is blockwise bi-hyperhyperimmune and $D \geq_T X$. We want to code D into a new blockwise bi-hhi set $Z \subseteq \omega$. First, let

$$Y = X \oplus X = \{2n \mid n \in X\} \cup \{2n+1 \mid n \in X\}.$$

We claim that Y is blockwise bi-hhi. Let $f\colon \omega \to \omega$ be a Δ^0_2 function. Applying the condition in part (3) of Lemma 2.1 to $n \mapsto f(2n)$, there is an $n \in \omega$ such that $[n, f(2n)] \cap X = \emptyset$. Therefore, $[2n, f(2n)] \cap Y = \emptyset$. So Y is blockwise hhi. Similarly, $\omega \smallsetminus Y$ is blockwise hhi, so Y is blockwise bi-hhi.

The advantage of Y is that every maximal (under $\subseteq$) interval contained in Y has even length. Let $\{[n_i, m_i]\}_{i\in\omega}$ be the maximal intervals contained in Y, in the order that they appear. Let $Z = \bigcup_{i\in\omega}[n_i, m_i - D(i)]$. It is not hard to see that $Z \equiv_T D$. Clearly, $Z \leq_T D$, and $i \in D$ if and only if the ith maximal interval contained in Z has odd length. We claim that Z is blockwise bi-hhi. Because $Z \subseteq Y$, it is clearly blockwise hhi. Let $f\colon \omega \to \omega$ be a Δ^0_2 function. Since $f(n)+1$ is also a Δ^0_2 function and $\omega \smallsetminus Y$ is blockwise hhi, by Lemma 2.1 there is an $n \in \omega$ such that $[n, f(n)+1] \subseteq Y$. This implies that $[n, f(n)] \subseteq Z$, so Z is blockwise bi-hhi. $\square$

It is easy to see that blockwise bi-hyperhyperimmunity is a comeager property.

Lemma 2.3. *Weak* 2*-generic sets are blockwise bi-hyperhyperimmune.*

Proof. Assume that $X \subseteq \omega$ is weakly 2-generic. If $\mathcal{A} = \{F_n\}_{n\in\omega}$ is a disjoint weak array, then $\{\sigma \in 2^{<\omega} \mid (\exists n, s)(\forall t \geq s)(\forall m)[m \in \widehat{F_{n,t}} \to \sigma(m) = 1]\}$ is a dense Σ^0_2 set of strings. So X must blockwise contain a member of $\mathcal{A}$. Because $\omega \smallsetminus X$ is weakly 2-generic too, X must also blockwise avoid a member of $\mathcal{A}$. Therefore, X is blockwise bi-hhi. $\square$

3. Δ^0_2 escaping functions

In this section we finish the proof of the main theorem by proving the necessary facts about Δ^0_2 escaping functions. There are Δ^0_2 hyperhyperimmune sets, so it is not the case that every hyperhyperimmune set computes a Δ^0_2 escaping function. However, it is easy to see that blockwise hyperhyperimmunity is sufficient.

Lemma 3.1. *Every blockwise hyperhyperimmune set computes a Δ^0_2 escaping function.*

Proof. Assume that $X = \{x_0 < x_1 < x_2 < \cdots\}$ is blockwise hhi. Consider the X-computable function $g\colon \omega \to \omega$ defined by $g(n) = x_n$ for all $n \in \omega$. We claim that g is Δ^0_2 escaping. Let $f\colon \omega \to \omega$ be a Δ^0_2 function. There is an $n \in \omega$ such that

$[n, f(n)] \cap X = \emptyset$. This implies that $g(n) = x_n > f(n)$. Therefore, no Δ^0_2 function majorizes g, hence no Δ^0_2 function dominates g. □

We can prove that a hyperhyperimmune set computes a Δ^0_2 escaping function if we make a small assumption about its complement. Let $\omega^{[n]} = \{\langle n, m\rangle\}_{m\in\omega}$.

Lemma 3.2. *Assume that X is hyperhyperimmune and $(\forall n)\ \omega^{[n]} \cap X \neq \emptyset$. Then X computes a Δ^0_2 escaping function. In particular, every bi-hyperhyperimmune set computes a Δ^0_2 escaping function.*

Proof. For each $n \in \omega$, let $g(n)$ be the least $m \in \omega$ such that $\langle n, m\rangle \in X$. By assumption, $g\colon \omega \to \omega$ is total, hence X-computable. We claim that g is Δ^0_2 escaping. It is sufficient to show that no Δ^0_2 function majorizes g. Any total Δ^0_2 function is majorized by a function that is computably approximable from below, so in fact, it is sufficient to show that no function that is computably approximable from below majorizes g. Let $f\colon \omega \to \omega$ be such a function. Consider the disjoint weak array $\{\{\langle n, m\rangle \mid m \leq f(n)\}\}_{n\in\omega}$. If X avoids the n-th set in this array, then $g(n) > f(n)$. Thus, g is Δ^0_2 escaping.

The second part follows because if $X \subseteq \omega$ is bi-hhi, then its complement is immune. So $\omega \smallsetminus X$ does not contain the infinite c.e. set $\omega^{[n]}$, for each $n \in \omega$. □

The next result is the last and most technically involved step in the proof of Theorem 1.1.

Theorem 3.3. *Every Δ^0_2 escaping function computes a weak 2-generic.*

Proof. Let $g\colon \omega \to \omega$ be a Δ^0_2 escaping function. We may assume that g is increasing. Let $\{U_k\}_{k\in\omega}$ be an effective list of $\Sigma^0_1[\emptyset']$ sets of strings (not necessarily dense) and $U_{k,s}$ a uniformly computable Σ_2 approximation, i.e., $\sigma \in U_k \iff (\exists s)(\forall t \geq s)(\sigma \in U_{k,s})$. The goal is to construct a weakly 2-generic sequence $X \in 2^\omega$.

We define $X(n)$ at stage $n \in \omega$ of the construction. We may inductively assume that at the start of stage n, for each $k < n$ there is a distinguished string σ_k associated to U_k. Furthermore, the distinguished strings form a chain comparable to $X \upharpoonright n$ (not necessarily ordered by their indices). If $\sigma_k \notin U_{k,s}$ for any stage $s \in (g(n-1), g(n)]$, then declare σ_k to no longer be distinguished.

Now for each $k \leq n$ with no distinguished string, we want to find it one. Let τ be the longest element in the chain of $X \upharpoonright n$ and the currently distinguished strings. Find the $k \leq n$ lacking a distinguished string and the $\sigma \succcurlyeq \tau$ that minimize $\max\{|\sigma|, s(\sigma, k)\}$, where $s(\sigma, k)$ is the last stage $s \leq g(n)$ such that $\sigma \notin U_{k,s}$. In other words, we are looking for a short extension of τ that has looked like it is in U_k for a long time. Make $\sigma_k = \sigma$ the distinguished string for U_k. Let $\tau = \sigma_k$ and repeat this process until every $k \leq n$ has a distinguished string.

Note that the distinguished strings still form a chain comparable to $X \upharpoonright n$. Define $X(n)$ to preserve comparability with this chain. This completes the construction.

Verification. Assume that U_k is dense. We define a Δ^0_2 function $f\colon \omega \to \omega$ such that if $n \in \omega$ is the least number greater than or equal to k such that $g(n) \geq f(n)$, then at the end of stage n of the construction above, the distinguished string σ_k for U_k is contained in $U_{k,s}$ for all $s \geq g(n)$. Therefore, it will remain distinguished and will be a prefix of X.

Fix $n \geq k$. Assume that we know $g \upharpoonright n$; we will remove this assumption below. We want to define $f(n)$ to be large enough to lock in the distinguished string for U_k. By assumption, we know the sequence of distinguished strings at the start of stage n. Let $s_0 = g(n-1)$ (or 0, if $n = 0$). Take s_1 large enough that for every $j < n$, if σ_j has not permanently entered U_j by stage s_0, then it will leave before stage s_1. So, if $g(n) \geq f(n) \geq s_1$, the only distinguished strings that remain at the start of stage n are those that will never be canceled.

Let τ be the longest element in the chain of $X \upharpoonright n$ and the remaining distinguished strings. Pick the shortest $\sigma \in U_k$ that extends τ and let $s_2 \geq \max\{s_1, |\sigma|\}$ be large enough that σ has permanently entered U_k. Pick $s_3 \geq s_2$ large enough that, for all $j \leq n$, all elements of U_{j,s_2} of length at most s_2 that have not permanently entered U_j by stage s_2 will leave by stage s_3. Now if $g(n) \geq s_3$, then the only thing that stops σ from becoming the distinguished string for U_k is the presence of a better candidate σ_j for U_j, with $j \leq n$. Note that $s(\sigma_j, j) \leq \max\{|\sigma|, s(\sigma, k)\} \leq s_2$, so σ_j must have appeared by s_2. The fact that σ_j did not leave U_j by s_3 tells us that it is locked in as the distinguished string for U_j. If $j \neq k$, then repeat the process just described starting with s_1 equal to s_3. This process can only repeat finitely many times. After the final repetition, s_3 will be large enough to guarantee that if $g(n) \geq s_3$, then the distinguished string σ chosen for U_k is permanent. Let $f(n) = s_3$. Note that this whole process can be carried out with a $\emptyset'$ oracle.

We must remove the assumption that we know $g \upharpoonright n$. Take $g \upharpoonright k$ as given. For each $n \geq k$, assume that $g \upharpoonright [k, n)$ has not yet exceeded $f \upharpoonright [k, n)$. This gives us a finite number of possibilities for $g \upharpoonright n$; apply the process above to each case to find a sufficiently large value of $f(n)$, and let $f(n)$ be the maximum of these. So if $n \geq k$ is least such that $g(n) \geq f(n)$, then at stage n of the construction of X, we lock in a distinguished string $\sigma_k \in U_k$. This guarantees that $\sigma_k \prec X$. □

We will show in Theorem 4.6 that there is a Δ^0_3 escaping function that does not compute a weak 3-generic. Compare that to the following result.

Theorem 3.4. *Every Δ^0_3 escaping function computes a 2-generic.*

Proof. Given a Δ^0_3 escaping function $g\colon \omega \to \omega$, the construction is the same as above. The verification is almost the same, except that we define a Δ^0_3 function $f\colon \omega \to \omega$ for every U_k, whether or not it is dense. At the point in the definition of $f(n)$ that we want the shortest $\sigma \in U_k$ that extends τ, there is no longer a guarantee that such a σ exists. We ask $\emptyset''$ if there is a $\sigma \in U_k$ that extends τ. If so, we proceed as before, trying to ensure that there is a $\sigma_k \in U_k$ such that $\sigma_k \prec X$. If not, then finish with the definition of $f(n)$. In the latter case, if $g(n) \geq f(n)$, then U_k is not dense along X. □

4. Consequences and counterexamples

The next result collects several observations about the bi-hyperhyperimmune degrees, all of which follow easily from the work above.

Proposition 4.1.

(1) *Every bi-hyperhyperimmune degree is array non-computable. (Hence no array computable degree computes a weak 2-generic.)*
(2) *No bi-hyperhyperimmune has minimal Turing degree.*
(3) *Every Δ^0_3 degree strictly above $\emptyset'$ is bi-hyperhyperimmune.*
(4) *Not every $\Sigma^0_2 \setminus \Delta^0_2$ degree is bi-hyperhyperimmune.*
(5) *Every degree strictly above $\emptyset'$ computes a non-Δ^0_2 degree that is not bi-hyperhyperimmune.*

(6) *There are downward cones of bi-hyperhyperimmune degrees.*

Proof. (1) We have shown that every bi-hhi computes a Δ^0_2 escaping function. On the other hand, Downey, Jockusch and Stob [5] proved that all functions of array computable degree are dominated by the modulus function of $\emptyset'$.

(2) No weak 2-generic (indeed, no 1-generic) is minimal.

(3) Miller and Martin [13] showed that every non-computable Δ^0_2 degree is hyperimmune. Relativizing to $\emptyset'$, every Δ^0_3-degree strictly above $\emptyset'$ is hyperimmune relative to $\emptyset'$, hence Δ^0_2 escaping.

(4) Shore [14] and Cooper, Lewis and Yang [2] independently proved the existence of minimal degrees in $\Sigma^0_2 \smallsetminus \Delta^0_2$.

(5) There is a Δ^0_2 function tree $T\colon 2^{<\omega} \to 2^{<\omega}$ such that every path through T has minimal Turing degree. Let $D >_T \emptyset'$ and consider $T[D]$, which is minimal, hence does not have bi-hhi degree. Note that $T[D] \leq_T T \oplus D \leq_T \emptyset' \oplus D \leq_T D$. But $D \leq_T T[D] \oplus T$ and D is not Δ^0_2, so $T[D]$ is not Δ^0_2.

(6) Based on a result of Martin, Jockusch [10] proved that the 2-generic degrees are downwards dense, meaning that every non-computable degree below a 2-generic computes a 2-generic. This implies that every non-computable degree below a 2-generic is bi-hyperhyperimmune. ☐

We finish the paper with three counterexamples. The first gives a negative answer to Question 6.6 in Csima and Kalimullin [3]. If $n \in \omega$ and $F \subseteq \omega$, they defined $\{n\} \oplus F$ to be the following infinite graph. It consists of an ω-chain with an $(n+5)$-cycle linked to 0. For each $m \in F$, there is a 3-cycle linked to m, and for each $m \notin F$, there is a 4-cycle linked to m. Consider the graph $\mathcal{H}$ that is the disjoint union of all $\{n\} \oplus F$ such that

- $n \in \omega$,
- $F \subseteq \omega$ is finite, and
- if $\{W_{\varphi_n(m)}\}_{m\in\omega}$ is a disjoint weak array, then $(\exists m)\, W_{\varphi_n(m)} \subseteq F$.

Csima and Kalimullin proved that every degree that computes a copy of $\mathcal{H}$ is bi-hhi. They suggested that the *spectrum* of $\mathcal{H}$, i.e., the collection of degrees computing a copy of $\mathcal{H}$, might be exactly the bi-hyperhyperimmune degrees. This is not the case.

Proposition 4.2. *If* $\mathbf{a}$ *is in the spectrum of* $\mathcal{H}$ *then* $\mathbf{a}$ *is high*$_2$ *(i.e.,* $\mathbf{a}'' \geq \mathbf{0}'''$*).*

Proof. It is routine to check that

$$U = \{n \in \omega \mid \{W_{\varphi_n(m)}\}_{m\in\omega} \text{ is a disjoint weak array with no empty members}\}$$

is Π^0_3 complete. (The real complexity comes from the fact that every member of the array is finite, not from the completeness of φ_n, the disjointness of the array, or the nonemptiness of the members.) Note that $\{n\} \oplus \emptyset$ is a component of $\mathcal{H}$ if and only if $n \notin U$. But if $\mathbf{a}$ computes a copy of $\mathcal{H}$, then $\mathbf{a}''$ can determine if $\{n\} \oplus \emptyset$ is a component of $\mathcal{H}$. Therefore, $\mathbf{a}'' \geq \mathbf{0}'''$. ☐

Corollary 4.3. *There is a bi-hyperhyperimmune degree that does not compute a copy of* $\mathcal{H}$*.*

Proof. It is not hard to see that there is a bi-hyperhyperimmune degree that is not high$_2$. For example, if $A \leq_T \emptyset''$ is 2-generic, then $A'' \equiv_T A \oplus \emptyset'' \equiv_T \emptyset''$ [10]. Such an A has bi-hhi degree, but does not compute a copy of $\mathcal{H}$. ☐

Question 4.4. Is the collection of bi-hyperhyperimmune Turing degrees the spectrum of a structure?

There is a Δ^0_2 hyperhyperimmune set. On the other hand, we have proved that every blockwise hyperhyperimmune computes a Δ^0_2 escaping function, hence cannot be Δ^0_2. Therefore, there is a hhi set that is not blockwise hhi. This simple argument does not work to prove that there is a bi-hhi set that is not blockwise bi-hhi, as we know that the two properties agree degree-wise.

Theorem 4.5. *There is a bi-hyperhyperimmune set $X \subseteq \omega$ such that neither X nor $\omega \setminus X$ is blockwise hyperhyperimmune.*

Proof. Let $\{\{V_{k,j}\}_{j\in\omega} \mid k \in \omega\}$ be an effective list of all uniform sequences of disjoint c.e. sets. Define a Δ^0_2 function $f\colon \omega \to \omega$ as follows. Fix $n \in \omega$. For each $k < n$, let $j(k)$ be least such that $V_{k,j(k)} \cap [0, n] = \emptyset$. Note that $j(k) \leq n + 1$, so $\emptyset'$ can find it. If $V_{k,j(k)} = \emptyset$, then let $g(k) = 0$. Otherwise, let $g(k)$ be an element of $V_{k,j(k)}$. It should be clear that $\emptyset'$ can compute $g(k)$. For each $k, m < n$, define $i(k, m)$ such that $m \in V_{k,i(k,m)}$, if such an index exists. Let $h(k, m)$ be a member of $V_{k,i(k,m)}$ larger than n. If $h(k, m)$ is otherwise undefined, let $h(k, m) = 0$. Again, $\emptyset'$ can compute $h(k, m)$. Let $f(n) = \max\{n + 2, g(k), h(k, m)\}_{k,m<n}$.

Our goal is to construct a bi-hyperhyperimmune set $X \subseteq \omega$ that neither contains nor avoids an interval of the form $[n, f(n)]$. By Lemma 2.1, this ensures that neither X nor its complement is blockwise hyperhyperimmune. We construct X by initial segments. Let $\tau_0 = 01$. Assume that, at the beginning of stage $k \in \omega$, we have a string $\tau_k \in 2^{<\omega}$ that neither contains nor avoids an interval of the form $[n, f(n)]$. If $\mathcal{A} = \{V_{k,j}\}_{j\in\omega}$ has an infinite member, then it is not a weak array, so let $\tau_{k+1} = \tau_k$. If $\mathcal{A}$ has an empty member, then X automatically contains and avoids this member, so again let $\tau_{k+1} = \tau_k$. Otherwise, we want to extend τ_k to σ so that σ contains a member of $\mathcal{A}$. The definition of f forces $[n, f(n)]$ to have length at least three, meaning that it is always safe to extend τ_k by adding alternating ones and zeros, starting with whichever differs from the last bit of τ_k. So we may assume that $|\tau_k| > k$.

Let $p = |\tau_k|$. Choose $F \in \mathcal{A}$ such that $q = \min(F) > p$. Extend τ_k to a string ρ of length $q - 1$ by adding alternating ones and zeros. Define σ extending $\rho 0$ such that σ only adds zeros except at the positions in F, where it adds ones, and $|\sigma| = \max(F) + 1$. We claim that σ does not contain an interval of the form $[n, f(n)]$. If it does, then we must have $n \geq q$ and $n \in F$. But $\mathcal{A}$ has no empty member, so $g(k)$ is in a member of $\mathcal{A}$ disjoint from $[0, n]$. Therefore, $n < g(k)$ and $g(k) \notin F$, so $f(n) \geq g(k)$ implies that $[n, f(n)] \not\subseteq \sigma$. We also claim that σ does not avoid an interval of the form $[n, f(n)]$. If it does, then it must be the case that $n \in (q, \max(F))$. But $q \in F$ and $k, q < n$, so we define $h(k, q)$ to be an element of F larger than n, if possible. This is possible because $n < \max(F)$, so $f(n) \geq h(k, q)$ implies that $[n, f(n)] \cap \sigma \neq \emptyset$.

The process to extend σ to τ_{k+1} to avoid a member of $\mathcal{A}$, while preserving the property that it neither contains nor avoids an interval of the form $[n, f(n)]$, is completely symmetric. Let $X = \bigcup_{k\in\omega} \tau_k$. The construction ensures that X is bi-hyperhyperimmune but that neither X nor its complement is blockwise hyperhyperimmune. □

As mentioned in the introduction, Kurtz [11] proved that every function that is not dominated by a computable function, i.e., every Δ^0_1 escaping function, computes a weak 1-generic. We have shown that every Δ^0_2 escaping function computes a weak 2-generic. Our final counterexample shows that this pattern does not extend, at least not in the most naïve way. Recall that, by Theorem 3.4, every Δ^0_3 escaping function *does* compute a 2-generic.

Theorem 4.6. *There is a Δ^0_3 escaping function that does not compute a weak 3-generic.*

Proof. We build a Δ^0_3 function tree $f\colon \omega^{<\omega} \to \omega^{<\omega}$ such that if $h \in \omega^\omega$, then $f[h]$ does not compute a weak 3-generic. Furthermore, for all $\tau \in \omega^{<\omega}$ and $n \in \omega$, it will be the case that $f(\tau n)$ extends $f(\tau)m$ for some $m \geq n$. So, there is an $h \in \omega^\omega$ such that $f[h]$ is Δ^0_3 escaping.

Let $\{\tau_s\}_{s\in\omega}$ be an effective enumeration of $\omega^{<\omega}$ such that each string is enumerated only after its proper prefixes have been enumerated. In particular, τ_0 is the empty string. The construction of f is done relative to $\emptyset''$. We begin stage $s \in \omega$ of the construction with f defined on $\tau_0, \dots, \tau_s$. For each $t \leq s$, we have an associated infinite c.e. set of strings $V_{t,s}$ extending $f(\tau_t)$. When we define $f(\tau_t n)$, it will be an extension of an element of $V_{t,s}$. During the construction, we build a sequence of dense $\Sigma^0_1[\emptyset'']$ sets of binary strings $\{U_e\}_{e\in\omega}$. We ensure that if $h \in \omega^\omega$ and $\varphi_e^{f[h]} \in 2^\omega$, then no prefix of $\varphi_e^{f[h]}$ is in U_e. So $f[h]$ does not compute a weak 3-generic by φ_e, for each $e \in \omega$.

Let $f(\tau_0)$ be the empty string and let $V_{0,0} = \{m\}_{m\in\omega}$. At stage $s =< e, i >$, we make sure that every string of length i has an extension in U_e. Let $k = i+s+1$. For each $t \leq s$, use $\emptyset''$ to determine if there is a binary string ρ_t of length k such that infinitely many $\sigma \in V_{t,s}$ can be extended to strings σ' such that $\varphi_e^{\sigma'} \restriction k = \rho_t$. If so, let $V_{t,s+1}$ be an infinite set of such extensions (each extending a different element of $V_{t,s}$). If not, let ρ_t be undefined and let $V_{t,s+1}$ be $V_{t,s}$ without the finitely many σ that can be extended to σ' to make $\varphi_e^{\sigma'} \restriction k$ converge to a binary string. Put every binary string of length k into U_e except for $\rho_0, \dots, \rho_s$. This ensures that every binary string of length i has an extension in U_e. We have done this while not adding a prefix of any possible $\varphi_e^{f[h]}$. Assume that $\tau_{s+1} = \tau_t n$. Choose $\sigma \in V_{t,s+1}$ such that σ extends $f(\tau_t)m$ for some $m \geq n$. Remove σ from $V_{t,s+1}$ and let $f(\tau_{s+1}) = \sigma$. Let $V_{s+1,s+1} = \{\sigma m\}_{m\in\omega}$. This completes stage s.

The construction ensures that if $h \in \omega^\omega$, then $f[h]$ does not compute a weak 3-generic. As noted, h can be chosen so that $f[h]$ is Δ^0_3 escaping. □

The construction actually shows that no amount of "non-domination strength" is enough to compute a weak 3-generic. In particular, if $\{g_i\}_{i\in\omega}$ is any countable collection of functions, we can choose h so that $f[h]$ is not dominated by any g_i. It is still the case that $f[h]$ does not compute a weak 3-generic.

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